Worked Out Homework 7

MA 303 Fall 2011 (Aaron N. K. Yip)

Friday, Dec. 9, in class, LAST DAY OF CLASS

(**Beware:** I am using the following expression for Fourier series for any $2L$-periodic function:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

while the textbook is using:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right).$$

Hence:

$$[\text{my } a_0] = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \quad \text{while} \quad [\text{textbook's } a_0] = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$\text{i.e.} \quad [\text{my } a_0] = \frac{[\text{textbook's } a_0]}{2}.$$

In the following, I am always using my convention.)

1. This problem is to produce some "interesting" formula for the value π.

   (a) Making use of the Fourier series of $f(x) = \begin{cases} 1, & 0 < x < \pi \\ -1, & -\pi < x < 0 \end{cases}$ and f is 2π-periodic to prove that

   $$\pi = 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right].$$

   (Hint: set $x = \frac{\pi}{2}$ into the Fourier Series of f.)

   (b) Making use of the Fourier series of $f(x) = |x|$ for $-\pi < x < \pi$ and f is 2π-periodic to prove that

   $$\pi^2 = 8\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right].$$

   (Hint: set $x = 0$ or $x = \pm\pi$ into the Fourier Series of f.)

   (c) Making use of the Fourier series of $f(x) = x^2$ for $-\pi < x < \pi$ and f is 2π-periodic to prove that

   $$\pi^2 = 12\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right].$$

   (Hint: set $x = 0$ into the Fourier Series of f.)

2. (a) Prove the Bessel-Parseval's Identity: for any $2L$-periodic function $f(x)$, then

$$\frac{1}{2L}\int_{-L}^{L} f^2(x)\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

where a_0 and a_n, b_n's are the Fourier coefficients of $f(x)$.

(Hint: take the square of the Fourier series expansion of $f(x)$, multiply out term by term and make use of the orthogonal property of the sin and cos functions.)

(b) Apply the above identity to the function $f(x) = x$ for $-\pi < x < \pi$ and write down an "interesting" identity.

3. Solve the following heat equation:

$$\begin{gathered} u_t(x,t) = u_{xx}(x,t) + 13\sin 5x, \quad t > 0,\ 0 < x < \pi; \\ u(0,t) = 0,\ u(\pi,t) = 0; \\ u(x,0) = \sin x - 5\sin 7x. \end{gathered}$$

Find also the behavior of $u(x,t)$ as $t \longrightarrow +\infty$.

4. Solve the following heat equation with Neumann boundary condition:

$$\begin{gathered} u_t(x,t) = u_{xx}(x,t), \quad t > 0,\ 0 < x < \pi; \\ u_x(0,t) = 0,\ u_x(\pi,t) = 0; \\ u(x,0) = f(x). \end{gathered}$$

