Worked Out Homework 3
MA 303 Fall 2011 (Aaron N. K. Yip)
Friday, Sept. 30, in class

1. For each of the following system, find the general solution and plot the phase plot:

   (a) $\dfrac{dX}{dt} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} X$

   (b) $\dfrac{dX}{dt} = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} X$

   (c) $\dfrac{dX}{dt} = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} X$

   (d) $\dfrac{dX}{dt} = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix} X$

2. (Textbook) p. 519, Section 9.3, # 26 (a) and (c) (only).

   (Hint: for (c), write down the differential equation for $r(t)$.)

3. (Textbook) p. 519, Section 9.3, # 27.

4. (Textbook) p. 519, Section 9.3, # 28.

MA303 Fall 2011 (Yip) HW3 (Solution)

#1 (a) $\dot{X} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} X$

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ 2 & 2-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda) - 2$$
$$= \lambda^2 - 3\lambda = 0$$

$\lambda = 0, 3$

$\boxed{\lambda = 0}$ $(A - 0I)V_1 = 0 \Longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ \cancel{2} & \cancel{2} & \cancel{0} \end{array}\right)$

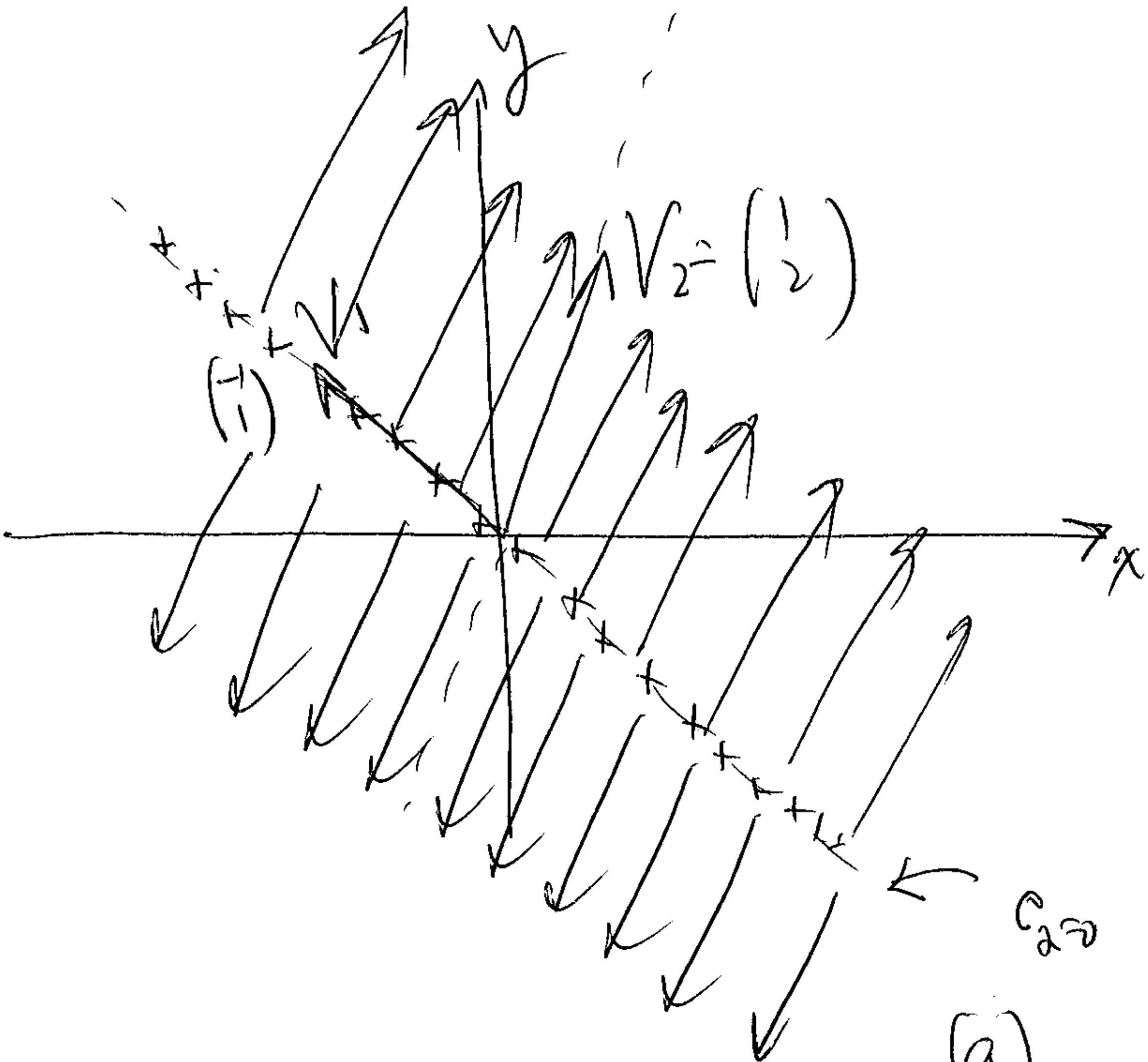
$V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\quad \alpha + \beta = 0$

