MA 303: Differential and Partial Differential Equations for Engineering and Sciences

Fall 2011, Test One

(Instructor: Aaron N. K. Yip)

V02

- This test booklet has TEN questions – EIGHT MULTIPLE CHOICE and TWO WRITTEN questions – totaling 100 points for the whole test. You have 60 minutes to do this test. **Plan your time well. Read the questions carefully. You do not need to attempt the questions in sequence.**
- This test is **closed book and closed note. No calculator nor any other electronic device** is allowed.
- For the multiple choice questions, no partial credit will be given.

  For the written questions, in order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.
- You can use both sides of the papers to write your answers. But please indicate so if you do.

Name: (Write Clearly) Key 02 (Major:)

Question	Score
#1 – 8.(max. 56 pts)	
#9.(max. 22 pts)	
#10.(max. 22 pts)	
Total(100 pts)	

MA 363, Fall 2011, Test One Solution (Yip)

VERSION 02

1. **Multiple choice question. Enter your answer in the computer scantron.**
Determine the values of r for which the following differential equation:

$$t^2y'' - 7\overset{t}{\cancel{y'}} \overset{-}{\cancel{+}} 20y = 0 \Longrightarrow \boxed{t^2y'' - 7ty' - 20y = 0}$$

has a solution of the form $y = t^r$ for $t > 0$.

(a) $r = 2$ or 5

(b) $r = -4$ or -5

(c) $r = 4$ or 5

(d) $r = 2$ or -10

(e) $r = -2$ or 10

$y = t^r$; $y' = rt^{r-1}$; $y'' = r(r-1)t^{r-2}$

$t^2 r(r-1)t^{r-2} - 7t\, r t^{r-1} - 20t^r = 0$

$[r(r-1) - 7r - 20]\cancel{t^r} = 0$

$r^2 - 8r - 20 = 0$

$(r-10)(r+2) = 0 \Rightarrow r = 10, -2$

2. **Multiple choice question. Enter your answer in the computer scantron.**
Find the *general* solution of the following differential equation:

$$ty' + 2y = t^2 + 2 \quad (t > 0)$$

(a) $y = \dfrac{C}{t^2} + \dfrac{t^2}{4} + 1$

(b) $y = \dfrac{C}{t^4} + \dfrac{t^2}{5} + 1$

(c) $y = t^3 + \dfrac{t^2}{5} + C$

(d) $y = \dfrac{C}{t^3} + \dfrac{t^2}{5} + 1$

(e) $y = t^2 + \dfrac{Ct^2}{4} + 1$

$y' + \frac{2}{t}y = t + \frac{2}{t}$

$I = e^{\int \frac{2}{t}dt} = t^2$

$y = I^{-1}\left[\int I\left(t + \frac{2}{t}\right)dt\right]$

$= \frac{1}{t^2}\left[\int (t^3 + 2t)\,dt\right]$

$= \boxed{\frac{t^2}{4} + 1 + \frac{C}{t^2}} \Longleftarrow = \frac{1}{t^2}\left[\frac{t^4}{4} + t^2 + C\right]$

2

3. **Multiple choice question. Enter your answer in the computer scantron.**
Find the solution of the following system of differential equation:

$$\dot{X} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

(a) $\begin{pmatrix} 5e^{-2t} - 4e^{-3t} \\ -10e^{-2t} + 12e^{-3t} \end{pmatrix}$

(b) $\begin{pmatrix} 4e^{-2t} - 4e^{-3t} \\ -9e^{-2t} + 12e^{-3t} \end{pmatrix}$

(c) $\begin{pmatrix} e^{-2t} - 4e^{-3t} \\ -e^{-2t} + 2e^{-3t} \end{pmatrix}$

(d) $\begin{pmatrix} 7e^{-2t} - 5e^{-3t} \\ -14e^{-2t} + 15e^{-3t} \end{pmatrix}$