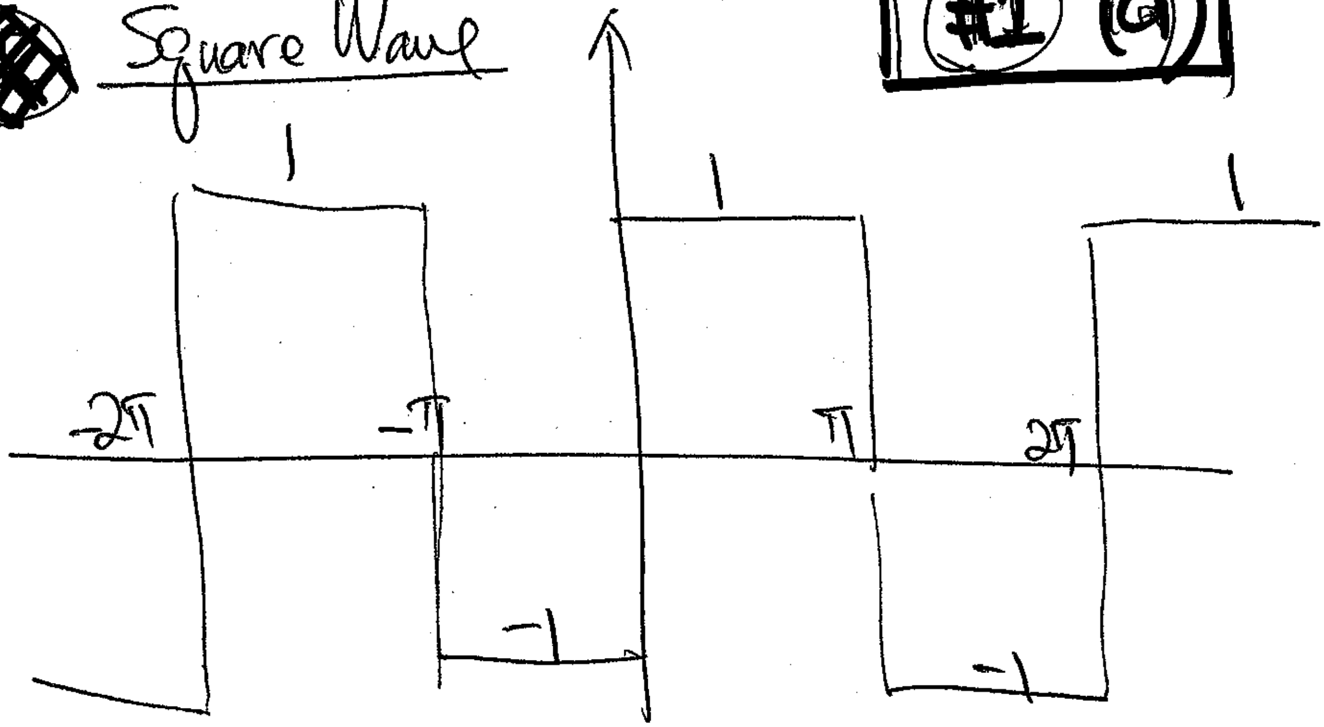
Find also the behavior of $u(x,t)$ as $t \longrightarrow +\infty$.

(Hint: Follow the same derivation as in the Dirichler boundary condition case. Note that $\lambda = 0$ IS a legitimate eigenvalue. Find its eigenfunction.)

MA303 Written HW 7 Solutions Fall 2011 (Xi)

#1 (a)

Square Wave



$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases}$$

(2π- periodic) $\qquad L = \pi$

Note: $f(x)$ is odd fct.

$$f(x) = a_0 + \sum_{n=} \left(a_n \cos nx + b_n \sin nx \right)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx \quad = 0$$

odd

even

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \quad = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) (\sin nx)\, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} (-1) \sin nx\, dx + \int_{0}^{\pi} (1) \sin nx\, dx \right]$$

$= \cdots$

$$\boxed{Q_2 \quad = \frac{2}{\pi}\int_0^{\pi} \sin nx \, dx}$$

$$= \frac{1}{\pi}\left[\frac{\cos nx}{n}\Bigg|_{-\pi}^{0} - \frac{\cos nx}{n}\Bigg|_{0}^{\pi} \right]$$

$$= \frac{1}{\pi}\left[\frac{1-\cos n(-\pi)}{\cancel{n}} - \frac{\cos n\pi - 1}{\cancel{n}} \right]$$

$$= \frac{1}{\pi n}\left[2 - 2\cos n\pi \right]$$

$$= \frac{2}{n\pi}\left[1 - \cos n\pi \right]$$

$$= \frac{2}{n\pi}\left[1 - (-1)^n \right] = \begin{cases} \frac{4}{n\pi} & n\text{-odd } n \\ 0 & n\text{-even } n \end{cases}$$

Hence

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right]\sin(nx)$$

$n = 2k-1$ $\qquad\qquad$ $n=2k-1 \quad k=1,2,\ldots$

$$= \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi}\sin(2k-1)x \qquad n=2k+1,\ k=0,1,2\ldots$$