$\boxed{\lambda = 3}$ $(A - 3I)V_2 = 0 \Longrightarrow \left(\begin{array}{cc|c} -2 & 1 & 0 \\ \cancel{2} & \cancel{-1} & \cancel{0} \end{array}\right)$

$V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad -2\alpha + \beta = 0$

General solution: $X(t) = c_1 e^{0t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\boxed{X(t) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$


y
x
$V_2 = \begin{pmatrix}1\\2\end{pmatrix}$
V_1
$\begin{pmatrix}-1\\1\end{pmatrix}$
$c_2 = 0$
(a)

(b) $\dot{X} = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} X$

$$\det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ -2 & -2-\lambda \end{pmatrix} = (1-\lambda)(-2-\lambda)+2$$

$$= \lambda^2 + \lambda = 0$$

$\lambda = 0, -1$

$\boxed{\lambda = 0}$ $\quad (A - 0I)V_1 = 0 \Longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ \cancel{-2} & \cancel{-2} & \cancel{0} \end{array}\right)$

$\alpha + \beta = 0$

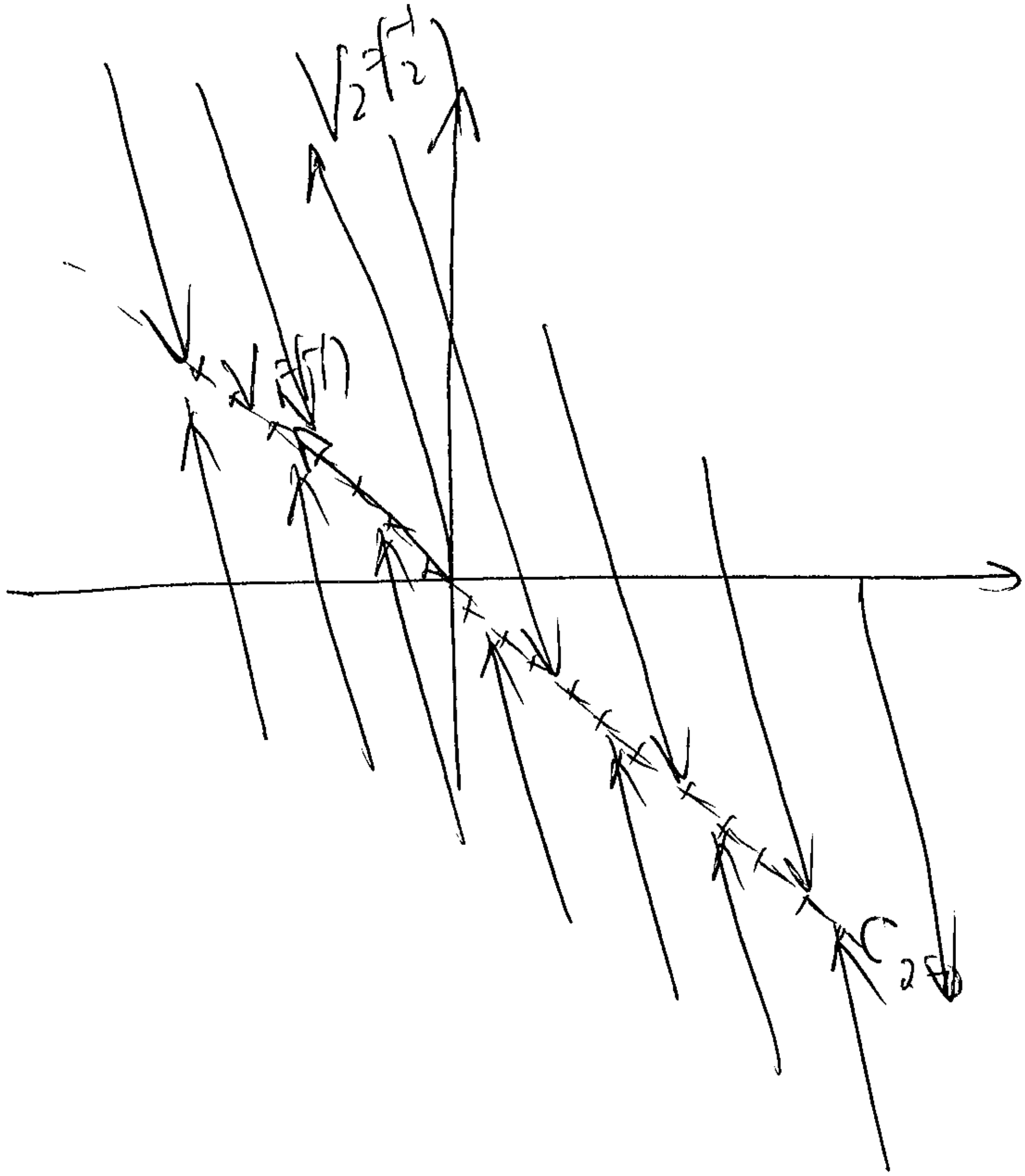
$V_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\boxed{\lambda = -1}$ $\quad (A + I)V_2 = 0 \Longrightarrow \left(\begin{array}{cc|c} 2 & 1 & 0 \\ \cancel{-2} & \cancel{-1} & \cancel{0} \end{array}\right)$

