Worked Out Homework 5
MA 303 Fall 2011 (Aaron N. K. Yip)
Monday, Nov. 7, in class

1. This question is to explore the Laplace Transform of periodic functions. A function $f(t)$ is call *periodic* with *period* T if for all $t > 0$, $f(t+T) = f(t)$, i.e. the function repeats itself every T time interval. (As we are considering Laplace transform, we will only consider $t \geq 0$.)

   (a) Show that the Laplace Transform of f is given by $\dfrac{\int_0^T f(t)e^{-st}\,dt}{1-e^{-sT}}$.
   (Hint: $\displaystyle\int_0^\infty f(t)e^{-st}\,dt = \int_0^T f(t)e^{-st}\,dt + \int_T^{2T} f(t)e^{-st}\,dt + \int_{2T}^{3T} f(t)e^{-st}\,dt + \cdots$
   and use the fact that $\displaystyle\int_{nT}^{(n+1)T} f(t)e^{-st}\,dt = e^{-snT}\int_0^T f(t)e^{-st}\,dt.$)

   (b) Find the Laplace Transform of the following function

   $$f(t) = \begin{cases} 1 & t \in [0,1], [2,3], [4,5], \ldots \\ -1 & t \in [1,2], [3,4], [5,6], \ldots \end{cases}$$

   (c) Find the Laplace Transform of the function g which equals t for $t \in [0,1]$ and $2-t$ for $t \in [1,2]$ and is periodic with period 2.

   (d) Find the Laplace Transform of the function h which equals $\sin t$ for $t \in [0,\pi]$ and is periodic with period π.

2. (This problem provides a second method to solve the problem from Hw 4, #5.) Find the inverse Laplace Transform of the following functions:

   $$\frac{1}{(s^2+1)^2} \quad \text{and} \quad \frac{s}{(s^2+1)^2}$$

   (Hint: Follow the following line of thoughts:

   (a) What is the inverse Laplace Transform of $F(s) = \frac{1}{s^2+1}$?

   (b) What is $F'(s)$?

   (c) What is the inverse Laplace Transform of $F'(s)$?

   (d) If $f(t)$ is the inverse Laplace Transform of $sG(s)$, what is the inverse Laplace Transform of $G(s)$?)

3. Use *Laplace transform method* to find the matrix exponential of the following matrices:

   $$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

4. Use *Laplace transform method* to solve the following system of differential equation:

$$\dot{X} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} X + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(The above is the problem from Test One, #9.)

5. This exercise is to demonstrate that basically *all* the Laplace transforms covered in class can be explained by using the *properties* of Laplace transform.

For the following, let $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$ and $G(s) = \mathcal{L}\{g(t)\} = \int_0^\infty e^{-st} g(t)\, dt$. In addition, a and b are arbitrary constants and n is some positive integer.

Using ONLY the facts that:

$$\begin{aligned}
\mathcal{L}\{1\} &= \frac{1}{s} \\
\mathcal{L}\{af + bg\} &= aF(s) + bG(s) \\
\mathcal{L}\{f'\} &= sF(s) - f(0) \\
\mathcal{L}\{f''\} &= s^2F(s) - sf(0) - f'(0) \\
\mathcal{L}\left\{f^{(n)}\right\} &= s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0) \\
\mathcal{L}\left\{e^{at}f(t)\right\} &= F(s-a) \\
\mathcal{L}\{f(t-a)u(t-a)\} &= e^{-as}F(s), \quad (a \geq 0) \\
\mathcal{L}\{tf(t)\} &= -F'(s)
\end{aligned}$$

prove or explain the following Laplace transforms pairs:

$$\begin{aligned}
\mathcal{L}\{\delta(t)\} &= 1 \quad (\text{hint}: \quad \delta(t) = u'(t)) \\
\mathcal{L}\{t^n\} &= \frac{n!}{s^{n+1}} \\
\mathcal{L}\left\{e^{at}\right\} &= \frac{1}{s-a} \\
\mathcal{L}\left\{t^n e^{at}\right\} &= \frac{n!}{(s-a)^{n+1}} \\
\mathcal{L}\{\cos bt\} &= \frac{s}{s^2+b^2} \quad (\text{hint}: \quad \cos(bt) = \frac{e^{ibt}+e^{-ibt}}{2}) \\
\mathcal{L}\{\sin bt\} &= \frac{b}{s^2+b^2} \quad (\text{hint}: \quad \sin(bt) = \frac{e^{ibt}-e^{-ibt}}{2i}) \\
\mathcal{L}\left\{e^{at}\cos bt\right\} &= \frac{s-a}{(s-a)^2+b^2} \\
\mathcal{L}\left\{e^{at}\sin bt\right\} &= \frac{b}{(s-a)^2+b^2}
\end{aligned}$$