VERSION 02

Look at Solution of Version 01

1. Solve the following equation:

$$\begin{aligned} u_t &= u_{xx}, \quad 0 < x < 1; \\ u(0,t) = 0, & \quad u(1,t) = 0; \\ u(x,0) &= \sin(4\pi x) - 3\sin(6\pi x). \end{aligned}$$

(a) $e^{-16\pi^2 t}\sin(4\pi x) - 3e^{-36\pi^2 t}\sin(6\pi x)$

(b) $e^{-4\pi^2 t}\sin(4\pi x) - 3e^{-6\pi^2 t}\sin(6\pi x)$

(c) $e^{-4t}\sin(4x) - 3e^{-6t}\sin(6x)$

(d) $e^{-16t}\sin(4x) - 3e^{-36t}\sin(6x)$

(e) $e^{-16t}\sin(4\pi x) - 3e^{-36t}\sin(6\pi x)$

2. Solve the following equation:

$$\begin{aligned} u_t &= u_{xx}, \quad 0 < x < \pi; \\ u(0,t) = 0, & \quad u(\pi,t) = 0; \\ u(x,0) &= \begin{cases} 1 & \text{for } 0 \le x \le \frac{\pi}{2} \\ 0 & \text{for } \frac{\pi}{2} \le x \le \pi \end{cases} \end{aligned}$$

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\pi n}\left[1 - \cos\frac{n\pi}{2}\right] e^{-n^2\pi^2 t}\sin(n\pi x)$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\pi n}\left[1 - \cos(n\pi)\right] e^{-n^2\pi t}\sin(n\pi x)$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\pi n}\left[1 - \cos\frac{n\pi}{2}\right] e^{-n^2 t}\sin(nx)$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\pi n}\left[1 - \cos(n\pi)\right] e^{-n^2 t}\sin(nx)$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\pi n}\left[1 - \cos\frac{n\pi}{2}\right] e^{-n^2 t}\sin(nx)$

3. Find the steady state for

$$\begin{aligned} u_t &= u_{xx} + 5\sin(7\pi x), \quad 0 < x < 1; \\ u(0,t) = 0, \quad & u(1,t) = 0; \\ u(x,0) &= x(1-x). \end{aligned}$$

(a) the zero function

(b) $-\frac{x^3}{6} + \frac{x^4}{12}$

(c) $\frac{25}{49}\sin(7\pi x)$

(d) $\frac{5}{49\pi^2}\sin(7\pi x)$

(e) $-\frac{5}{49\pi^2}\sin(7\pi x)$

4. Find the steady state for

$$\begin{aligned} u_t &= u_{xx}, \quad 0 < x < 1; \\ u_x(0,t) = 0, \quad & u_x(1,t) = 0; \\ u(x,0) &= x(1-x). \end{aligned}$$

(a) $-\frac{x^3}{6} + \frac{x^4}{12}$

(b) $\frac{x^3}{6} - \frac{x^4}{12}$

(c) the constant function $\frac{1}{6}$

(d) the constant function $\frac{5}{6}$

(e) the zero function

5. Find the steady state for

$$\begin{aligned} u_t &= u_{xx}, \quad 0 < x < 1; \\ u(0,t) = 10, & \quad u(1,t) = 5; \\ u(x,0) &= x(1-x). \end{aligned}$$

(a) $10 - 5x + \dfrac{x^3}{6} - \dfrac{x^4}{12}$

(b) $10 - 5x + x(1-x)$

(c) $10 - 5x$

(d) the constant function 7.5

(e) the zero function

6. Solve the following equation:

$$\begin{aligned} u_{tt} &= u_{xx}, \quad 0 < x < 1; \\ u(0,t) = 0, & \quad u(1,t) = 0; \\ u(x,0) &= \sin(\pi x) \\ u_t(x,0) &= \sin(7\pi x) \end{aligned}$$