(e) $\begin{pmatrix} e^{-2t} - 5e^{-3t} \\ -12e^{-2t} + 13e^{-3t} \end{pmatrix}$

$A = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}$

$\det(A-\lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ -6 & -5-\lambda \end{pmatrix}$

$= \lambda^2+5\lambda+6 = 0$

$= (\lambda+2)(\lambda+3) = 0$

$\lambda = -2, -3$

$(A+2I)V_1 = 0 \Rightarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ -6 & -3 & 0 \end{array}\right] \qquad V_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$(A+3I)V_2 = 0 \Rightarrow \left[\begin{array}{cc|c} 3 & 1 & 0 \\ -6 & -2 & 0 \end{array}\right] \qquad V_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$X(t) = c_1 e^{-2t}\begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 e^{-3t}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = -7e^{-2t}\begin{bmatrix} -1 \\ 2 \end{bmatrix} + 5e^{-3t}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$X(0) = c_1\begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 7e^{-2t} - 5e^{-3t} \\ -14e^{-2t} + 15e^{-3t} \end{bmatrix}$ #

$\left.\begin{array}{l} -c_1 - c_2 = 2 \\ 2c_1 + 3c_2 = 1 \end{array}\right\} \Rightarrow \begin{array}{l} c_2 = 5 \\ c_1 = -7 \end{array}$

4. **Multiple choice question. Enter your answer in the computer scantron.**
Find the solution of the following system of differential equation:

$$\dot{X} = \begin{pmatrix} -2 & 4 \\ -4 & -2 \end{pmatrix} X, \quad X(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

(Hint: you are given that one of the eigenvalue is $-2+4i$.)

(a) $e^{4t}\begin{pmatrix} \cos 2t - \sin 2t \\ -\cos 2t - 2\sin 2t \end{pmatrix}$

(b) $e^{-2t}\begin{pmatrix} 2\cos 4t + 2\sin 4t \\ 2\cos 4t - \sin 4t \end{pmatrix}$

(c) $e^{-2t}\begin{pmatrix} \cos 4t + 2\sin 4t \\ 2\cos 4t - \sin 4t \end{pmatrix}$

(d) $e^{4t}\begin{pmatrix} \cos 2t + 2\sin 2t \\ 2\cos 2t - 3\sin 2t \end{pmatrix}$

(e) $e^{-2t}\begin{pmatrix} 2\cos 4t - \sin 4t \\ -\cos 4t - 2\sin 4t \end{pmatrix}$

$(A + (2-4i)I)V = 0$

$\to \left(\begin{array}{cc|c} -4i & 4 & 0 \\ -4 & -4i & 0 \end{array}\right)$

$\to \left(\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right)$ $i^2+1=0$

$\to \left(\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\lambda_1 = -2+4i$

$V_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$e^{-2t}\left[C_1\begin{pmatrix} \sin 4t \\ \cos 4t \end{pmatrix} + C_2\begin{pmatrix} -\cos 4t \\ \sin 4t \end{pmatrix}\right]$

$\lambda_2 = -2-4i,$

$V_2 = \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - i\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

at $t=0$, $X(0) = \begin{pmatrix} -c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\boxed{c_2 = -1,\ c_1 = 2}$

$X(t) = c_1 e^{-2t}\left[\cos 4t\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sin 4t\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right] + c_2 e^{-2t}\left[\sin 4t\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \cos 4t\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right]$

5. **Multiple choice question. Enter your answer in the computer scantron.**

Find the *general* solution of the following system of differential equation:

$$\dot{X} = \begin{pmatrix} -4 & 0 & -1 \\ 0 & -4 & -1 \\ 0 & 0 & 3 \end{pmatrix} X$$

$\lambda = -4, -4, 3$

(a) $c_1 e^{4t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{-3t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(b) $c_1 e^{-4t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} -1 \\ -1 \\ 7 \end{pmatrix}$

(c) $c_1 e^{-4t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{-4t} \left[t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 7 \end{pmatrix} \right] + c_3 e^{3t} \begin{pmatrix} -1 \\ -1 \\ 7 \end{pmatrix}$

