MA 303 Review (Fall 2011, Yip)

Note Title 12/9/2011

From 1st course in differential equation:

(1) Characteristic polynomial:

$$ay'' + by' + cy = 0 \implies ar^2 + br + c = 0$$
$$\implies r_1, r_2$$

r_1, r_2 real, $r_1 \neq r_2 \implies y = Ae^{r_1x} + Be^{r_2x}$

$r_1 = r_2 = r \implies y = Ae^{rx} + Bre^{rx}$

$r_1, r_2 = \alpha \pm \beta i \implies y = Ae^{rx}\cos\beta x + Be^{rx}\sin\beta x$

(2) Separation of variable: $\frac{dy}{dx} = f(x)g(y)$

$$\Rightarrow \int \frac{dy}{g(y)} = \int f(x)\,dx$$

(3) Exponential function:

$$y' = ay \Longrightarrow y(x) = y(0)e^{ax}$$

(4) Integrating factor:

$$y' = ay + b(t)$$

$$y(t) = y(0)e^{at} + e^{at}\int_0^t e^{-as} b(s)\,ds$$

(5) Generalization of (4)

$$y' = p(t)\, y + q(t),$$

$$y(t) = e^{\int_0^t p(r)\,dr} y(0) + e^{\int_0^t p(r)\,dr} \int_0^t e^{-\int_0^s p(r)\,dr} q(s)\, ds$$

$$y(t) = e^{\int_0^t p(r)\,dr} y(0) + \int_0^t e^{\int_s^t p(r)\,dr} q(s)\, ds$$

Euler Identity:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Solving Systems of Differential Equations

$$\frac{dX}{dt} = AX, \qquad X(0) = X_0 \quad \text{(Homog Eqn)}$$

$$\frac{dX}{dt} = AX + b(t), \qquad X(0) = X_0 \quad \text{(Inhomog. Eqn)}$$

(Higher order eqn $\Longrightarrow$ 1st order system)

Eigenvector expansion: (A – real matrix)

$$AV_i = \lambda_i V_i$$

If λ_i – distinct, real $\qquad \boxed{c_i e^{\lambda_i t} V_i}$

If $\lambda_i = \alpha + \beta i$ $\qquad V_i = \vec{u} + i\vec{w}$

(paired with $\lambda_j = \alpha - \beta i$), $\qquad V_j = \vec{u} - i\vec{w}$

$$\boxed{e^{\alpha t}\left\{c_1\left[\cos\beta t\,\vec{u} - \sin\beta t\,\vec{w}\right] + c_2\left[\sin\beta t\,\vec{u} + \cos\beta t\,\vec{w}\right]\right\}}$$

If λ_i repeats twice, (defective case)

then corresponding general solution is:

$$\boxed{\begin{gathered} c_1 e^{\lambda_i t} V_i + c_2 e^{\lambda_i t} \left[t V_i + \tilde{\eta} \right] \\ \text{where } (A - \lambda I)\tilde{\eta} = V_i \end{gathered}}$$

Phase Plot & Stability (2x2 system)

$\lambda_1 > \lambda_2 > 0 \implies$ source (unstable)

$\lambda_1 > 0 > \lambda_2 \implies$ saddle pt

$0 > \lambda_1 > \lambda_2 \implies$ sink (asym. stable)

$\lambda_1, \lambda_2 = \alpha \pm \beta i$; $\alpha > 0 \implies$ spiral out (unstable)

$\lambda_1, \lambda_2 = \alpha \pm \beta i$; $\alpha < 0 \implies$ spiral in (asym. stable)

$\lambda_1, \lambda_2 = \pm \beta i$, $(\alpha = 0) \implies$ center (stable)

Other cases:

$\lambda_1 = \lambda_2 > 0$ source (improper node)

(unstable)

$\lambda_1 = \lambda_2 < 0$ sink (improper node)

(asymptotically stable)

Matrix exponential e^{At}

- $e^{At} = \mathcal{L}^{-1}\left[(sI-A)^{-1}\right]$

where $Av_i = \lambda_i v_i$

- Non-defective A:

$$e^{At} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^{-1}$$

- $e^{At} = \left[\Phi(t)\right]\left[\Phi(0)\right]^{-1}$

where $\left[\Phi(t)\right] = \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \end{bmatrix}$, $x_i(t)$ - general solution

Solution formula using e^{At}:

$$\frac{dX}{dt} = AX + \vec{b}(t), \quad X(0) = X_0$$

$$\boxed{X(t) = e^{At}X_0 + e^{At}\int_0^t e^{-As}\,\vec{b}(s)\,ds}$$

Nonlinear System.

