MA 303: Differential and Partial Differential Equations for Engineering and Sciences
Fall 2011, Test Two

(Instructor: Aaron N. K. Yip)

- This test booklet has TEN questions – EIGHT MULTIPLE CHOICE and TWO WRITTEN questions – totaling 100 points for the whole test. You have 60 minutes to do this test. **Plan your time well. Read the questions carefully. You do not need to attempt the questions in sequence.**
- This test is **closed book and closed note**. **No calculator nor any other electronic device** is allowed.
- For the multiple choice questions, no partial credit will be given.

  For the written questions, in order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.
- You can use both sides of the papers to write your answers. But please indicate so if you do.

Name: (Write Clearly) Answer Key V02 (Major:)

Question	Score
#1 – 8.(max. 56 pts)	
#9.(max. 20 pts)	
#10.(max. 24 pts)	
Total(100 pts)	

Properties and Formulas for Laplace Transform

Let $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ and $G(s) = \mathcal{L}\{g(t)\} = \int_0^\infty e^{-st} g(t)\,dt$. In the following a and b are arbitrary constants and n is some positive integer.

$$\mathcal{L}\{af + bg\} = aF(s) + bG(s) \qquad (1)$$

$$\mathcal{L}\{f'\} = sF(s) - f(0) \qquad (2)$$

$$\mathcal{L}\{f''\} = s^2F(s) - sf(0) - f'(0) \qquad (3)$$

$$\mathcal{L}\left\{f^{(n)}\right\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0) \qquad (4)$$

$$\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a) \qquad (5)$$

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s) \qquad (6)$$

$$\mathcal{L}\{tf(t)\} = -F'(s) \qquad (7)$$

$$\mathcal{L}\{f * g(t)\} = F(s)G(s) \qquad (8)$$

$$\mathcal{L}\{\delta(t)\} = 1 \qquad (9)$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad (10)$$

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a} \qquad (11)$$

$$\mathcal{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad (12)$$

$$\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2} \qquad (13)$$

$$\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2} \qquad (14)$$

$$\mathcal{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2+b^2} \qquad (15)$$

$$\mathcal{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2} \qquad (16)$$

VERSION 02

1. **Multiple choice question. Enter your answer in the computer scantron.**
   Find the inverse Laplace of the following function:

$$\frac{4s+2}{s^2+2s+5} = \frac{4s+2}{(s+1)^2+2^2}$$

$$= \frac{4(s+1)-2}{(s+1)^2+2^2}$$

$$= 4\left[\frac{s+1}{(s+1)^2+2^2}\right] - \frac{2}{(s+1)^2+2^2}$$

$$\xrightarrow{\mathcal{L}^{-1}} 4e^{-t}\cos 2t - e^{-t}\sin 2t$$

   (a) $e^{-t}\cos 2t + 2e^{-t}\sin 2t$

   (b) $e^{-t}\cos 2t - 4e^{-t}\sin 2t$

   (c) $2e^{-t}\cos 2t - 4e^{-t}\sin 2t$

   (d) $4e^{-t}\cos 2t + e^{-t}\sin 2t$

   (e) $4e^{-t}\cos 2t - e^{-t}\sin 2t$

2. **Multiple choice question. Enter your answer in the computer scantron.**
   Find the inverse Laplace of the following function:

$$\frac{s-1}{s(s^2+1)}$$

   (a) $1 - 2\cos t$

   (b) $3 - \sin t$

   (c) $-1 + \cos t + \sin t$

   (d) $1 - \cos t + \sin t$

   (e) $2 - \cos t + 2\sin t$

$$\frac{s-1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

$$s-1 = A(s^2+1) + (Bs+C)s$$

$$s=0 \Rightarrow A=-1$$

$$s^2 \Rightarrow B=1$$

$$s \Rightarrow C=1$$

$$\Rightarrow -\frac{1}{s} + \frac{s+1}{s^2+1} \Rightarrow -1+\cos t+\sin t$$

3. **Multiple choice question. Enter your answer in the computer scantron.**
Find the solution of the following differential equation:

$$y''(t) + 4y'(t) + 4y(t) = g(t), \quad y(0) = 4, \;\; y'(0) = 2$$

where $g(t)$ is some given function.

