Worked Out Homework 6

MA 303 Fall 2011 (Aaron N. K. Yip)

Monday, Nov. 28, in class

1. (a) Consider the following differential equation:

$$u''(x) + \lambda u(x) = 0; \qquad 0 < x < \pi;$$
$$u(0) = 0, \qquad u'(\pi) = 0.$$

   (b) Consider the following differential equation:

$$u''(x) + \lambda u(x) = 0; \qquad 0 < x < \pi;$$
$$u'(0) = 0, \qquad u(\pi) = 0.$$

   For both of the above problems, find all the (real-valued) λ such that there is a *non-trivial* solution $u(x)$. Find the $u(x)$ also.

   Note: for the above problems, it can be shown (from more advanced technique) that the eigenalue λ must be real number.

2. Find the Fourier series of the 2π-periodic function $f(x)$ which equals x^2 on $[-\pi, \pi]$.

3. Find the Fourier series of the 2π-periodic function $f(x)$ which equals $(x^2 - \pi^2)^2$ on $[-\pi, \pi]$.

MA303 Written Hw 6 Solution (Fall 2011, Yip)

#1 (a) $$u''(x) + \lambda u(x) = 0 \qquad 0 < x < \pi$$

$$u(0) = 0, \qquad u'(\pi) = 0$$

$\underline{\underline{\lambda < 0}}$, $\lambda = -\mu^2$, $\mu > 0$

$$u''(x) - \mu^2 u(x) = 0$$

Char. poly. : $r^2 - \mu^2 = 0 \Rightarrow r = \pm\mu$

$$u(x) = Ae^{\mu x} + Be^{-\mu x}$$

$u(0) = 0 \Rightarrow A + B = 0, \Rightarrow B = -A$

$u'(\pi) = 0$: $u'(x) = \mu[Ae^{\mu x} - Be^{-\mu x}]$

$$u'(\pi) = \mu[Ae^{\mu\pi} - Be^{-\mu\pi}]$$

$$= \mu A[e^{\mu\pi} + e^{-\mu\pi}] = 0$$

As $\mu \neq 0$, $e^{\mu\pi} + e^{-\mu\pi} \neq 0 \Rightarrow A = 0$

$$\Rightarrow B = 0$$

$\boxed{\text{Hence } u(x) = 0, \text{ trivial solution}}$

$\underline{\underline{\lambda = 0}}$, $\quad u''(x) = 0$

$$u(x) = A + Bx$$

$$\left.\begin{aligned} u(0) = 0 &\Rightarrow A = 0 \\ u'(\pi) = 0 &\Rightarrow B = 0 \end{aligned}\right\} \Rightarrow u(x) = 0,$$

<u>trivial solution.</u>

$\underline{\underline{\lambda > 0}}$, $\quad \lambda = \mu^2, \quad \mu > 0.$

$$u''(x) + \mu^2 u(x) = 0$$

Char. poly: $r^2 + \mu^2 = 0 \Rightarrow r = \pm \mu i$

General solution: $u(x) = A\cos\mu x + B\sin\mu x.$

$$u(0) = 0 \Longrightarrow \boxed{A = 0}$$

$u'(\pi) = 0$: $\quad u'(x) = B\mu\cos\mu x$

$$u'(\pi) = B\mu\cos\mu\pi = 0$$

We need $B \neq 0 \Longrightarrow \cos\mu\pi = 0$

i.e. $\mu\pi =$ odd multiples of $\pi/2$

i.e. $\mu\pi = \left(\frac{2n+1}{2}\right)\pi$

i.e. $\mu = \frac{2n+1}{2}, \quad n = 0, 1, 2, \ldots$

Hence <u>non-trivial solution</u> :

$$\left\{ \begin{array}{l} \lambda_n \left(= \mu_n^2\right) = \left(\frac{2n+1}{2}\right)^2, \\ \\ u_n(x) = \sin \mu_n x = \sin\left(\frac{2n+1}{2} x\right) \end{array} \right\}$$

$n = 0, 1, 2, \ldots$

(b) $u''(x) + \lambda u(x) = 0 \quad 0 < x < \pi$

$u'(0) = 0, \quad u(\pi) = 0$

Similar to (a), $\lambda < 0$ & $\lambda = 0$ do not lead to non-trivial solutions

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Hence just need to consider $\lambda > 0$