(a) $\cos(\pi t)\sin(\pi x) + \sin(7\pi t)\sin(7\pi x)$

(b) $\sin(\pi t)\sin(\pi x) + \cos(7\pi t)\sin(7\pi x)$

(c) $\cos(\pi t)\sin(\pi x) + \dfrac{1}{7\pi}\sin(7\pi t)\sin(7\pi x)$

(d) $\sin(\pi t)\sin(\pi x) + \dfrac{1}{7\pi}\cos(7\pi t)\sin(7\pi x)$

(e) $\cos(\pi t)\sin(\pi x) + 7\pi\sin(7\pi t)\sin(7\pi x)$

7. Consider the following equation (wave equation) with friction:

$$\begin{aligned} u_{tt} &= u_{xx} - u_t, \quad 0 < x < 1; \\ u(0,t) = 0, &\quad u(1,t) = 0; \\ u(x,0) &= \sin(\pi x) \\ u_t(x,0) &= 0 \end{aligned}$$

Find the steady state of the solution.

(a) $-\frac{1}{\pi^2}\sin(\pi x)$

(b) $\pi^2 \sin(\pi x)$

(c) $-\pi \sin(\pi x)$

(d) The zero function.

(e) There is no steady state as the elastic string will keep oscillating.

8. Find the Fourier series for the following function:

$$f(x) = \begin{cases} 1, & 0 < x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < x < 2\pi \end{cases}$$

and f is 2π-periodic.

(a) $\frac{1}{2} + \sum_{n=1}^{\infty} \left[\frac{1}{n}\sin(n\pi)\cos n\pi x + \frac{1}{n}(1-\cos(n\pi))\sin n\pi x\right]$

(b) $\frac{1}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty} \left[\frac{1}{n}\sin\frac{n\pi}{2}\cos n\pi x + \frac{1}{n}\left(1-\cos\frac{n\pi}{2}\right)\sin n\pi x\right]$

(c) $\frac{1}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty} \left[\frac{1}{n}\sin\frac{n\pi}{2}\cos nx + \frac{1}{n}\left(1-\cos\frac{n\pi}{2}\right)\sin nx\right]$

(d) $\frac{1}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty} \left[\frac{1}{n}\sin\frac{n\pi}{2}\cos nx + \frac{1}{n}\left(1-\cos\frac{n\pi}{2}\right)\sin nx\right]$

(e) $\frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \left[\frac{1}{n}\sin(n\pi)\cos nx + \frac{1}{n}(1-\cos(n\pi))\sin nx\right]$