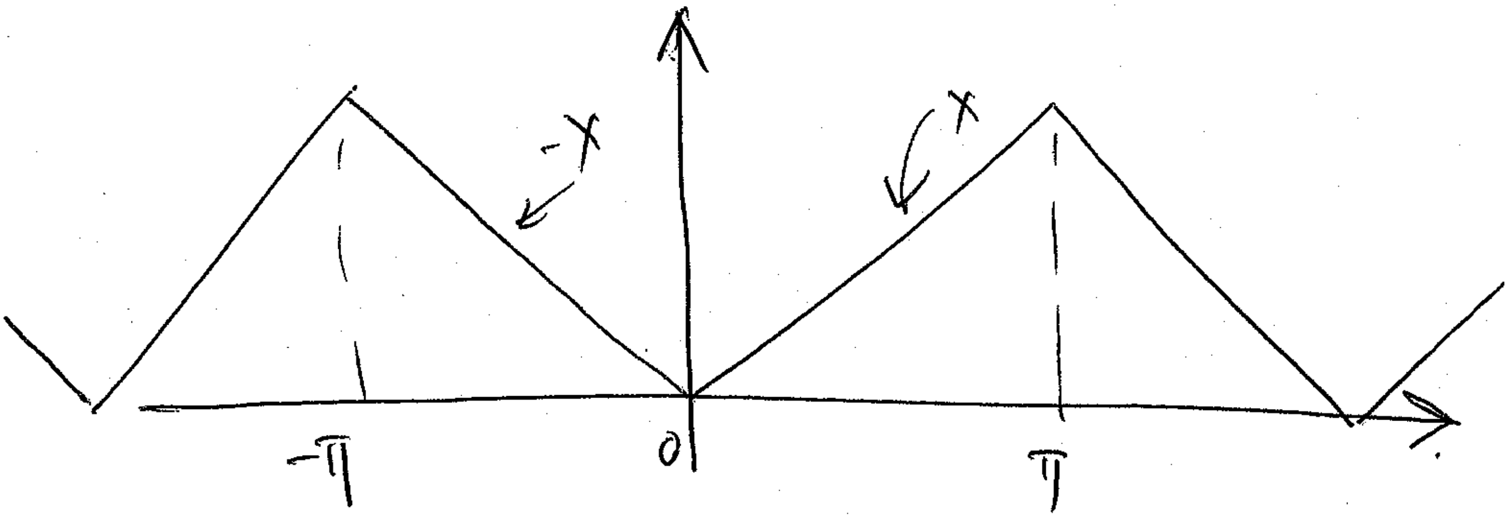
$$\boxed{f(x) = \frac{4}{\pi}\Big[\underset{(k=1)}{\sin x} + \underset{(k=2)}{\tfrac{1}{3}\sin 3x} + \underset{k=3}{\tfrac{1}{5}\sin 5x} + \cdots}$$

$\boxed{\text{Set } x = \pi/2} \Longrightarrow$

$$1 = \frac{4}{\pi}\left[\sin\pi/2 + \frac{\sin 3\pi/2}{3} + \frac{1}{5}\sin 5\pi/2 + \cdots\right]$$

$$\boxed{\pi = 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right]}$$

Triangular Wave $\boxed{\#1(b)}$



$$f(x) = |x| \qquad -\pi < x < \pi \ , \ 2\pi\text{-period}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + \underbrace{\cancel{b_n \sin nx}}_{0} \right)$$

$$\boxed{b_n = 0 \quad \text{for even fct}}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{1}{2\pi}\left[2\,\frac{\pi^2}{2}\right]$$

$$= \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x)\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\int x\,d\,\frac{\sin nx}{n}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\cancel{\frac{x\sin nx}{n}\Big|_0^{\pi}} - \frac{1}{n}\int_0^{\pi} \sin nx \, dx \right]$$

$$= \frac{2}{\pi}\left[\frac{1}{n} \frac{\cos nx}{n}\Big|_0^{\pi} \right]$$

$$= \frac{2}{\pi n^2}\left[\cos(n\pi) - 1 \right]$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1 \right]$$

$$= \begin{cases} 0 & n\text{-even} \\ -\dfrac{4}{n^2\pi} & n\text{-odd} \end{cases}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left[(-1)^n - 1\right] \cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{1}{5^2}\cos 5x + \cdots\right.$$