$\lambda = \mu^2, \quad \mu \neq 0$

$$u''(x) + \mu^2 u(x) = 0$$

Char. poly: $r^2 + \mu^2 = 0 \Rightarrow r = \pm \mu i$

$$u(x) = A \cos \mu x + B \sin \mu x$$

$$u'(x) = -A\mu \sin \mu x + B\mu \cos \mu x$$

$u'(0) = 0 \Rightarrow \underline{\underline{B = 0}}$

Hence $u(x) = A\cos \mu x$

$u(\pi) = 0 \Rightarrow A \cos \mu\pi = 0$

We need $A \neq 0$, hence $\cos \mu \pi = 0$

i.e. $\mu\pi =$ odd multiples of $\pi/2$

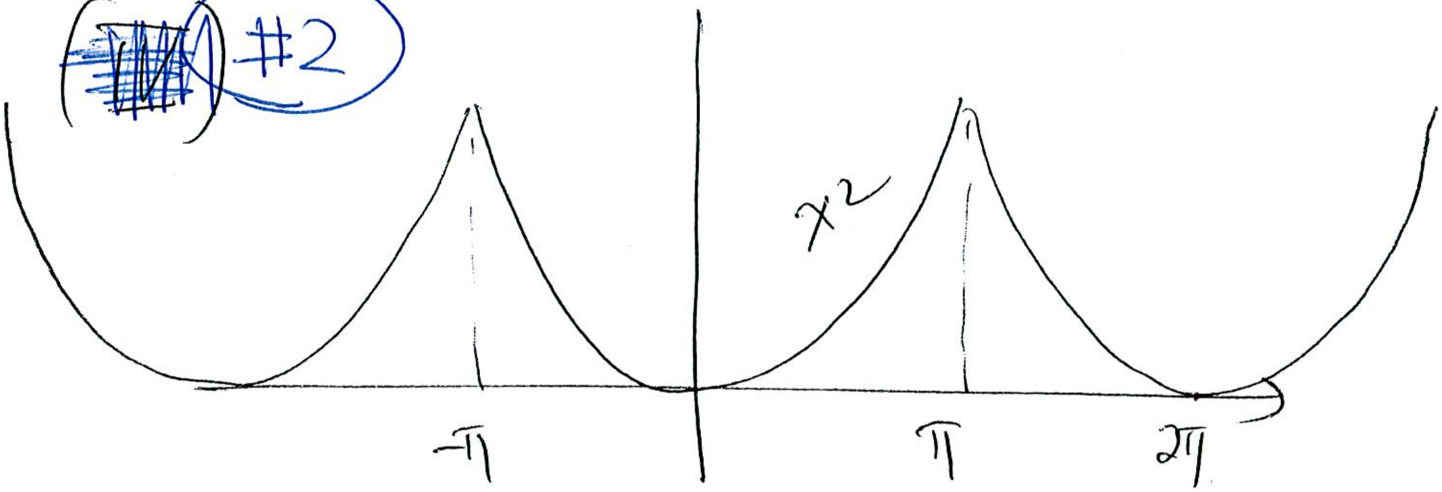
$$\mu\pi = \frac{2n+1}{2}\pi$$

$$\mu = \frac{2n+1}{2}, \quad n = 0, 1, 2, \ldots$$

Hence <u>non-trivial solution</u>:

$$\left\{ \begin{aligned} \lambda_n &= (\mu_n)^2 = \left(\frac{2n+1}{2}\right)^2 \\ u_n(x) &= (\cos \mu_n x) = \cos\left(\frac{2n+1}{2}x\right) \end{aligned} \right\}_{n=0,1,2,\ldots}$$

#2



$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

(only sine series as the function is even.)

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{3\pi}\pi^3 = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx\,dx$$

$$a_n = \frac{2}{\pi}\left[x^2\, d\,\frac{\sin nx}{n}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\cancelto{0}{\frac{x^2 \sin nx}{n}\Big|_0^{\pi}} - \frac{1}{n}\int_0^{\pi}(\sin nx)\,2x\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n}\int x\, d\,\frac{\cos nx}{n}\Big|_0^{\pi}\right]$$

$$= \frac{4}{\pi n^2}\left[x\cos nx\Big|_0^{\pi} - \int_0^{\pi}\cos nx\,dx\right]$$

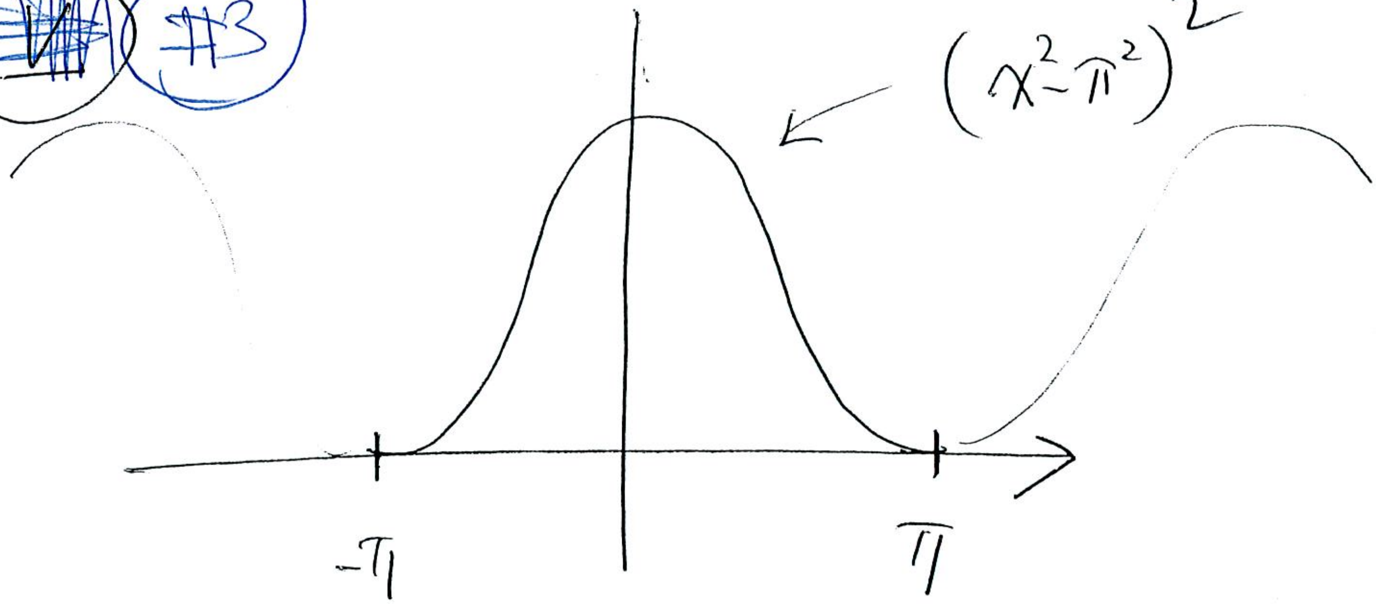
$$= \frac{4}{\pi n^2}\left[\pi\cos n\pi + \cancelto{0}{\frac{\sin nx}{n}\Big|_0^{\pi}}\right]$$