9. Solve the following equation:

$$y'' + 2y' + 2y = \delta(t-1), \quad y(0) = 0, \quad y'(0) = 0.$$

(a) $e^{-(t-1)}\sin(t-1)u(t-1)$

(b) $e^{-(t-1)}\cos(t-1)u(t-1)$

(c) $\left(e^{-(t-1)} + \cos(t-1)\right)u(t-1)$

(d) $\left(e^{t-1} + \sin(t-1)\right)u(t-1)$

(e) $e^{t-1}\sin(t-1)u(t-1)$

10. Solve the following equation:

$$y' + 8y = 1 + e^{-6t}, \quad y(0) = 0.$$

(a) $\frac{1}{8} + \frac{1}{2}e^{6t} - \frac{5}{8}e^{8t}$

(b) $4 - e^{2t} + 3e^{8t}$

(c) $\frac{1}{8} + \frac{1}{2}e^{-6t} - \frac{5}{8}e^{-8t}$

(d) $8 - 2e^{-6t} + 6e^{-8t}$

(e) $4 - e^{-2t} + 3e^{-8t}$

11. Solve the following equation:

$$ty' + 3y = t + 3, \;\; y(1) = 7.$$

(a) $\dfrac{23t^3}{4} + 1 + \dfrac{1}{4t}$

(b) $\dfrac{23}{4t^3} + 1 + \dfrac{t}{4}$

(c) $\ln t + \dfrac{5}{4} + \dfrac{23}{4t}$

(d) $\ln t - \dfrac{5}{4} + \dfrac{33}{4t}$

(e) $t^3 \ln t - \dfrac{5}{4} + \dfrac{33}{4t}$

12. Solve the following equation:

$$y'' + 4y' + 4y = 0, \quad y(-1) = 0, \;\; y'(-1) = 3.$$

(a) $3e^{-2x} \cos 2x + e^{-2x} \sin 2x - 3$

(b) $3e^{2x} \cos 2x + e^{2x} \sin 2x - 3$

(c) $3e^{-2x} - 3e^{2x}$

(d) $3(1 + x)e^{-2x}$

(e) $3e^{-2}(1 + x)e^{-2x}$

13. Consider the following system with a parameter α:

$$X' = \begin{pmatrix} 3 & \alpha \\ 8 & -5 \end{pmatrix} X$$

Which of the following statement is true?

(a) As α increases and passes through -2,
the system changes from a saddle to a sink.

(b) As α increases and passes through -2,
the system changes from a spiral in to a spiral out.

(c) As α increases and passes through -2,
the system changes from a spiral in to a source.

(d) As α increases and passes through $-\frac{15}{8}$,
the system changes from a sink to a saddle point.

(e) As α increases and passes through $-\frac{15}{8}$,
the system changes from a sink to a source point.

14. Consider the following system of differential equation:

$$X' = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} X + \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

You are given that $e^{At} = \begin{pmatrix} 1-t & t \\ -t & t+1 \end{pmatrix}$, find the solution $X(t)$.

(a) $\begin{pmatrix} t \\ t^2+t \end{pmatrix}$

(b) $\begin{pmatrix} t+t^2 \\ t^2+\frac{t}{2} \end{pmatrix}$

(c) $\begin{pmatrix} t^2 \\ \frac{t^2}{2}-t \end{pmatrix}$

(d) $\begin{pmatrix} t-t^2 \\ -t^2-t \end{pmatrix}$

(e) $\begin{pmatrix} t^2+t \\ -t^2-t \end{pmatrix}$

15. Find the Laplace transform of the function $f(t)$ which equals $(t-1)^2$ for $0 \leq t \leq 1$ and 0 otherwise.

(a) $\dfrac{1-e^{-s}}{s^3} - \dfrac{2}{s^2} + \dfrac{1}{s}$.

(b) $\dfrac{2}{s^3} - \dfrac{(1-e^{-s})}{s^2} + \dfrac{1}{s}$.

(c) $\dfrac{2}{s^3} - \dfrac{2}{s^2} + \dfrac{1-e^{-s}}{s}$.

(d) $\dfrac{2(1-e^{-s})}{s^3} - \dfrac{2}{s^2} + \dfrac{1}{s}$.

(e) $\dfrac{2(1-e^{-s})}{s^3} + \dfrac{2}{s^2} - \dfrac{1}{s}$.