(d) $c_1 e^{-4t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix}$

(e) $c_1 e^{-4t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{-4t} \left[t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 7 \end{pmatrix} \right] + c_3 e^{3t} \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix}$

$\underline{\lambda = -4}$ $(A + 4I)V = 0 \Rightarrow \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\alpha \quad \beta \quad \gamma$

$\gamma = 0$, α, β free

$\alpha = 1, \beta = 0, \gamma = 0$ $\quad V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\alpha = 0, \beta = 1, \gamma = 0$ $\quad V_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\lambda = 3$ $(A - 3I)V = 0 \Rightarrow \left[\begin{array}{ccc|c} -7 & 0 & -1 & 0 \\ 0 & -7 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 7 & 0 & 1 & 0 \\ 0 & 7 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$V_3 = \begin{bmatrix} -1 \\ -1 \\ 7 \end{bmatrix}$

6. **Multiple choice question. Enter your answer in the computer scantron.**
Find all the critical points of the given system of equations:

$$\begin{aligned} \dot{x} &= x - x^2 - xy \\ \dot{y} &= y - 3xy - 2y^2 \end{aligned}$$

(a) $(0,0)$, $(0,\frac{3}{2})$, $(1,0)$

(b) $(0,0)$, $(0,2)$, $(\frac{1}{2},\frac{1}{2}))$

(c) $(0,0)$, $(0,\frac{3}{2})$, $(1,0)$, $(-1,2)$

(d) $(0,0)$, $(0,\frac{3}{2})$, $(1,0)$, $(\frac{1}{2},\frac{1}{2})$

(e) $(0,0)$, $(0,\frac{1}{2})$, $(1,0)$, $(-1,2)$

$x(1-x-y)=0$

& $y(1-3x-2y)=0$

$(x=0 \text{ or } x+y=1)$ & $(y=0 \text{ or } 3x+2y=1)$

$(x=0, y=0)$

$(x=0, y=1/2)$

$(x=1, y=0)$

+ Solve $\left.\begin{array}{l} x+y=1 \\ 3x+2y=1 \end{array}\right\} \Rightarrow y=2, x=-1$

$(-1,2)$

7. **Multiple choice question. Enter your answer in the computer scantron.**
What can you say about the eigenvalues and eigenvectors of the following matrix:

Everybody gets credit.

$$A = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 3 & 4 \end{pmatrix}$$

(a) 5 is an eigenvalue which repeats two times and has only one eigenvector; 4 is an eigenvalue which repeats once and has only one eigenvector.

(b) 5 is an eigenvalue which repeats two times and has two eigenvector; 4 is an eigenvalue which repeats two times and has two eigenvectors.

(c) 5 is an eigenvalue which repeats three times and has only one eigenvector; 4 is an eigenvalue which repeats once and has only one eigenvector.

(d) 5 is an eigenvalue which repeats three times and has only two eigenvectors; 4 is an eigenvalue which repeats once and has one eigenvector.

(e) 5 is an eigenvalue which repeats three times and has only two eigenvectors; 4 is an eigenvalue which repeats two times and has one eigenvector.

(Note: For eigenvectors, it is always assumed that they are linearly independent.)

