Worked Out Homework 1
MA 303 Fall 2011 (Aaron N. K. Yip)
Friday, Sept. 2, in class

1. (a) Do Textbook (Boyce-DiPrima, 9th-ed.) section 3.3, page 165, #34 (on Euler Equation)

   (b) Use the above result to find the *general solutions* $y(t)$ of the the following differential equations:

   i. $t^2y'' + ty' + y = 0$

   ii. $t^2y'' + 4ty' + 2y = 0$

   iii. $t^2y'' - 3ty' - 6y = 0$

   iv. $t^2y'' - ty' + 5y = 0$

   v. $t^2y'' + 3ty' + y = 0$

   vi. $t^2y'' - 3ty' + 4y = 0$

2. Find the eigenvalues and eigenvectors of the following matrices:

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

In each case and for each eigenvalue, determine the algebraic multiplicity (i.e. the number of times the eigenvalue repeats) and the geometric multiplicity (i.e. the number of linearly independent eigenvectors).

MA303 Fall 2011 (Yip) HW 1 Solution

#1 1a) $t^2\dfrac{d^2y}{dt^2} + \alpha t\dfrac{dy}{dt} + \beta y = 0$

Let $x = \ln t$, change variable t to x:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \qquad \text{(Change rule)}$$

$$= \frac{dy}{dx}\frac{d}{dt}\ln t = \frac{dy}{dx}\frac{1}{t}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right)$$

$$= \frac{d}{dt}\left[\frac{dy}{dx}\frac{1}{t}\right]$$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{1}{t} + \frac{dy}{dx}\left(\frac{d}{dt}\frac{1}{t}\right)$$

$$= \frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{dt}\frac{1}{t} + \frac{dy}{dx}\left(-\frac{1}{t^2}\right)$$

$$= \frac{d^2y}{dx^2}\left(\frac{1}{t}\right)^2 - \frac{1}{t^2}\frac{dy}{dx}$$

Hence $\cancel{t^2}\left[\frac{d^2y}{dx^2}\cancel{\frac{1}{t^2}} - \cancel{\frac{1}{t^2}}\frac{dy}{dx}\right]$

$$+\alpha\cancel{t}\left(\frac{dy}{dx}\cancel{\frac{1}{t}}\right) + \beta y = 0$$

$$\frac{d^2y}{dx^2} + (\alpha-1)\frac{dy}{dx} + \beta y = 0$$

(b) (i) $t^2y'' + ty' + y = 0$ $\quad y = y(t)$

$\alpha = 1,\ \beta = 1$

$\boxed{\text{in term of } x}$:

$$y''(x) + (1-1)\,y'(x) + y(x) = 0$$

$$y''(x) + y(x) = 0$$

$$r^2 + 1 = 0 \Rightarrow r = \pm i$$

$$y(x) = c_1\cos x + c_2\sin x$$

$$\boxed{y(t) = c_1\cos \ln t + c_2\sin \ln t}$$

(ii) $t^2y'' + 4ty' + 2y = 0$ $\quad \alpha = 4,\ \beta = 2$

$$\Downarrow$$

$$y''(x) + 3y'(x) + 2y = 0$$

$r^2 + 3r + 2 = 0 \Rightarrow (r+2)(r+1) = 0$

$r = -1, -2$

$$y(x) = c_1 e^{-x} + c_2 e^{-2x}$$

$$y(t) = c_1 e^{-\ln t} + c_2 e^{-2\ln t}$$

$$\boxed{y(t) = \frac{c_1}{t} + \frac{c_2}{t^2}}$$

$$\boxed{e^{c\ln t} = t^c}$$

(iii) $t^2y'' - 3ty' - 6y = 0$

$$\Downarrow$$

$$y''(x) - 4y'(x) - 6y = 0$$

$r^2 - 4r - 6 = 0, \quad r = \frac{4 \pm \sqrt{16+24}}{2} = 2 \pm \sqrt{10}$

$$y(x) = c_1 e^{(2+\sqrt{10})x} + c_2 e^{(2-\sqrt{10})x}$$

$$y(t) = c_1 e^{(2+\sqrt{10})\ln t} + c_2 e^{(2-\sqrt{10})\ln t}$$

$$= \boxed{c_1 t^{(2+\sqrt{10})} + c_2 t^{(2-\sqrt{10})}}$$

(iv) $$t^2 y'' - t y' + 5y = 0$$

$$\Downarrow$$

$$y''(x) - 2y'(x) + 5y = 0$$

$$r^2 - 2r + 5 = 0 \qquad r = \frac{2 \pm \sqrt{4-20}}{2}$$

$$= 1 \pm 2i$$

$$y(x) = c_1 e^x \cos 2x + c_2 e^x \sin 2x$$

$$y(t) = c_1 e^{\ln t} \cos 2\ln t + c_2 e^{\ln t} \sin 2\ln t$$

$$= \boxed{c_1 t \cos 2\ln t + c_2 t \sin 2\ln t}$$

(V) $t^2y'' + 3ty' + y = 0 \qquad y = y(t)$

$$\Downarrow$$

$$y''(x) + 2y'(x) + y = 0$$

$$r^2 + 2r + 1 = 0 \Rightarrow r = -1, -1$$

$$y(x) = c_1 e^{-x} + c_2 e^{-x} x$$

$$y(t) = c_1 e^{-\ln t} + c_2 e^{-\ln t} \ln t$$

$$\boxed{= \frac{c_1}{t} + \frac{c_2 \ln t}{t}}$$

(Vi) $t^2 y'' - 3ty' + 4y = 0$

$$\Downarrow$$

$$y''(x) - 4y'(x) + 4y = 0$$

$$r^2 - 4r + 4 = 0 \qquad (r-2)^2 = 0 \qquad r = 2, 2$$

$$y(x) = c_1 e^{-2x} + c_2 x e^{-2x}$$

$$y(t) = c_1 e^{-2\ln t} + c_2 (\ln t) e^{-2\ln t} = \boxed{\frac{c_1}{t^2} + \frac{c_2 \ln t}{t^2}}$$

#2 $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$\det[A-\lambda I] = \begin{bmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} = (3-\lambda)^3 = 0$

$\lambda = 3, 3, 3$

$m=3$

Solve $(A-3I)V = 0$ $\quad V = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$

$\Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \beta = 0$

α, γ - free

$\boxed{\begin{matrix} \lambda=3 \\ m=3 \\ k=2 \end{matrix}}$

$\alpha=1, \gamma=0 \Rightarrow V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\alpha=0, \gamma=1 \Rightarrow V_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad$ geom mult. $(k) = 2$

Note: Algebraic mult. (m) = # of times λ repeats

(k) Geometric Mult = # of linearly independent eigenvectors

(also = # of free variables.)

$$B = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\det(B-\lambda I) = \det\begin{bmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{bmatrix} = (3-\lambda)^3 = 0$$

$\lambda = 3, 3, 3$

$m = 3$

Solve $(B-3I)V = 0$

$$\Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{matrix} \beta = 0 \\ \gamma = 0, \end{matrix} \quad \alpha = \text{free}$$

$$V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (\alpha = 1)$$

$$\boxed{\begin{matrix} \lambda = 3 \\ m = 3 \\ k = 1 \end{matrix}}$$

$$C = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\det(C - \lambda I) = \det \begin{bmatrix} 3-\lambda & 0 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

$$= (3-\lambda)^2 (2-\lambda)$$

For $\lambda = 2$, Solve $(C - 2I)V = 0$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} \alpha = 0 \\ \beta + \gamma = 0 \\ \gamma - \text{free} \end{array}$$

$$V = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad (\gamma = 1, \ \beta = -1)$$

$$\boxed{\begin{array}{c} \lambda = 2 \\ m = 1 \\ k = 1 \end{array}}$$

For $\lambda = 3$, Solve $(C - 3I)V = 0$

$$\Rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \gamma = 0$$

α, β - free

$$V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (\alpha = 1, \beta = 0), \qquad V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad (\alpha = 0, \beta = 1)$$

$$\boxed{\begin{matrix} \lambda = 3 \\ m = 2 \\ k = 2 \end{matrix}}$$

$$D = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\det(D - \lambda I) = \det \begin{bmatrix} 3-\lambda & 1 & 2 \\ 0 & 3-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{bmatrix} = (3-\lambda)^2 (2-\lambda)$$

$\lambda = 2$ Solve $(D - 2I)V = 0$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\alpha = \gamma, \quad \beta + 3\gamma = 0, \quad \gamma = \text{free}$

$$V = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \quad (\gamma = 1,\ \alpha = 1,\ \beta = -3)$$

$$\boxed{\begin{matrix} \lambda = 2 \\ m = 1 \\ k = 1 \end{matrix}}$$

$\lambda = 3$ Solve $(D - 3I)V = 0$

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \begin{cases} \beta = 0 \\ \gamma = 0 \\ \alpha \text{ free} \end{cases}$$

$$V = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (\alpha = 1)$$

$$\boxed{\begin{matrix} \lambda = 3 \\ m = 2 \\ k = 1 \end{matrix}}$$