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases}$$

Critical pt $P = (x_0, y_0)$: $\quad f(x_0, y_0) = 0$

$\qquad g(x_0, y_0) = 0$

Linearized System at $P = (x_0, y_0)$:

$(\bar{x} = x - x_0,\ \bar{y} = y - y_0)$:

$$\frac{d}{dt}\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}\Bigg|_P \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}$$

Nonlinear stability = Linearized stability

except when some eigenvalues equal zero

or some eigenvalues lie on the purely imaginary axis

In these cases, need further more detailed analysis.

Laplace Transform (Properties !!!)

$$\mathcal{L}(f^{(n)}) = s^n \mathcal{L}(f) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

$$\mathcal{L}((-t) f(t)) = F(s)' \qquad F(s) = \mathcal{L}(f)$$

$$\mathcal{L}(e^{at} f(t)) = F(s-a)$$

$$\mathcal{L}(f(\underline{t-a})\, u(\underline{t-a})) = e^{at} F(s) \leftarrow (a > 0)$$

$$\begin{cases} (f * g)(t) = \displaystyle\int_0^t f(\tau)\, g(t-\tau)\, d\tau \\ \qquad\qquad = \displaystyle\int_0^t f(t-\tau)\, g(\tau)\, d\tau \\ \mathcal{L}(f * g) = \mathcal{L}(f)\,\mathcal{L}(g) \end{cases}$$

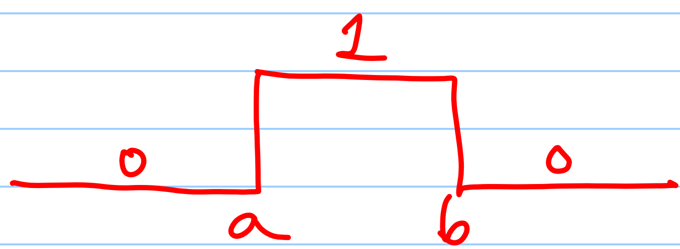
<u>Note</u>

$$\mathcal{L}(f(t)\, g(t)) \neq \mathcal{L}(f)\,\mathcal{L}(g)$$

To compute Laplace Transform:

- Use Table
- Make use of Properties
- Partial fraction.
- Window fct: $\boxed{0 < a < b}$

$$W_{a,b}(t) = \begin{cases} 1 & t \in (a,b) \\ 0 & t \notin (a,b) \end{cases}$$



$$= u(t-a) - u(t-b)$$

<u>Numerical Method</u> : $y' = f(t, y)$

1^{st} order $\underbrace{\text{Euler}}_{\text{forward}}$ Scheme: $\boxed{y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n)}$

1^{st} order backward Euler Scheme:

$$\boxed{y_{n+1} = y_n + f(t_{n+1}, y_{n+1})(t_{n+1} - t_n)}$$

Improved Euler:

$$\boxed{\begin{aligned} \tilde{y}_{n+1} &= y_n + f(t_n, y_n)(t_{n+1} - t_n) \\ y_{n+1} &= y_n + \frac{1}{2}\left[f(t_n, y_n) + f(t_{n+1}, \tilde{y}_{n+1})\right](t_{n+1} - t_n) \end{aligned}}$$

Fourier Series (2L-periodic function)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \qquad \left(\int_{-L}^{L} 1\cdot 1\,dx = 2L\right)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad \left(\int_{-L}^{L} \cos^2\frac{n\pi x}{L}\,dx = L\right)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad \left(\int_{-L}^{L} \sin^2\frac{n\pi x}{L}\,dx = L\right)$$