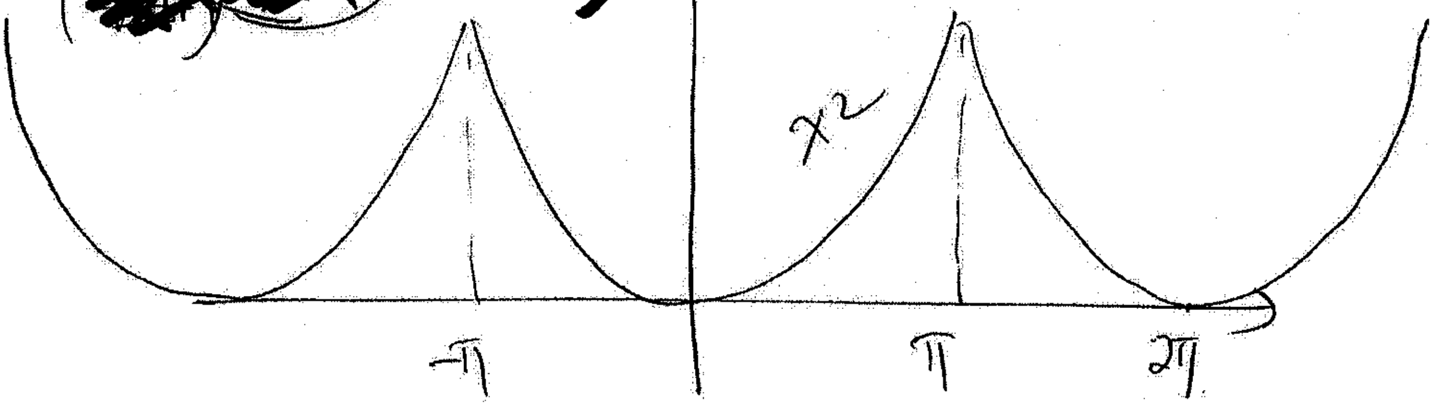
$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2}$$

$\boxed{x=0} \Longrightarrow$

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right]$$

$$\Longrightarrow \boxed{\pi^2 = 8\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right]}$$

[redacted] #1(c)



$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

(only sine seri as the function is even.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{3\pi}\pi^3 = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx\,dx$$

$$a_n = \frac{2}{\pi}\left[\int x^2 \, d\frac{\sin nx}{n}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\underbrace{\cancel{\frac{x^2 \sin nx}{n}\Big|_0^{\pi}}}_{0} - \frac{1}{n}\int_0^{\pi} (\sin nx) 2x \, dx\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n}\int x \, d\frac{\cos nx}{n}\Big|_0^{\pi}\right]$$

$$= \frac{4}{\pi n^2}\left[x\cos nx\Big|_0^{\pi} - \int_0^{\pi} \cos nx \, dx\right]$$

$$= \frac{4}{\pi n^2}\left[\pi \cos n\pi + \underbrace{\cancel{\frac{\sin nx}{n}\Big|_0^{\pi}}}_{0}\right]$$

$$\boxed{= \frac{4}{n^2}(-1)^n}$$

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$$

$$= \frac{\pi^2}{3} + 4\left[-\cos x + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \cdots\right]$$

$\boxed{x=0} \Longrightarrow$

$$\boxed{\pi^2 = 12\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \cdots\right]}$$

#2

(a) $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$

$$f(x)^2 = \left(a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$\left(a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

= (like terms)

$$a_0^2 + \sum_{n=1}^{\infty} a_n^2 \cos^2\frac{n\pi x}{L} + b_n^2 \sin^2\frac{n\pi x}{L}$$

+ (cross terms)