(a) $y(t) = (18t + 4)e^{-2t} + \int_0^t g(t-\tau)\tau e^{-2\tau}\, d\tau$

(b) $y(t) = (18t + 4)e^{2t} + \int_0^t g(t-\tau)\tau e^{2\tau}\, d\tau$

(c) $y(t) = (10t + 4)e^{-2t} + \int_0^t g(t-\tau)(t-\tau) e^{2(t-\tau)}\, d\tau$

(d) $y(t) = (10t + 4)e^{-2t} + \int_0^t g(t-\tau)\tau e^{-2\tau}\, d\tau$

(e) $y(t) = (10t + 4)e^{2t} + \int_0^t g(t-\tau)\tau e^{2\tau}\, d\tau$

$s^2 Y - 4s - 2$
$+ 4[sY - 4] + 4Y = G(s)$

$(s^2 + 4s + 4)Y$
$= 4s + 18 + G(s)$

$Y = \frac{4s+18}{s^2+4s+4} + \frac{G(s)}{s^2+4s+4}$

$= \frac{4(s+2)+10}{(s+2)^2} + \frac{G(s)}{(s+2)^2}$

$= \frac{4}{s+2} + \frac{10}{(s+2)^2} + \frac{G(s)}{(s+2)^2}$

$\downarrow$

$4e^{-2t} + 10te^{-2t} + te^{-2t} * g(t)$

4. **Multiple choice question. Enter your answer in the computer scantron.**
Solve the following integral-differential equation:

$$\varphi'(t) - \frac{1}{2}\int_0^t (t-\xi)^2 \varphi(\xi)\, d\xi = t, \quad \varphi(0) = 1$$

(a) $\frac{1}{2}\left(e^t - e^{-t}\right)$

(b) $\frac{1}{2}\left(e^t + e^{-t}\right)$

(c) $\sin t$

(d) $\cos t$

(e) $\cos 2t - \sin 2t$

$s\Phi - 1 - \frac{1}{2}\frac{2}{s^3}\Phi = \frac{1}{s^2}$

$\left(s - \frac{1}{s^3}\right)\Phi = 1 + \frac{1}{s^2} = \frac{s^2+1}{s^2}$

$\Phi = \frac{(s^2+1)s}{s^4-1} = \frac{s}{s^2-1}$

$= \frac{1}{2}\left[\frac{1}{s+1} + \frac{1}{s-1}\right] \rightarrow \frac{1}{2}\left[e^{t} + e^{-t}\right]$

5. **Multiple choice question. Enter your answer in the computer scantron.**
Solve the following integral-differential equation:

$$y''(t) + 3y'(t) + 2y(t) = g(t), \quad y(0) = y'(0) = 0$$

where $g(t)$ is the function which is equal to one for $1 \le t \le 2$ and zero otherwise.

(a) $\left(e^{-(t-1)} - e^{-2(t-1)}\right) u(t-1) - \left(e^{-(t-2)} - e^{-2(t-2)}\right) u(t-2)$

(b) $\left(\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}\right) u(t-1) - \left(\frac{1}{2} - e^{t} + \frac{1}{2}e^{-2t}\right) u(t-2)$

(c) $\left(\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\right) u(t-1) - \left(\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\right) u(t-2)$

(d) $\left(e^{-(t-1)} - \frac{1}{2}e^{-2(t-1)}\right) u(t-1) - \left(e^{-(t-2)} - \frac{1}{2}e^{-2(t-2)}\right) u(t-2)$