$2\alpha + \beta = 0$

$V_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

General solution is $\boxed{X(t) = \underbrace{c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{\lambda = 0} + \underbrace{c_2 e^{-t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}}_{\lambda = -1}}$



(b)

(c) $\dot{X} = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} X$

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda)+1$$
$$= \lambda^2 - 4\lambda + 4 = 0$$

$\underline{\lambda = 2, 2}$

Solve $(A - 2I)V_1 = 0 \Rightarrow \left(\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right)$

$V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad -\alpha + \beta = 0$

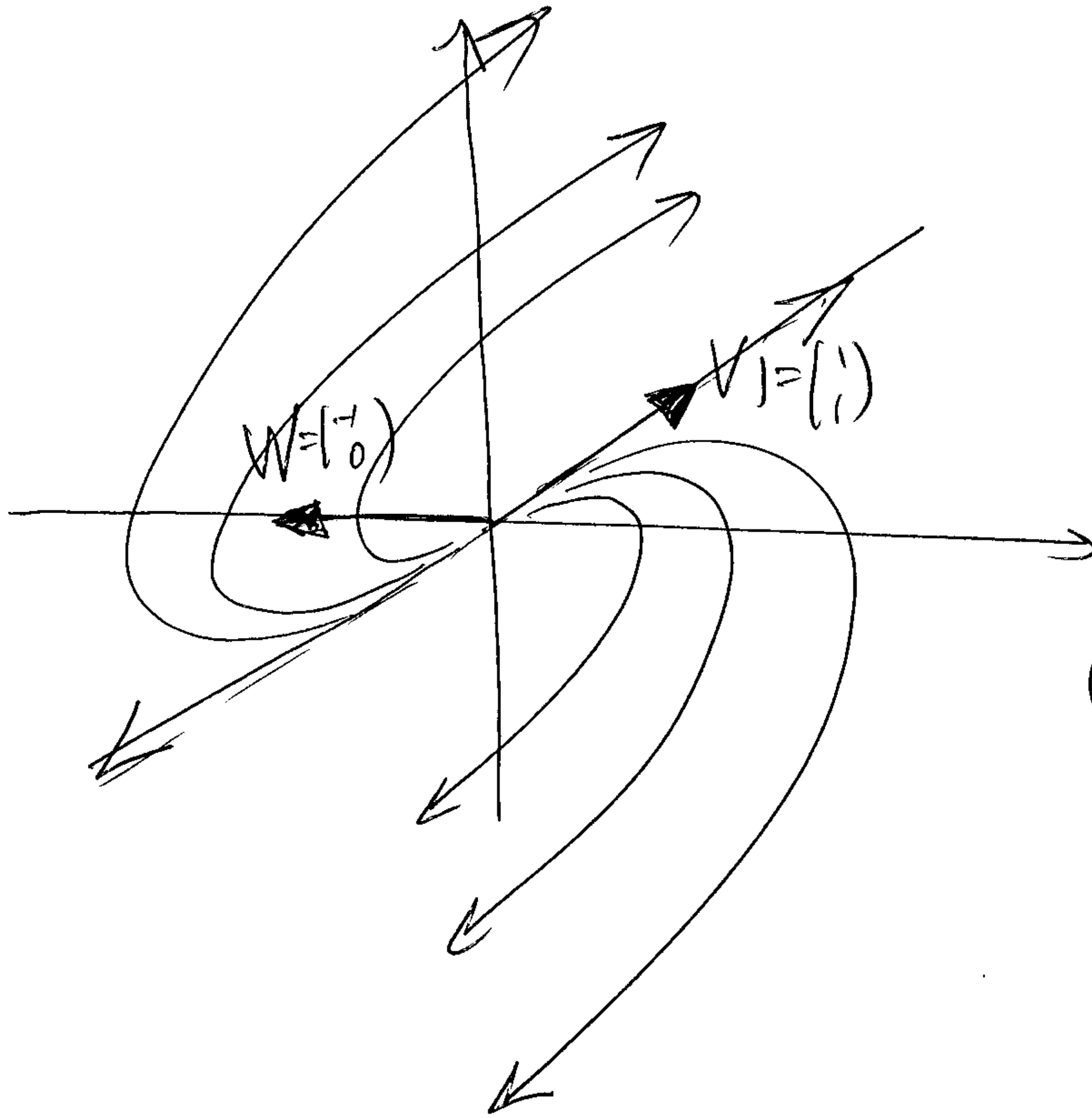
(Defective). Find W s.t.