$$\boxed{= \frac{4}{n^2}(-1)^n}$$

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$$

$$= \frac{\pi^2}{3} + 4\left[-\cos x + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \cdots\right]$$

~~VII~~ #3



$$f(x) = a_0 + \sum_{n=[redacted]}^{\infty} a_n \cos nx$$

(only sine series on the function is even)

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} (x^2-\pi^2)^2 \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} (x^4 - 2\pi^2 x^2 + \pi^4) \, dx$$

$$= \frac{1}{\pi}\left[\frac{\pi^5}{5} - \frac{2\pi^5}{3} + \pi^5\right]$$

$$= \boxed{\frac{8\pi^4}{15}}$$

$$\frac{1}{5} - \frac{2}{3} + 1$$

$$\frac{3-10+15}{15}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\left(x^4 - 2\pi^2 x^2 + \pi^4\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\underbrace{\int_0^{\pi} x^4 \cos nx\,dx} - 2\pi^2\int_0^{\pi} x^2\cos nx\,dx + \pi^4\underbrace{\int_0^{\pi}\cos nx\,dx}_{0}\right]$$

$$\boxed{\int_0^{\pi} x^4 \cos nx\,dx =} \int_0^{\pi} x^4\, d\left(\frac{\sin nx}{n}\right)$$

$$= \underbrace{\frac{x^4 \sin nx}{n}\Big|_0^{\pi}}_{0} - \frac{4}{n}\int_0^{\pi} x^3 \sin nx\,dx$$

$$= \frac{4}{n}\int_0^{\pi} x^3\, d\left(\frac{\cos nx}{n}\right)$$

$$= \frac{4}{n}\left[\frac{x^3 \cos nx}{n}\Big|_0^{\pi} - \frac{3}{n}\int_0^{\pi} x^2 \cos nx \, dx \right]$$

$$= \frac{4}{n^2}\left[\pi^3 \cos n\pi - 3\int_0^{\pi} x^2 \cos nx \, dx \right]$$

$$= \frac{4\pi^3}{n^2}\cos(n\pi) - \frac{12}{n^2}\int_0^{\pi} x^2 \cos nx \, dx$$

$$\boxed{= \frac{4\pi^3(-1)^n}{n^2} - \frac{12}{n^2}\int_0^{\pi} x^2 \cos(nx) \, dx}$$

$$\boxed{\int_0^{\pi} x^2 \cos nx \, dx} = \int_0^{\pi} x^2 \, d\,\frac{\sin nx}{n}$$

$$= \cancel{\frac{x^2 \sin nx}{n}\Big|_0^{\pi}} - \frac{2}{n}\int_0^{\pi} x \sin nx \, dx$$

$$0$$

$$= \frac{2}{n}\int_0^{\pi} x \, d\frac{\cos nx}{n}$$

$$= \frac{2}{n}\left[\frac{x\cos nx}{n}\Big|_0^{\pi} - \underbrace{\cancel{\frac{1}{n}\int_0^{\pi}\cos nx\,dx}}_{0}\right]$$

$$= \frac{2\pi\cos n\pi}{n^2} = \boxed{\frac{2\pi(-1)^n}{n^2}}$$

<u>Hence</u>

$$a_n = \frac{2}{\pi}\left[\cancel{\frac{4\pi^3(-1)^n}{n^2}} - \frac{12}{n^2}\,\frac{2\pi(-1)^n}{n^2} \cancel{- 2\pi^2\,\frac{2\pi(-1)^n}{n^2}}\right]$$

$$= -\frac{48}{n^4}(-1)^n$$

Hence

$$\boxed{\left(x^2-\pi^2\right)^2 = \frac{8\pi^4}{15} - 48 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \cos nx}$$