eg. $a_0 a_n \cos\frac{n\pi x}{L}$, $a_0 b_n \sin\frac{n\pi x}{L}$

$+ a_n a_m \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L} + \cdots$

<u>Note</u>

$$\int_{-L}^{L} a_0^2\,dx = a_0^2\,(2L)$$

$$\int_{-L}^{L} \cos^2\frac{n\pi x}{L}\,dx = L$$

$$\int_{-L}^{L} \sin^2\frac{n\pi x}{L}\,dx = L$$

$$\int_{-L}^{L} (\text{cross terms}) = 0$$

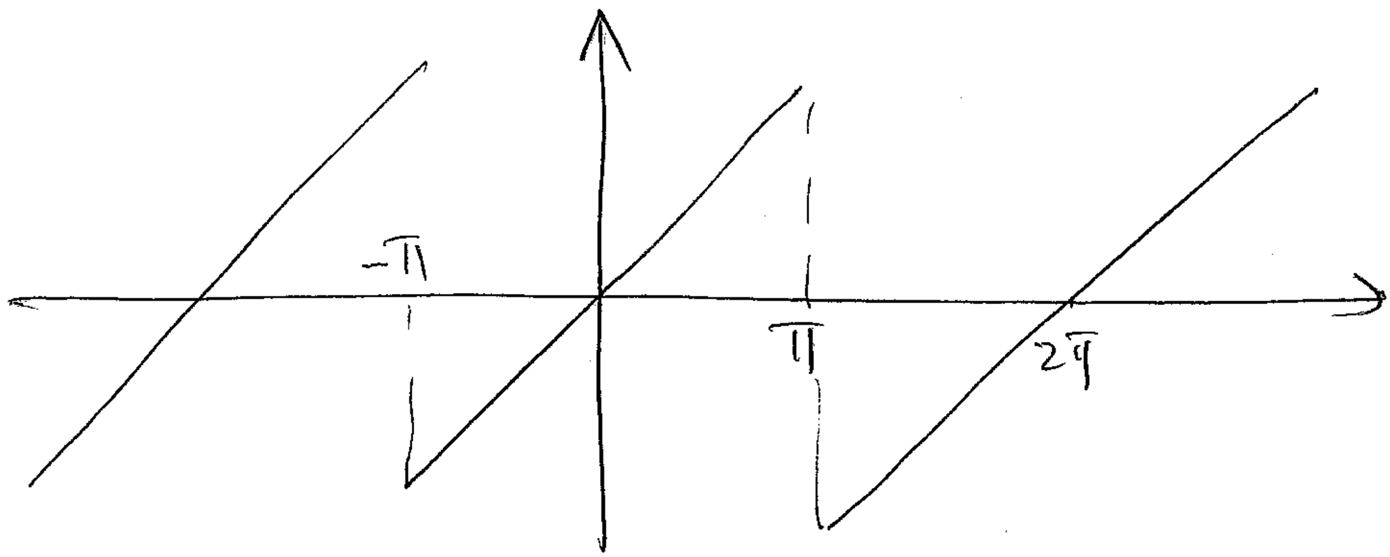
by orthogonality property

Hence

$$\int_{-L}^{L} f^2(x)\,dx = 2L a_0^2 + L\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$\frac{1}{2L}\int_{-L}^{L} f^2(x)\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

(b) Sawtooth



$f(x) = x \quad -\pi \leq x \leq \pi$, 2π-periodic

— odd fct.

$$f(x) = \cancel{a_0} + \sum_{n=1}^{\infty} \left(\cancel{a_n \cos nx} + b_n \sin nx \right)$$

0 0

$\boxed{a_n = 0 \text{ for odd fct}}$

odd × ~~even~~ odd = even

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi} x\sin nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\int x\,d\left(-\frac{\cos nx}{n}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{x\cos nx}{n}\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}(\cos nx)\,dx\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{n}\cos n\pi + \cancel{\frac{1}{n^2}\sin nx\Big|_0^{\pi}}\right]$$

$$= -\frac{2}{n}\cos n\pi = \boxed{-\frac{2}{n}(-1)^n} = \frac{2}{n}(-1)^{n+1}$$

or $b_n = \frac{1}{\pi}\left[-\frac{x\cos nx}{n}\Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi} \cos nx\, dx \right]$

$$= \frac{1}{\pi n}\left[-\pi \cos n\pi + (-\pi)\cos n(-\pi) \right]$$

$$= -\frac{2}{n}\cos n\pi = \boxed{-\frac{2}{n}(-1)^n}$$

$$= \boxed{\frac{2}{n}(-1)^{n+1}}$$

$$f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin nx$$

$$= 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} \cdots \right]$$

[redacted]) $f(x) = x \qquad -\pi < x < \pi$ (Sawtooth fct)

$$(L = \pi)$$

$$x = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right]$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = 2^2\left[\frac{1}{2}\left(\underset{b_1^2}{1} + \underset{b_2^2}{\frac{1}{2^2}} + \underset{b_3^2}{\frac{1}{3^2}} + \underset{b_4^2}{\frac{1}{4^2}} + \cdots\right)\right]$$