(e) $\left(e^{-t} - e^{-2t}\right) u(t-1) - \left(e^{-t} - e^{-2t}\right) u(t-2)$

$$(s^2+3s+2)Y = \mathcal{L}(u(t-1) - u(t-2)) = \frac{e^{-s} - e^{-2s}}{s}$$

$$Y = \frac{1}{s(s+1)(s+2)}\left[e^{-s} - e^{-2s}\right]$$

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

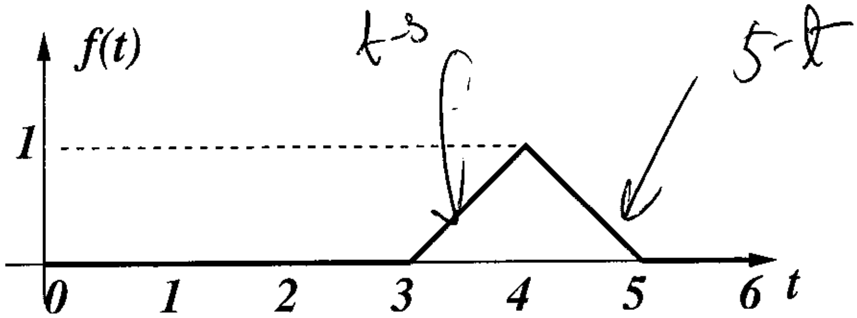
$$1 = A(s+1)(s+2) + Bs(s+2) + C(s)(s+1)$$

$s=0 \Rightarrow A = 1/2$

$s=-1 \Rightarrow B = -1 \Rightarrow \frac{1}{s(s+1)(s+2)} \xrightarrow{\mathcal{L}^{-1}} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$

$s=-2 \Rightarrow C = 1/2$

6. **Multiple choice question. Enter your answer in the computer scantron.**
Find the Laplace transform of the following function f:



(a) $\dfrac{e^{-3s} + 2e^{-4s} - e^{-5s}}{s^3}$

(b) $\dfrac{e^{3s} - 2e^{4s} - e^{5s}}{s^3}$

(c) $\dfrac{e^{-3s} - 2e^{-4s} + e^{-5s}}{s^2}$

(d) $\dfrac{e^{-3s} + 2e^{-4s} - e^{-5s}}{s^2}$

(e) $\dfrac{e^{3s} + 2e^{4s} - e^{5s}}{s^2}$

$$f(t) = (t-3)\left[u(t-3) - u(t-4)\right]$$
$$+ (5-t)\left[u(t-4) - u(t-5)\right]$$

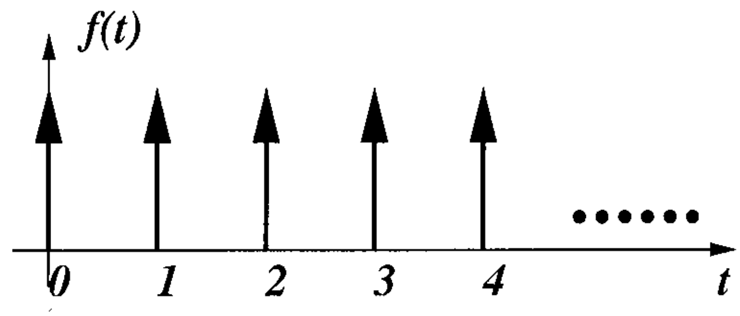
$8-2t$

$$= (t-3)\,u(t-3) + (5-t-t+3)\,u(t-4) + (t-5)\,u(t-5)$$
$$= (t-3)(u(t-3)) - 2(t-4)\,u(t-4) + (t-5)\,u(t-5)$$

$$\Big\downarrow \mathcal{L}$$

$$\frac{e^{-3s} - 2e^{-4s} + e^{-5s}}{s^2}$$

7. **Multiple choice question. Enter your answer in the computer scantron.**
Find the Laplace transform of the following function f:



(sum of an infinite sequence of delta functions location at $0, 1, 2, 3, 4, \ldots$ and so forth)