Textbook:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

Expansion using orthogonal functions $\{e_n\}$

$$f = c_1 e_1 + c_2 e_2 + \cdots$$

$$c_n = \frac{\langle f, e_n \rangle}{\langle e_n, e_n \rangle}$$

Usually, $\langle f, g \rangle = \int_{[\ ]} f(x) g(x)\, dx$

Even function : $\boxed{f(x) = f(-x)}$

$$f(x) = a_0 + \sum_n a_n \cos \frac{n\pi x}{L}$$

Odd function : $\boxed{f(x) = -f(-x)}$

$$f(x) = \sum_n a_n \sin \frac{n\pi x}{L}$$

Heat Equation

$$u_t = D u_{xx} \qquad 0 < x < L$$

$$u(x,0) = f(x)$$

Boundary condition:

Dirichlet $u(0,t)=0, \quad u(L,t)=0$

Neumann : $u_x(0,t)=0, \quad u_x(L,t)=0$

Mixed : $u(0,t)=0, \quad u_x(L,t)=0$

or $u_x(0,t)=0, \quad u(0,t)=0$

Expand $u(x,t)$ using eigenfunction:

Dir: $\left\{ \sin \frac{n\pi x}{L} \right\}_{n=1}^{\infty}$

Neumann: $\left\{ \cos \frac{n\pi x}{L} \right\}_{n=0}^{\infty} = \left\{ 1, \cos \frac{n\pi x}{L} \right\}_{n=1}^{\infty}$

Mixed: $\left\{ \sin \left(\frac{2n+1}{2}\right) \frac{\pi x}{L} \right\}_{n=0}^{\infty}$ or $\left\{ \cos \left(\frac{2n+1}{2}\right) \frac{\pi x}{L} \right\}_{n=0}^{\infty}$

Solution of heat equation

$$u_t = D u_{xx} \qquad 0 < x < L$$

$$u(x,0) = f(x)$$

$$u(0,t) = 0, \quad u(L,t) = 0$$

Dir. B.C.

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\frac{Dn^2\pi^2}{L^2}t} \boxed{\sin\frac{n\pi x}{L}}$$

$t=0$

$$f(x) = \sum_n c_n \sin\frac{n\pi x}{L}$$

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Note:

$$\int_0^L \sin^2\frac{n\pi x}{L}\,dx = L/2$$

Neumann B.C. $u_x(0,t)=0, \quad u_x(L,t)=0$

$$u(x,t)=\sum_{n=0}^{\infty} c_n e^{-\frac{Dn^2\pi^2}{L^2}t} \boxed{\cos\frac{n\pi x}{L}}$$

$$= c_0 \cdot 1 + \sum_{n=1}^{\infty} c_n e^{-\frac{Dn^2\pi^2}{L^2}t} \cos\frac{n\pi x}{L}$$

$t=0 \Rightarrow f(x)= c_0 \cdot 1 + \sum_{n=1}^{\infty} c_n \cos\frac{n\pi x}{L}$

$c_0 = \frac{1}{L}\int_0^L f(x)\cdot 1\,dx, \qquad c_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$

$$L = \int_0^L 1 \cdot 1 \, dx$$

$$\frac{L}{2} = \int_0^L \cos^2 \frac{n\pi x}{L} \, dx$$

Inhomogeneous Heat Equation

① Inhomogeneous B.C. : use off-set function

② Inhomogeneous equation : $u_t = D u_{xx} + h(x,t)$

expand $h(x,t)$ using eigenfunctions.

Wave Equation (Dir B.C.)

$$u_{tt} = a^2 u_{xx} \qquad 0 < x < L$$

$$u(0,t) = 0, \quad u(L,t) = 0$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

Solution:

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\frac{n\pi a t}{L} + B_n \sin\frac{n\pi a t}{L} \right) \sin\frac{n\pi x}{L}$$

$$A_n = \frac{\int_0^L f(x) \sin\frac{n\pi x}{L}\,dx}{\int_0^L \sin^2\frac{n\pi x}{L}\,dx = L/2}; \qquad B_n = \frac{1}{\left(\frac{n\pi a}{L}\right)} \frac{\int_0^L g(x) \sin\frac{n\pi x}{L}\,dx}{L/2 = \int_0^L \sin^2\frac{n\pi x}{L}\,dx}$$