$\|$ $\quad (a_0 = 0,\ a_n = 0)$

$$\frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{\pi^3}{3\pi} = \frac{\pi^2}{3}$$

Hence $\dfrac{\pi^2}{3} = 2\left[1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \cdots\right]$

$$\boxed{\pi^2 = 6\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right]}$$

~~#13~~ #3

$$\begin{cases} u_t = u_{xx} + 13 \sin 5x & 0 < x < \pi \\ u(0,t) = 0 \quad \& \quad u(\pi,t) = 0 \\ u(x,0) = \sin x - 5 \sin 7x \end{cases}$$

Let $\quad u(x,t) = \sum_{n=1}^{\infty} C_n(t) \sin nx \qquad$ (Dir. B.C.)

Sub. the above into the equation:

$$\sum_{n=1}^{\infty} \dot{c}_n(t) \sin nx = \sum_{n=1}^{\infty} -n^2 c_n(t) \sin nx$$

$$+ 13 \sin 5x$$

Hence,
$$\begin{cases} \dot{c}_n = -n^2 c_n(t) & n \neq 5 \\ \dot{c}_5 = -25\, c_5(t) + 13 & (n=5) \end{cases}$$

$$\boxed{\hat{C}_n(t) = \hat{C}_n(0)\,e^{-n^2 t} \qquad n \neq 5}$$

$$\hat{C}_5'(t) = -25\,\hat{C}_5(t) + 13 \qquad n = 5$$

$$C_5(t) = C_5(0)\,e^{-25t} + e^{-25t}\int_0^t e^{25s}\,13\,ds$$

$$= C_5(0)\,e^{-25t} + e^{-25t}\,13\left[\frac{e^{25t}-1}{25}\right]$$

<u>Hence</u> $u(x,t)$

$$= \left(C_5(0)\,e^{-25t} + e^{-25t}\,\frac{13}{25}\left(e^{25t}-1\right)\right)(\sin 5x)$$

$$+ \sum_{n\neq 5}^{\infty} C_n(0)\,e^{-n^2 t}\sin nx$$

To find $\{\hat{C}_n(0)\}$:

$t=0$

$$\boxed{\begin{aligned}\sin x - 5\sin 7x = {} & C_5(0)\sin 5x \\ & + \sum_{n\neq 5}^{\infty} C_n(0)\sin nx\end{aligned}}$$

Hence $C_1(0)=1$, $C_7(0)=-5$

all other $C_n(0)=0$ (including $C_5(0)$)

Finally,

$$\boxed{\begin{aligned}u(x,t) = {} & e^{-t}\sin x - 5e^{-49t}\sin 7x \\ & + \frac{13}{25}\left(1-e^{-25t}\right)\sin 5x\end{aligned}}$$

#

as $t\to\infty$, $\boxed{u(x,t) \longrightarrow -\frac{13}{25}\sin 5x}$

#4

$$u_t = u_{xx} \qquad t>0, \quad 0<x<\pi$$

$$u_x(0,t)=0, \quad u_x(\pi,t)=0$$

$$u(x,0) = f(x)$$

Neumann B.C.

1. Separation of Variables:

$$u(x,t) = X(x)\,T(t)$$

$$X(x)\,T'(t) = X''(x)\,T(t)$$

$$\frac{T'(t)}{T(t)} = \underbrace{\frac{X''(x)}{X(x)} = -\lambda}$$

$$\boxed{\begin{array}{c} X''(x) + \lambda X(x) = 0 \\ X'(0)=0, \quad X'(\pi)=0 \end{array}}$$

PLUS

(2) Finding of λ + X

$\underline{\lambda < 0}$ $\quad \lambda = -\mu^2$

$$X''(x) - \mu^2 X(x) = 0 \qquad r^2 - \mu^2 = 0$$

$$r = \pm\mu$$

$$\Rightarrow X(x) = Ae^{\mu x} + Be^{-\mu x}$$

$$X'(x) = A\mu e^{\mu x} - B\mu e^{-\mu x}$$

$$X'(0) = 0 \Rightarrow \quad A - B = 0 \Rightarrow A = B$$

$$X'(\pi) = 0 \Rightarrow \mu\left(Ae^{\mu\pi} - Be^{-\mu\pi}\right) = 0$$

$$\Rightarrow A\left(e^{\mu\pi} - e^{-\mu\pi}\right) = 0$$

$$\Rightarrow A = 0$$

Hence NO nontrivial solution.