16. Find the inverse Laplace transform of the following function:

$$\frac{s+1}{s^2+4s+13}$$

(a) $e^{-2t}\cos(3t) + e^{-2t}\sin(3t)$

(b) $e^{-2t}\cos(3t) + 3e^{-2t}\sin(3t)$

(c) $e^{-2t}\cos(3t) - \frac{1}{3}e^{-2t}\sin(3t)$

(d) $e^{3t}\cos(2t) - 3e^{3t}\sin(2t)$

(e) $\frac{1}{3}e^{3t}\cos(2t) + e^{3t}\sin(2t)$

17. Find the inverse Laplace transform of the following function:

$$\frac{1}{s^2(s+1)^2}$$

(a) $2 + t + te^{-t}$

(b) $-2 + t + (2+t)e^{-t}$

(c) $t + (1-t)e^{-t}$

(d) $\frac{t}{2} + (1-t)e^{-t}$

(e) $\frac{t}{2} + (1+t)e^{-t}$

18. The convolution of t and e^{-t} equals:

(a) $t + te^{-t}$

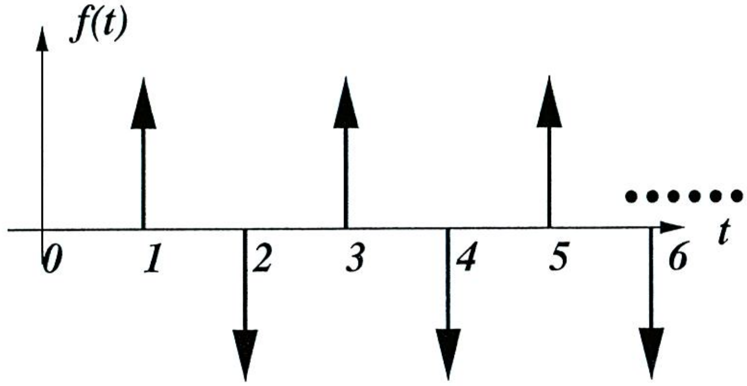
(b) $1 + te^{-t}$

(c) $(t-1)e^{-t}$

(d) $-1 + t + e^{-t}$

(e) $-1 + te^{-t}$

19. Find the Laplace transform of f which consists of an infinite sequence of positive delta functions at $t = 1, 2, 3, \ldots$ and negative delta functions at $t = 2, 4, 6, \ldots$:



(a) $\dfrac{1}{1+e^{-2s}}$

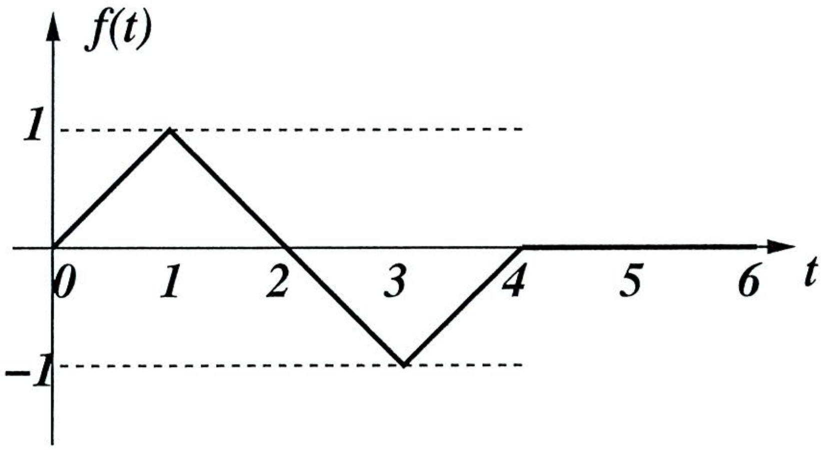
(b) $\dfrac{1}{1-e^{-2s}}$

(c) $\dfrac{1-e^{-s}}{1+e^{-2s}}$

(d) $\dfrac{e^{-s}}{1+e^{-s}}$

(e) $\dfrac{1}{1-e^{-s}} - \dfrac{1}{1+e^{-s}}$

20. Find the Laplace transform of the following function:



(a) $\left(1+\frac{1}{s^2}\right)e^{-s}-\frac{1}{s^2}e^{-3s}-e^{-4s}$

(b) $\left(1-\frac{1}{s^2}\right)e^{-s}+\frac{1}{s^2}e^{-3s}+e^{-4s}$

(c) $\frac{1}{s^2}\left[1+2e^{-s}-2e^{-3s}-e^{-4s}\right]$

(d) $\frac{1}{s^2}\left[1-2e^{-s}+2e^{-3s}-e^{-4s}\right]$

(e) $\frac{1}{s^2}\left[1-3e^{-s}+3e^{-3s}-e^{-4s}\right]$

21. Solve the following differential equation:

$$y' + y = \begin{cases} 1 & \text{for } 0 < t < 1 \\ 0 & \text{for } t > 1 \end{cases}, \quad y(0) = 0$$