$\lambda = 5, 5, 4, 4$

$$(A-5I)V=0 \Longrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

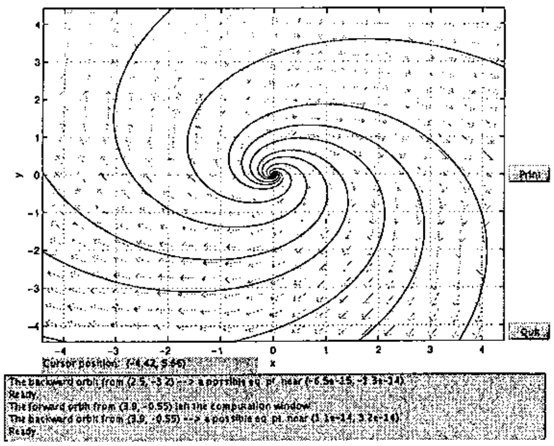
2 free variables

$$(A-4I)V=0 \Longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

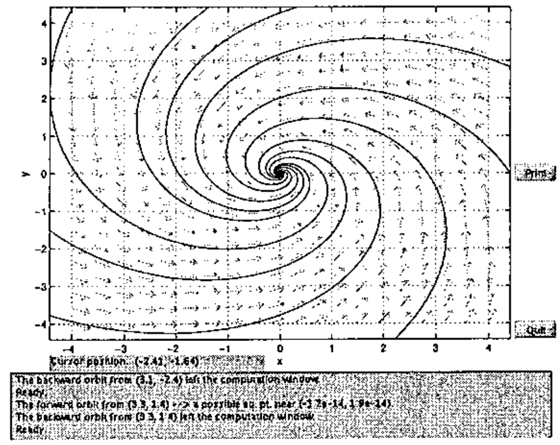
2 free variables.

8. **Multiple choice question. Enter your answer in the computer scantron.**

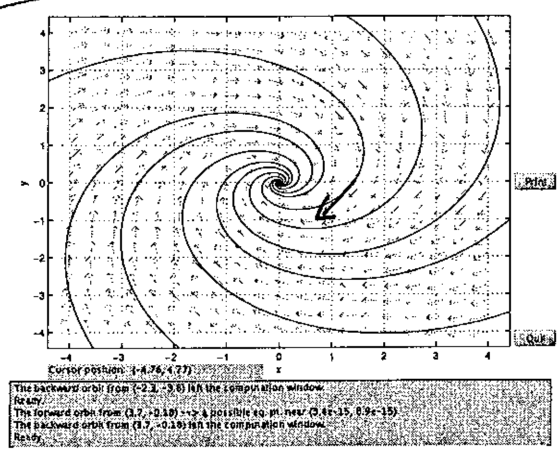
The phase plot of the system of differential equation $\dfrac{dX}{dt} = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} X$ is given by:



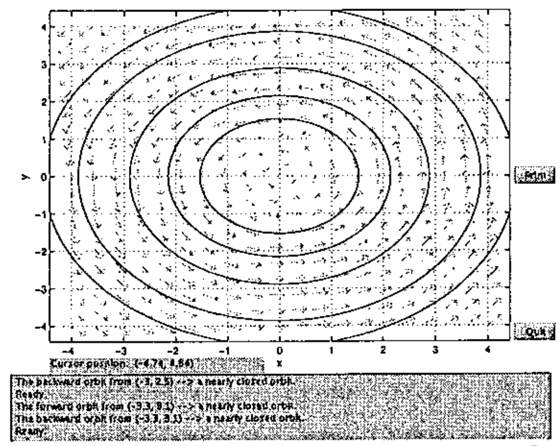
(a)



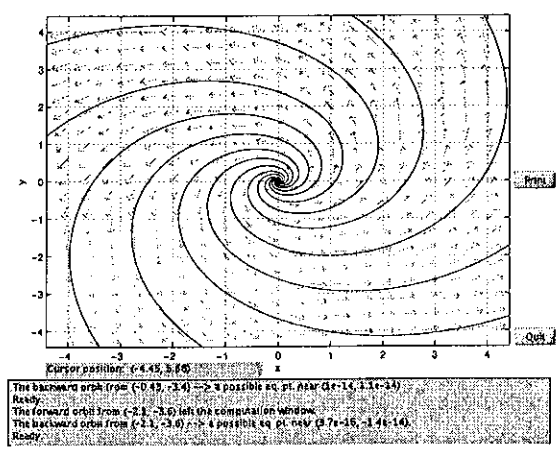
(b)



(c)



(d)



(e)

$$\det(A-\lambda I) = \det\begin{bmatrix} -1-\lambda & 2 \\ -2 & -1-\lambda \end{bmatrix}$$

$$= (\lambda+1)^2+4=0$$

$\lambda = -1 \pm 2i$, spiral in

$(-1<0)$.

at $X=\begin{pmatrix}1\\0\end{pmatrix}$, $\dot{X} = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1\\-2\end{bmatrix}$

9. Consider the following system of differential equation:

$$\dot{X} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} X + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(a) Find the matrix exponential e^{At} where A is the matrix of the above system.