$\underline{\lambda=0}$ $\quad X''(x)=0$

$\Rightarrow \quad X(x)=A+Bx$

$\quad\quad X'(x)=B$

$X'(0)=0 \Rightarrow B=0 \quad$ (automatically $X'(\pi)=0$)

$\Rightarrow \quad X(x)=A=1 \quad$ (A is free)

$\lambda>0 \quad X''(x)+\lambda X(x)=0 \Rightarrow r^2+\lambda=0$

$r=\pm\sqrt{\lambda}\,i$

$X(x)=A\cos\sqrt{\lambda}x+B\sin\sqrt{\lambda}x$

$X'(x)=-A\sqrt{\lambda}\sin\sqrt{\lambda}x+B\sqrt{\lambda}\cos\sqrt{\lambda}x$

$X'(0)=0 \Rightarrow B=0$

$X'(\pi)=0 \Rightarrow -A\sqrt{\lambda}\sin\sqrt{\lambda}\pi=0$

$A\neq 0$ (otherwise, only trivial soln)

$$\Rightarrow \sin(\sqrt{\lambda}\pi) = 0$$

$$\sqrt{\lambda}\pi = n\pi$$

$$\lambda = n^2$$

$$\Rightarrow X(x) = A \cos nx$$

$$= \cos nx \qquad (A = 1, \text{free})$$

Hence eigenvalues & eigenvectors:

$$\left\{ \lambda_n = n^2, \ \cos nx \right\}_{n=0}^{\infty}$$

③ Solution of $u(x,t)$:

$$u(x,t) = \underbrace{C_0(t) \cdot 1}_{n=0} + \sum_{n=1}^{\infty} C_n(t) \cos nx$$

$$\underbrace{\left(\dot{C}_0(t) + \sum_{n=1}^{\infty} \dot{C}_n(t) \cos nx\right)}_{u_t} = \underbrace{\sum_{n=1}^{\infty} -n^2 C_n(t) \cos nx}_{u_{xx}}$$

Compare coefficient :

$n=0$ $\qquad \dot{C}_0(t) = 0 \implies C_0(t) = C_0$

$n \geq 1$ $\qquad \dot{C}_n(t) = -n^2 C_n(t)$

$$\implies C_n(t) = C_n(0) e^{-n^2 t}$$

$$\boxed{u(x,t) = C_0 + \sum_{n=1}^{\infty} C_n(0) e^{-n^2 t} \cos nx}$$

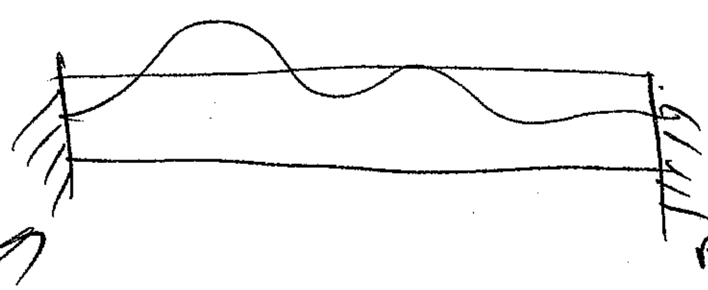
(5) As $t \to +\infty$

$$u(x,t) = C_0 + \underbrace{\sum_{n=1}^{\infty} C_n(0) e^{-n^2 t} \cos nx}_{\downarrow \, t \to \infty \atop 0}$$

$$u(x,t) \longrightarrow C_0 = \frac{1}{\pi} \int_0^{\pi} f(x)\, dx$$

$= \text{average of } f(x)$

(which makes physical sense for Neuman B.C.



insulator

insulator