$(A - 2I)W = V_1 \Rightarrow \left(\begin{array}{cc|c} -1 & 1 & 1 \\ -1 & 1 & 1 \end{array}\right)$

$W = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ $\qquad -\alpha + \beta = 1$

$\beta = 0, \quad \alpha = -1$

General solution: $\boxed{X(t) = C_1 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{2t}\left[t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right]}$


$W = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$
$V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(c).

1 d) $\dot{X} = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix} X$

$$\det(A-\lambda I) = \det\begin{pmatrix} -3-\lambda & 1 \\ -1 & -1-\lambda \end{pmatrix} = (3+\lambda)(1+\lambda)+1$$
$$= \lambda^2+4\lambda+4 = 0$$

$\underline{\lambda = -2, -2}$

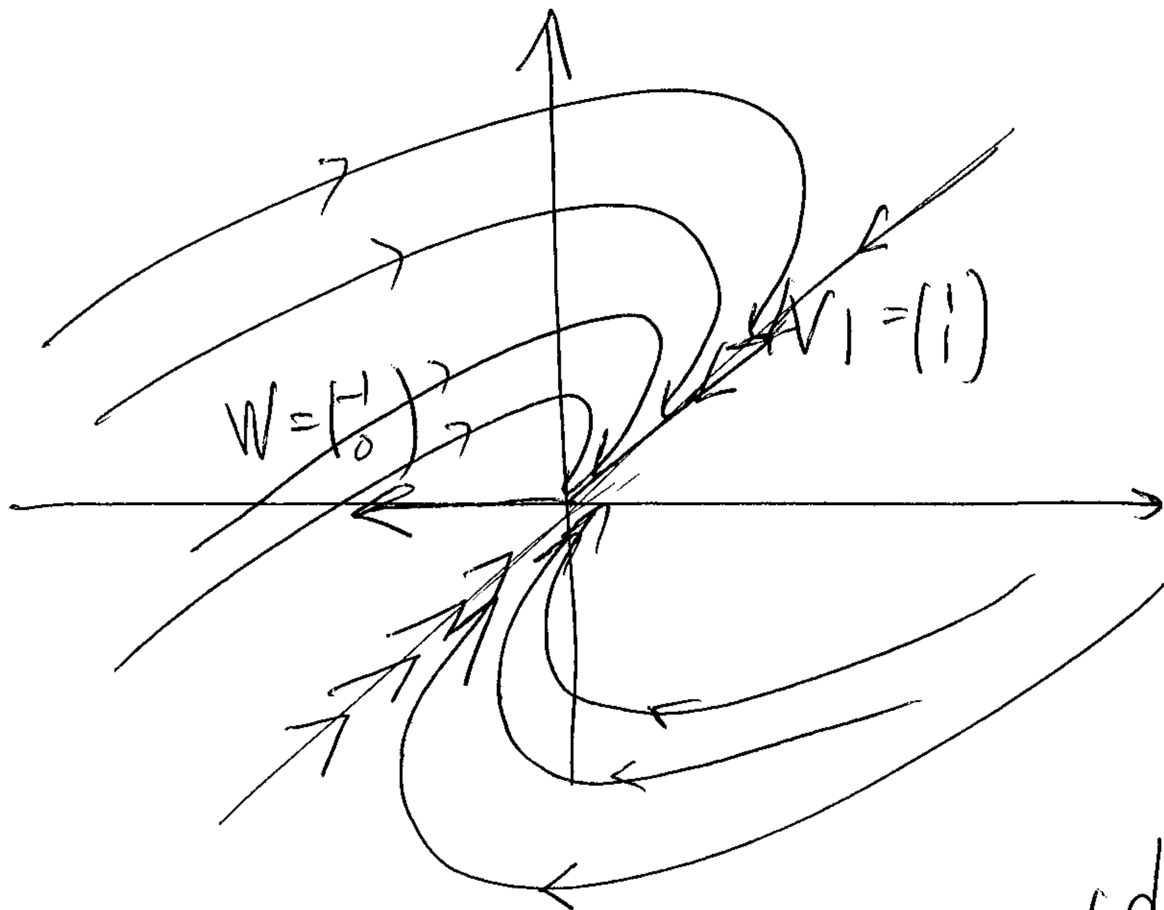
Solve $(A+2I)V_1 = 0 \Rightarrow \left(\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right)$