(b) Find the solution when $X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(a) (12 pts)

$$\det(A - \lambda I) = \det\begin{pmatrix} -1-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix} = (\lambda+1)(\lambda-1) + 1$$
$$= \lambda^2 = 0$$

$\lambda = 0, 0$

$$(A - 0I)V_1 = 0 \Rightarrow \left(\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ only 1 eigenvector, hence defective.

General solution $X_1(t) = e^{0t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$X_2(t) = e^{0t}\left[t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + W\right]$$

where $(A - 0I)W = V_1 \Rightarrow \left[\begin{array}{cc|c} -1 & 1 & 1 \\ -1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 0 & 0 \end{array}\right]$

$\alpha - \beta = -1$, $\beta = 0$, $\alpha = -1$, $W = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

Hence $X_2(t) = \begin{bmatrix} t-1 \\ t \end{bmatrix}$

This is a scrap paper.

$$\Phi(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & t-1 \\ 1 & t \end{bmatrix}$$

$$e^{At} = \Phi(t)\,\Phi(0)^{-1} = \begin{bmatrix} 1 & t-1 \\ 1 & t \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & t-1 \\ 1 & t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1-t & t \\ -t & t+1 \end{bmatrix} \quad \#$$

(b) $$X(t) = e^{At}X_0 + e^{At}\int_0^t e^{-As}\, b(s)\, ds$$

(10 pts)

$$= \begin{bmatrix} 1-t & t \\ -t & t+1 \end{bmatrix} \int_0^t \begin{bmatrix} 1+s & -s \\ s & -s+1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} ds$$

$$= \begin{bmatrix} 1-t & t \\ -t & t+1 \end{bmatrix} \int_0^t \begin{bmatrix} 1+s \\ s \end{bmatrix} ds$$

$$= \begin{bmatrix} 1-t & t \\ -t & t+1 \end{bmatrix} \begin{bmatrix} t+t^2/2 \\ t^2/2 \end{bmatrix} = \begin{bmatrix} (1-t)(t+t^2/2) + t^3/2 \\ -t(t+t^2/2) + (t+1)t^2/2 \end{bmatrix}$$

$$= \begin{bmatrix} t - t^2/2 \\ -t^2/2 \end{bmatrix} \quad \# \quad 10 \Longleftarrow \begin{bmatrix} t + t^2/2 - t^2 - \cancel{t^3/2} + \cancel{t^3/2} \\ -t^2 - \cancel{t^3/2} + \cancel{t^3/2} + t^2/2 \end{bmatrix}$$

10. Consider the system of linear differential equations given by:

$$\dot{X} = \begin{pmatrix} 0 & -4 \\ 1 & \alpha \end{pmatrix} X$$

where α is some real-valued parameter.

(a) Determine the range of α such that the phase plot has the type of *saddle, source, sink, spiral out, spiral in, and center.*

Indicate your answer on a number line for α.

(b) Plot the phase plot when $\alpha = -5$. *Show clearly the important qualitative feature(s).*

(a) $\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & -4 \\ 1 & \alpha - \lambda \end{pmatrix} = \lambda^2 - \lambda\alpha + 4 = 0$

$$\lambda_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 - 16}}{2}$$

$\lambda_1 = \frac{\alpha + \sqrt{\alpha^2 - 16}}{2}$, $\lambda_2 = \frac{\alpha - \sqrt{\alpha^2 - 16}}{2}$

When $\alpha^2 < 16 \Rightarrow$ Complex roots.

