Worked Out Homework 7
MA 303 Fall 2011 (Aaron N. K. Yip)
Friday, Dec. 9, in class, LAST DAY OF CLASS

(**Beware:** I am using the following expression for Fourier series for any $2L$-periodic function:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

while the textbook is using:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right).$$

Hence:

$$[\text{my } a_0] = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \quad \text{while} \quad [\text{textbook's } a_0] = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$\text{i.e.} \quad [\text{my } a_0] = \frac{[\text{textbook's } a_0]}{2}.$$

In the following, I am always using my convention.)

1. This problem is to produce some "interesting" formula for the value π.

   (a) Making use of the Fourier series of $f(x) = \begin{cases} 1, & 0 < x < \pi \\ -1, & -\pi < x < 0 \end{cases}$ and f is 2π-periodic to prove that

   $$\pi = 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right].$$

   (Hint: set $x = \frac{\pi}{2}$ into the Fourier Series of f.)

   (b) Making use of the Fourier series of $f(x) = |x|$ for $-\pi < x < \pi$ and f is 2π-periodic to prove that

   $$\pi^2 = 8\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right].$$

   (Hint: set $x = 0$ or $x = \pm\pi$ into the Fourier Series of f.)

   (c) Making use of the Fourier series of $f(x) = x^2$ for $-\pi < x < \pi$ and f is 2π-periodic to prove that

   $$\pi^2 = 12\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right].$$

   (Hint: set $x = 0$ into the Fourier Series of f.)

2. (a) Prove the Bessel-Parseval's Identity: for any $2L$-periodic function $f(x)$, then

$$\frac{1}{2L}\int_{-L}^{L} f^2(x)\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

where a_0 and $a_n,\ b_n$'s are the Fourier coefficients of $f(x)$.

(Hint: take the square of the Fourier series expansion of $f(x)$, multiply out term by term and make use of the orthogonal property of the sin and cos functions.)

(b) Apply the above identity to the function $f(x) = x$ for $-\pi < x < \pi$ and write down an "interesting" identity.

3. Solve the following heat equation:

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t) + 13\sin 5x, \quad t > 0,\ 0 < x < \pi; \\ u(0,t) &= 0,\ \ u(\pi,t) = 0; \\ u(x,0) &= \sin x - 5\sin 7x. \end{aligned}$$

Find also the behavior of $u(x,t)$ as $t \longrightarrow +\infty$.

4. Solve the following heat equation with Neumann boundary condition:

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t), \quad t > 0,\ 0 < x < \pi; \\ u_x(0,t) &= 0,\ \ u_x(\pi,t) = 0; \\ u(x,0) &= f(x). \end{aligned}$$

Find also the behavior of $u(x,t)$ as $t \longrightarrow +\infty$.

(Hint: Follow the same derivation as in the Dirichler boundary condition case. Note that $\lambda = 0$ IS a legitimate eigenvalue. Find its eigenfunction.)