$V_1 = \begin{pmatrix} 1 \\ +1 \end{pmatrix}$ $\quad -\alpha+\beta = 0$

defective. Find W s.t.

$(A+2I)W = V_1 \Rightarrow \left(\begin{array}{cc|c} -1 & 1 & 1 \\ -1 & 1 & +1 \end{array}\right)$

$W = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ $\quad -\alpha+\beta = 1, \quad \beta = 0, \quad \alpha = -1$



(d)

#2

(i) $\begin{cases} \dot{x} = y + x(x^2+y^2) \\ \dot{y} = -x + y(x^2+y^2) \end{cases}$

(ii) $\begin{cases} \dot{x} = y - x(x^2+y^2) \\ \dot{y} = -x - y(x^2+y^2) \end{cases}$

(a) Note that $(0,0)$ makes the right-hand-sides of (i) & (ii) to be zero. Hence $(0,0)$ is a critical pt. for both (i) & (ii).

Linearize (i) around $(0,0)$

$f(x,y) = y + x(x^2+y^2)$ | $g = -x + y(x^2+y^2)$

$f_x = x^2+y^2 + 2x^2 = 3x^2+y^2,$ | $g_x = -1 + 2xy$

$f_y = 1 + 2xy$ | $g_y = x^2+y^2+2y^2$
$= x^2+3y^2$

$$\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}\Bigg|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\det(A-\lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2+1=0, \quad \lambda = \pm i$$

$$\boxed{\begin{array}{c}\underline{\underline{\text{Center}}}\,! \\ \text{(for linearized system)}\end{array}}$$

Linearize (ii) around $(0,0)$

$f(x,y) = y - x(x^2+y^2)$

$f_x = -(x^2+y^2) - 2x^2$

$= -3x^2 - y^2$

$f_y = 1 - 2xy$

$g(x,y) = -x - y(x^2+y^2)$

$g_x = -1 - 2xy$

$g_y = -(x^2+y^2) - 2y^2$

$= -x^2 - 3y^2$

$$\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}\Bigg|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

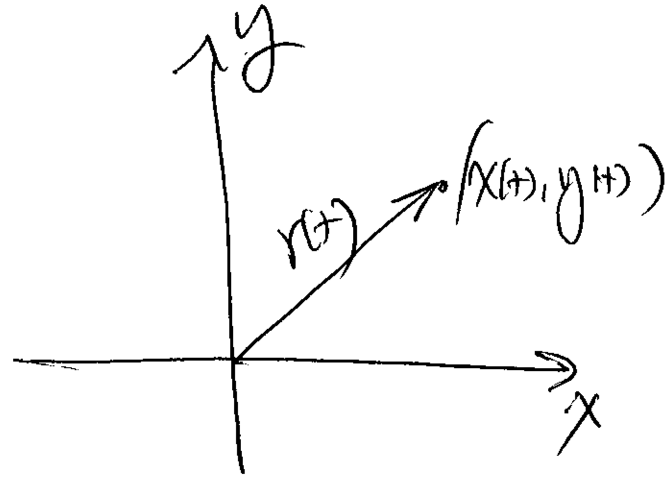
Similarly, $\lambda = \pm i$, $\boxed{\text{center for linearized system.}}$