(a) $\dfrac{1+s}{1-e^{-s}}$

(b) $\dfrac{1-s}{1+e^{-s}}$

(c) $\dfrac{1}{1+e^{-s}}$

(d) $\dfrac{1}{1-e^{-s}}$

(e) $\dfrac{1+e^{-s}}{1-e^{-s}}$

$f(t) = \delta(t) + \delta(t-1) + \delta(t-2) + \cdots$

$\downarrow \mathcal{L}$

$1 + e^{-s} + e^{-2s} + e^{-3s} \cdots$

$= \dfrac{1}{1-e^{-s}}$

8. **Multiple choice question. Enter your answer in the computer scantron.**
Apply the following *first order backward Euler scheme*:

$$y' = f(t, y): \quad y_{n+1} = y_n + f(t_{n+1}, y_{n+1})(t_{n+1} - t_n)$$

to solve:

$$y' = 1 - t - y, \quad y(0) = 1.$$

Consider uniform time stepping size: $t_{n+1} - t_n = h$. Let $y_0 = y(0)$. The explicit formula for y_2 is:

(a) $\dfrac{1 - 2h - 2h^2 + 2h^3}{(1+h)^2}$

(b) $\dfrac{1 + 2h - 2h^2 - 2h^3}{(1+h)^2}$

(c) $\dfrac{2 - 3h^2 + 2h^3}{(1+h)^2}$

(d) $\dfrac{1 - h + 3h^2}{(1+h)^2}$

(e) $\dfrac{1 + h + h^2}{(1+h)^2}$

$$y_1 = y_0 + (1 - h - y_1) h$$

$$(1+h) y_1 = 1 + h - h^2$$

$$y_1 = \frac{1+h-h^2}{1+h}$$

$$y_2 = y_1 + (1 - t_2 - y_2) h$$

$$(1+h) y_2 = \frac{1+h-h^2}{1+h} + (1-2h) h$$

$$= \frac{1+h-h^2 + (h-2h^2)(1+h)}{1+h}$$

$$= \frac{1+h-\cancel{h^2} + h + \cancel{h^2} - 2h^2 - 2h^3}{1+h}$$

$$= \frac{1+2h-2h^2-2h^3}{(1+h)} \Rightarrow \boxed{y_2 = \frac{1+2h-2h^2-2h^3}{(1+h)^2}}$$

9. Find the Laplace Transforms of the following function:

   (a) $t^2 e^t \sin 4t$. (Hint: make use of *properties of Laplace Transform*! There are many ways you can do this problem.)

   (b) $g(t) = \int_0^t f(\xi)\, d\xi$. (Express the Laplace Transform of g in terms of that of f.) (Hint: try differentiate!)

(a) <u>M1</u>:

$$e^t \sin 4t \xleftrightarrow[\text{(formula (15))}]{\mathcal{L}} \frac{4}{(s-1)^2+16}$$

$$-t\, e^t \sin 4t \xleftrightarrow[\text{(formula (7))}]{\mathcal{L}} \left(\frac{4}{(s-1)^2+16}\right)'$$

$$= \frac{-8(s-1)}{[(s-1)^2+16]^2}$$

$$\boxed{t^2 e^t \sin 4t} = (-t)^2 e^t \sin 4t \longleftrightarrow \left(\frac{-8(s-1)}{[(s-1)^2+16]^2}\right)'$$

$$= \frac{((s-1)^2+16)^{\cancel{2}}(-8) + 8(s-1)\, 2\, \cancel{[(s-1)^2+16]}\,(2)(s-1)}{[(s-1)^2+16]^4}$$

$$= \frac{-8[(s-1)^2+16] + 32(s-1)^2}{[(s-1)^2+16]^3} = \boxed{\frac{24(s-1)^2 - 128}{[(s-1)^2+16]^3}} \quad \#$$

This is a scrap paper.