(a) $\frac{1}{2}(1 - e^{-t})u(t) - \frac{1}{2}(1 - e^{-(t-1)})u(t-1)$

(b) $\frac{1}{2}(1 - e^{-t})u(t) + \frac{1}{2}(1 - e^{-(t-1)})u(t-1)$

(c) $(1 + e^{t})u(t) - (1 + e^{(t-1)})u(t-1)$

(d) $(1 - e^{t})u(t) - (1 - e^{(t-1)})u(t-1)$

(e) $(1 - e^{-t})u(t) - (1 - e^{-(t-1)})u(t-1)$

22. Find the Laplace transform of $te^{t}\sin(2t)$.

(a) $\frac{4s-2}{(s-1)^2(s^2+4)}$

(b) $\frac{4s+4}{(s-1)^2(s^2+4)}$

(c) $-\frac{4}{(s-1)^2(s^2+4)}$

(d) $\frac{4(s-1)}{((s-1)^2+4)^2}$

(e) $-\frac{4(s-1)}{((s-1)^2+4)^2}$

23. Consider the following improved Euler scheme for solving: $y' = f(t, y)$:

given y_n, the next value y_{n+1} is obtained by the following procedure:

$$
\begin{aligned}
\tilde{y}_{n+1} &= y_n + f(t_n, y_n)(t_{n+1} - t_n) \\
y_{n+1} &= y_n + \frac{1}{2}\Big(f(t_n, y_n) + f(t_{n+1}, \cancel{\tilde{y}_n})\Big)(t_{n+1} - t_n)
\end{aligned}
$$

$\tilde{y}_{n+1}$

If you use the above scheme to solve $y' = y$ with $y(0) = 1$ and uniform time steps: $t_{n+1} - t_n = h$, then what is the exact value of y_1?

(a) $y_1 = \dfrac{1}{1+h}$

(b) $y_1 = \dfrac{1}{1-h} + \dfrac{h^2}{2}$

(c) $y_1 = 1 + h + \dfrac{h^2}{2}$

(d) $y_1 = \dfrac{1}{1+h} + 2h + \dfrac{h^2}{2}$

(e) $y_1 = 1 + h + \dfrac{h^2}{2} + \dfrac{h^3}{6}$

24. Consider the following matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Which of the following statement is true?

(a) 2 is an eigenvalue with three linearly independent eigenvectors and 1 is an eigenvalue with only one eigenvector.

(b) 2 is an eigenvalue with only one eigenvector and 1 is an eigenvalue with three linearly independent eigenvectors.

(c) 2 is an eigenvalue with two linearly independent eigenvectors and 1 is an eigenvalue with two linearly independent eigenvectors.

(d) 2 is an eigenvalue with two linearly independent eigenvectors and 1 is an eigenvalue with only one eigenvector.

(e) 2 is an eigenvalue with only one eigenvector and 1 is an eigenvalue with two linearly independent eigenvectors.

25. Find the general solution of the following system:

$$X' = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} X$$

(a) $c_1 e^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 e^t \begin{pmatrix} \frac{1}{3} \\ 0 \\ 0 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(b) $c_1 e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{3t} \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} \frac{1}{3} \\ 0 \\ 0 \end{pmatrix} \right]$

(c) $c_1 e^{3t} \begin{pmatrix} \frac{1}{3} \\ 0 \\ 0 \end{pmatrix} + c_2 e^{3t} \left[t \begin{pmatrix} \frac{1}{3} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] + c_3 e^t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(d) $c_1 e^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 e^t \left[t \begin{pmatrix} \frac{1}{3} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] + c_3 e^{3t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(e) $c_1 e^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 e^t \left[t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ 0 \\ 0 \end{pmatrix} \right] + c_3 e^{3t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$