$0 < \alpha < 4 \Rightarrow \lambda_1, \lambda_2 = \underbrace{\frac{\alpha}{2}}_{+ve} \pm \frac{\sqrt{16 - \alpha^2}}{2} i$

$\Rightarrow$ spiral out

$-4 < \alpha < 0 \Rightarrow \lambda_1, \lambda_2 = \underbrace{\frac{\alpha}{2}}_{-ve} \pm \frac{\sqrt{16 - \alpha^2}}{2} i$

$\Rightarrow$ Spiral in

This is a scrap paper.

$\alpha = 0 \Rightarrow \lambda_1, \lambda_2 = \pm 2i \Rightarrow$ center.

$\alpha \geq 4 \Rightarrow \lambda_1 = \dfrac{\alpha + \sqrt{\alpha^2 - 16}}{2} > 0$ real root

$\lambda_2 = \dfrac{\alpha - \sqrt{\alpha^2 - 16}}{2} > 0$ real root.

$(\because \alpha > \sqrt{\alpha^2 - 16})$

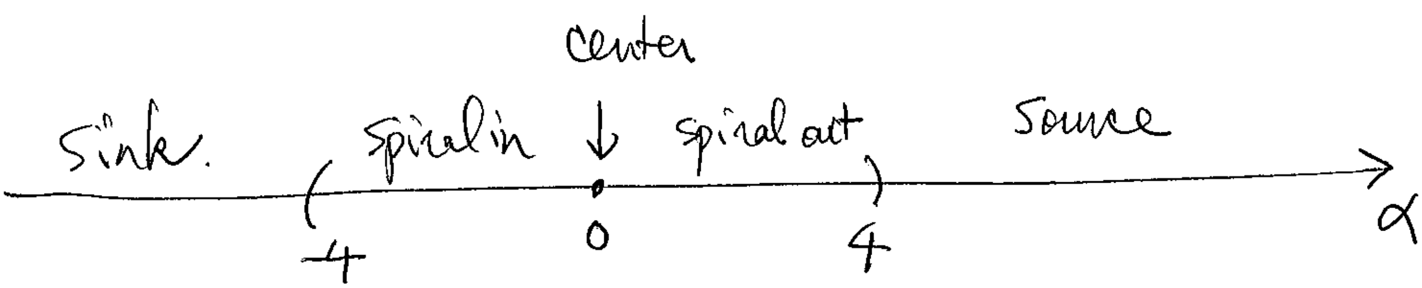
$\Rightarrow$ Source

$\alpha \leq -4 \Rightarrow \lambda_1 = \dfrac{\alpha + \sqrt{\alpha^2 - 16}}{2} < 0$ (real root)

$(|\alpha| > \sqrt{\alpha^2 - 16})$

$\lambda_2 = \dfrac{\alpha - \sqrt{\alpha^2 - 16}}{2} < 0$ (real root)

$\Rightarrow$ Sink.