(c) For nonlinear system, consider $r^2 = x^2 + y^2$

$$\frac{d}{dt}\left(r^2 = x^2 + y^2\right)$$

$$2r\dot{r} = 2x\dot{x} + 2y\dot{y}$$

$$\boxed{r\dot{r} = x\dot{x} + y\dot{y}}$$



for (i) $\quad r\dot{r} = x\left[y + x(x^2+y^2)\right] + y\left[-x + y(x^2+y^2)\right]$

$$= x^2(x^2+y^2) + y^2(x^2+y^2)$$

$$= (x^2+y^2)(x^2+y^2) = r^4$$

Hence $\dot{r} = r^3 > 0$, distance from the origin increases in time.

Solve for $r(t)$: $\quad \dfrac{dr}{dt} = r^3$ $\qquad$ ~~unstable pt~~

$$\int \frac{dr}{r^3} = \int dt$$

$$-\frac{1}{2r^2} = t + C$$

$$r^2 = \frac{-1}{2(t+C)}$$

$\underline{t=0}$ $\qquad r_0^2 = -\dfrac{1}{2C} \qquad C = -\dfrac{1}{2r_0^2}$

i.e. $\qquad r^2 = \dfrac{-1}{2\left(t - \frac{1}{2r_0^2}\right)} = \dfrac{1}{2\left[\frac{1}{2r_0^2} - t\right]}$

$\boxed{\text{As } t \to \frac{1}{2r_0^2}, \quad r \longrightarrow +\infty.}$ $\quad \boxed{\text{unstable pt.}}$

For (ii) $r\dot{r} = x[-y - x(x^2+y^2)]$
$+ y[x - y(x^2+y^2)]$

$= -(x^2+y^2)^4 = -r^4$

$\dot{r} = -r^3 < 0$, i.e. ~~radius~~ the distance decreases in time.

Solve for $r(t)$
$$\frac{dr}{dt} = -r^3$$

$$\int -\frac{dr}{r^3} = \int dt$$

$$\frac{1}{2r^2} = t + C$$

$t=0 \Rightarrow \quad \frac{1}{2r_0^2} = C$

Hence $\frac{1}{2r^2} = t + \frac{1}{2r_0^2}$

$$\rightarrow \quad r^2 = \frac{1}{2\left[t + \frac{1}{2r_0^2}\right]}$$

as $t \to +\infty$

$r \longrightarrow 0$

Asymptotically stable pt.

#3 Textbook pg 519 #27

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\det(A-\lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2+1=0 \Rightarrow \boxed{\lambda = \pm i}$$

$$A(\varepsilon) = \begin{bmatrix} \varepsilon & 1 \\ -1 & \varepsilon \end{bmatrix}$$

$$\det(A(\varepsilon)-\lambda I) = \det\begin{bmatrix} \varepsilon-\lambda & 1 \\ -1 & \varepsilon-\lambda \end{bmatrix}$$

$$= (\varepsilon-\lambda)^2+1=0$$

$$\lambda-\varepsilon = \pm i$$

$$\boxed{\lambda = \varepsilon \pm i}$$

if $\varepsilon<0$, $\Rightarrow$ spiral in; if $\varepsilon>0 \Rightarrow$ spiral out.

④ Textbook pg 519 # 28

$$B = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \qquad \lambda = -1, -1$$

$$B(\varepsilon) = \begin{bmatrix} -1 & 1 \\ -\varepsilon & -1 \end{bmatrix}$$

$$\det(B(\varepsilon) - \lambda I) = \det \begin{bmatrix} -1-\lambda & 1 \\ -\varepsilon & -1-\lambda \end{bmatrix}$$

$$= (\lambda+1)^2 + \varepsilon = 0$$

if $\varepsilon > 0$, small, $\underbrace{\lambda = -1 \pm \sqrt{\varepsilon}\, i}_{\boxed{\text{spiral in}}}$

if $\varepsilon < 0$, small, $(\lambda+1)^2 = -\varepsilon = |\varepsilon|$

$$\lambda = -1 \pm \sqrt{|\varepsilon|}$$

$\lambda_1 = -1 + \sqrt{|\varepsilon|} < 0$, $\lambda_2 = -1 - \sqrt{|\varepsilon|} < 0 \Rightarrow \boxed{\text{asymptotically stable.}}$