Notes on Partial Fractions

$$\frac{P(x)}{Q(x)} \qquad (\deg P < \deg Q).$$

Factor $Q(x) = (x-r_1)^{k_1}(x-r_2)^{k_2}\cdots\cdots$

Then
$$\frac{P(x)}{Q(x)} = \frac{A_1^{(1)}}{x-r_1} + \frac{A_2^{(1)}}{(x-r_1)^2} + \cdots + \frac{A_{k_1}^{(1)}}{(x-r_1)^{k_1}}$$
$$+ \frac{A_1^{(2)}}{x-r_2} + \frac{A_2^{(2)}}{(x-r_2)^2} + \cdots + \frac{A_{k_2}^{(2)}}{(x-r_2)^{k_2}} + \cdots$$
$$+ \cdots\cdots$$

If $Q(x)$ has no repeated roots $(k_1 = k_2 = \cdots = 1)$ $\Rightarrow$ simplify:

$$\frac{P(x)}{Q(x)} = \frac{A^{(1)}}{x-r_1} + \frac{A^{(2)}}{x-r_2} + \frac{A^{(3)}}{x-r_3} + \cdots$$

eg $\dfrac{6x+7}{(x+2)(x-1)^2} = \dfrac{A}{x+2} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$

$6x+7 = A(x-1)^2 + B(x+2)(x-1) + C(x+2)$

$\underline{x=1} \Rightarrow 13 = 3C \Rightarrow \boxed{C = 13/3}$

$\underline{x=-2} \Rightarrow -5 = 9A \Rightarrow \boxed{A = -5/9}$

Use any other value, such as $x=0$

$\Rightarrow \quad 7 = A - 2B + 2C$

$\Rightarrow \quad B = \dfrac{A+2C-7}{2} = \dfrac{1}{2}\left[-\dfrac{5}{9} + \dfrac{26}{3} - 7\right]$

$= \boxed{+\dfrac{5}{9}}$

the same

Or simply compare coeff, such as x^2:

$0 = A + B, \Rightarrow \boxed{B = -A = 5/9}$

In case of quadratic factors in $Q(x)$

$$Q(x) = (x^2 + a_1 x + b_1)^{k_1} (x^2 + a_2 x + b_2)^{k_2} \cdots$$

Then $(\deg P < \deg Q)$

$$\frac{P(x)}{Q(x)} = \frac{A_1^{(1)} x + B_1^{(1)}}{(x^2 + a_1 x + b_1)} + \frac{A_2^{(1)} x + B_2^{(1)}}{(x^2 + a_1 x + b_1)^2} + \cdots$$

$$\cdots + \frac{A_{k_1}^{(1)} x + B_{k_1}^{(1)}}{(x^2 + a_1 x + b_1)^{k_1}}$$

$$+ \frac{A_1^{(2)} x + B_1^{(2)}}{(x^2 + a_2 x + b_2)} + \frac{A_2^{(2)} x + B_2^{(2)}}{(x^2 + a_2 x + b_2)^2} + \cdots$$

$$\cdots + \frac{A_{k_2}^{(2)} x + B_{k_2}^{(2)}}{(x^2 + a_2 x + b_2)^{k_2}} + \cdots$$

$$+ \cdots$$

eg

$$\frac{6x+7}{(x^2+1)^2(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{E}{x-1} + \frac{F}{(x-1)^2}$$

$$\Rightarrow (6x+7) = (Ax+B)(x^2+1)(x-1)^2 + (Cx+D)(x-1)^2 + E(x^2+1)^2(x-1) + F(x^2+1)^2$$

$x=1 \Rightarrow \quad 13 = 4F \quad \boxed{F = 13/4}$

$$x=i \Rightarrow \quad 6i+7 = (Ci+D)(i-1)^2 = (Ci+D)(\cancel{-1}-2i+\cancel{1}) = (-2i)(Ci+D)$$

$Ci+D = -3 + \frac{7}{2}i \Rightarrow \boxed{C = \frac{7}{2}, D=-3}$

From here, you can either choose some other convenient values of x, and also compare coefficients.

For example, we need to find A, B, E, need 3 equations.

Compare $x^5 \Longrightarrow \quad 0 = A + E$

Compare constant term $\Longrightarrow \quad 7 = B + D - E + F$

Let $x = -1 \Longrightarrow -6 + 7 = (-A+B)(2)(4)$
$+(-C+D)(4)$
$+E(4)(-2)$
$+F(4)$

$$\Longrightarrow \left\{ \begin{aligned} A + E &= 0 \\ B - E &= 7 + 3 - 13/4 = \frac{27}{4} \\ B - A - E &= \frac{14}{8} = \frac{7}{4} \end{aligned} \right\} \Longrightarrow \boxed{\begin{aligned} E &= -5 \\ A &= 5 \\ B &= \frac{7}{4} \end{aligned}}$$

If $\deg P \geq \deg Q$, then use

long division

eg $\dfrac{x^3 - x^2 - 5x + 1}{x^2 + 3x + 2}$

quotient

$$\begin{array}{r l}
 & x - 4 \\
x^2 + 3x + 2 \, \big) & x^3 - x^2 - 5x + 1 \\
 & x^3 + 3x^2 + 2x \\
\hline
 & -4x^2 - 7x + 1 \\
 & -4x^2 - 12x - 8 \\
\hline
 & 5x + 9
\end{array}$$

remainder

So $\dfrac{x^3 - x^2 - 5x + 1}{x^2 + 3x + 2} = (x - 4) + \dfrac{5x + 9}{x^2 + 3x + 2}$

$\downarrow$

apply partial fraction