(b) $\underline{\underline{\alpha=5}}$, $\lambda_1 = -1$, $\lambda_2 = -4$

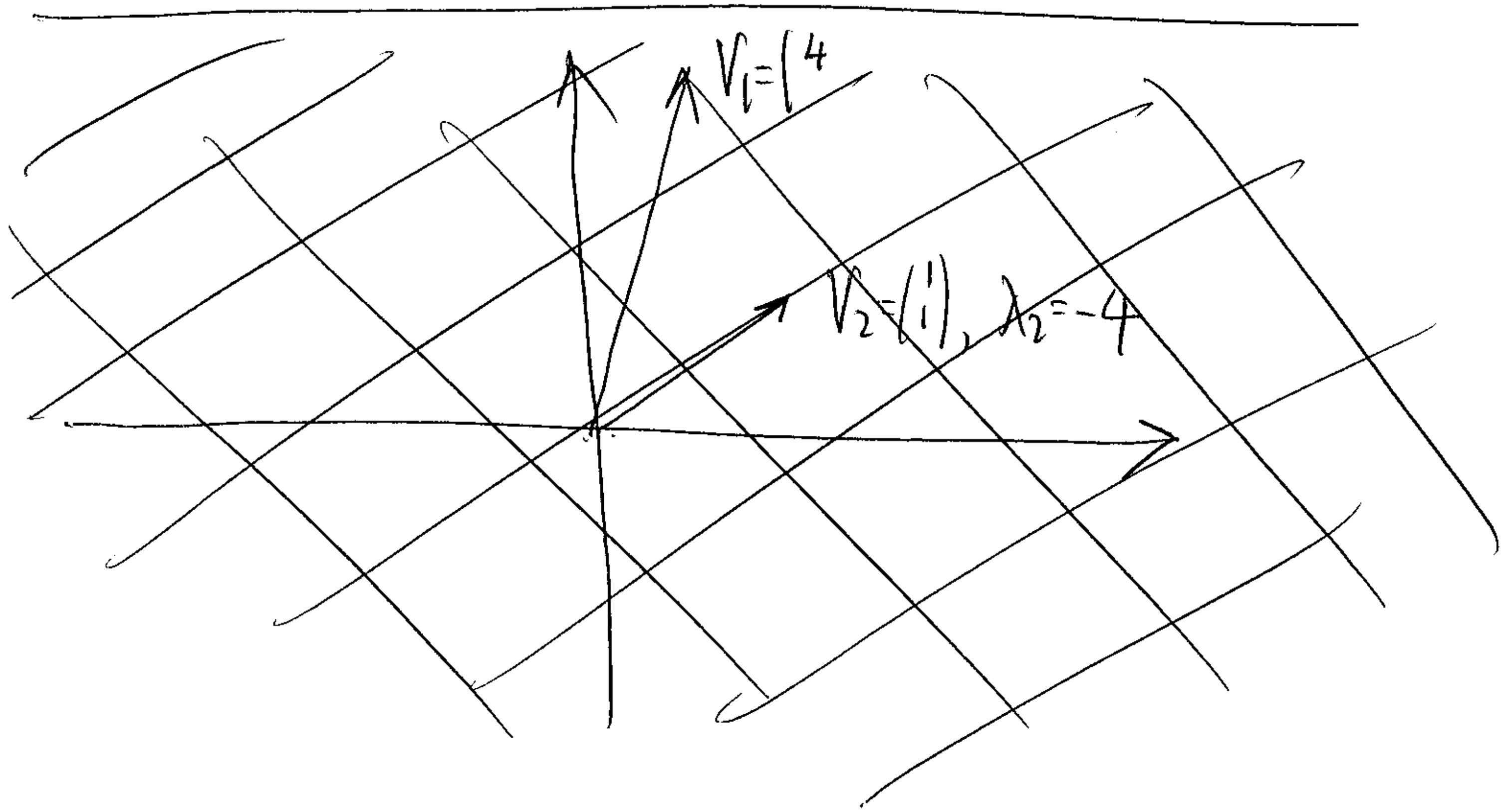
$\boxed{\lambda_1=-1}$ $A = \begin{pmatrix} 0 & -4 \\ 1 & -5 \end{pmatrix}$

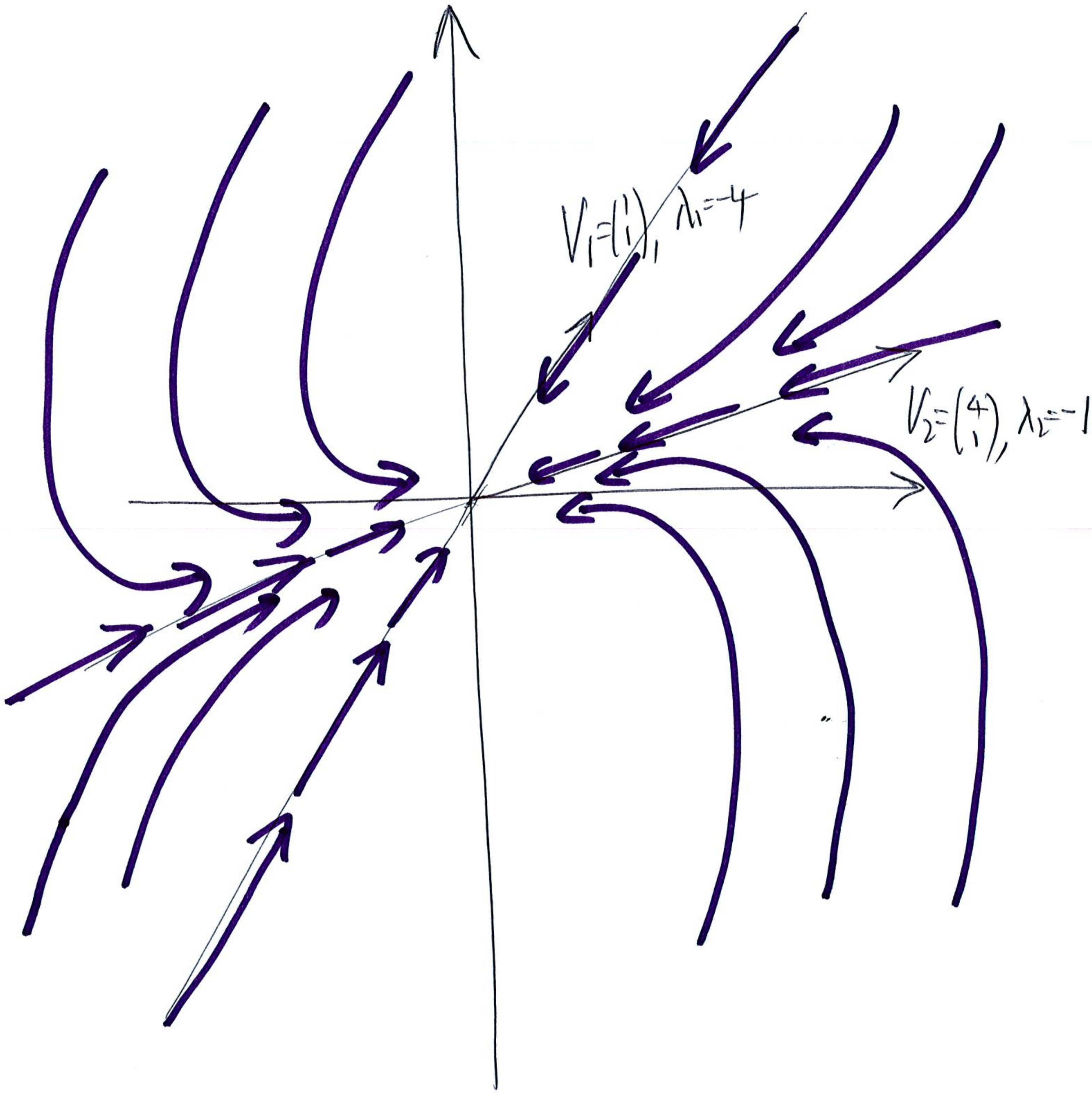
$(A+I)V=0 \Longrightarrow \left(\begin{array}{cc|c} 1 & -4 & 0 \\ 1 & -4 & 0 \end{array}\right) \Rightarrow V_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$\lambda_2=-4$

$(A+4I)V=0 \Longrightarrow \left(\begin{array}{cc|c} 4 & -4 & 0 \\ 1 & -1 & 0 \end{array}\right) \Rightarrow V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

General solution $X(t) = c_1 e^{-t}\begin{pmatrix} 4 \\ 1 \end{pmatrix} + c_2 e^{-4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$





$V_1=\begin{pmatrix}1\\1\end{pmatrix}, \lambda_1=-4$
$V_2=\begin{pmatrix}4\\1\end{pmatrix}, \lambda_2=-1$