M2:

$$t^2 e^t \xleftrightarrow[\text{formula (12)}]{} \frac{2}{(s-1)^3}$$

$$t^2 e^t \sin 4t = t^2 e^t \left[\frac{e^{4ti} - e^{-4ti}}{2i}\right]$$

$$= \frac{1}{2i}\left[t^2 e^t e^{4ti} - t^2 e^t e^{-4ti}\right]$$

$$\Big\downarrow \text{formula (5)}$$

$$= \frac{1}{2i}\left[\frac{2}{(s-4i-1)^3} - \frac{2}{(s+4i-1)^3}\right]$$

$$= \frac{1}{i}\left[\frac{(s-1+4i)^3 - (s-1-4i)^3}{(s-1-4i)^3 (s-1+4i)^3}\right]$$

$$= \frac{\frac{1}{i}\left[\begin{array}{l}\cancel{(s-1)^3} + 3(s-1)^2(4i) + \cancel{3(s-1)(4i)^2} + (4i)^3 \\ -\left(\cancel{(s-1)^3} - 3(s-1)^2(4i) + \cancel{3(s-1)(4i)^2} - (4i)^3\right)\end{array}\right]}{\left[(s-1)^2 - (4i)^2\right]^2}$$

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$$= \frac{1}{i}\left[\frac{24(s-1)^2 i + 128 i^3}{\left((s-1)^2+16\right)^3}\right]$$

$$\boxed{= \frac{24(s-1)^2 - 128}{\left[(s-1)^2+16\right]^3}}$$

(b) $g(t) = \int_0^t f(\xi)\,d\xi$, **Note** $g(0)=0$

$g'(t) = f(t)$

Hence $\mathcal{L}(g'(t)) \overset{\text{formula (2)}}{=} s\,\mathcal{L}(g(t)) - \cancel{g(0)}$

$$\mathcal{L}(f) = s\,\mathcal{L}(g(t)) \Longrightarrow \boxed{\mathcal{L}(g(t)) = \frac{\mathcal{L}(f(t))}{s}}$$

10. Given the following system of differential equation:

$$\begin{aligned}\dot{x} &= -x + y + x(x^2 + y^2)\\ \dot{y} &= -x - y + y(x^2 + y^2)\end{aligned}$$

Consider the polar coordinate (r, θ) representation of (x, y), i.e. $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

(a) Find $\frac{d}{dt}r$. Express your answer in terms of r and θ *only*.

(b) Find $\frac{d}{dt}\theta$. Express your answer in terms of r and θ *only*.

(c) Plot in the *same graph* the trajectories of the solution of the system when (i) $(x(0), y(0)) = (0.5, 0)$; (ii) $(x(0), y(0)) = (1, 0)$; (iii) $(x(0), y(0)) = (2, 0)$;

(Hint: $\frac{d}{d\xi}\tan^{-1}(\xi) = \frac{1}{1+\xi^2}$; CHAIN RULE, CHAIN RULE, CHAIN RULE...)

(a) $r = \sqrt{x^2+y^2}$

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{\sqrt{x^2+y^2}} = \left(x\left[-x + \cancel{y} + x r^2\right] + y\left[-\cancel{x} - y + y r^2\right] \right) \Big/ r$$

$$= -x^2 - y^2 + (x^2+y^2) r^2 \Big/ r = -r + r^3$$

$$\boxed{\dot{r} = -r + r^3}$$

(b) $$\dot{\theta} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left[\frac{x\dot{y} - y\dot{x}}{x^2}\right]$$

$$\dot{\theta} = \frac{1}{x^2+y^2}\begin{bmatrix} x\,(\cancel{-x}-y+\cancel{yr^2}) \\ -y\,(\cancel{-x}+y+\cancel{xr^2}) \end{bmatrix}$$

$$= \frac{-x^2-y^2}{x^2+y^2} = -1 \qquad \boxed{\dot{\theta} = -1}$$

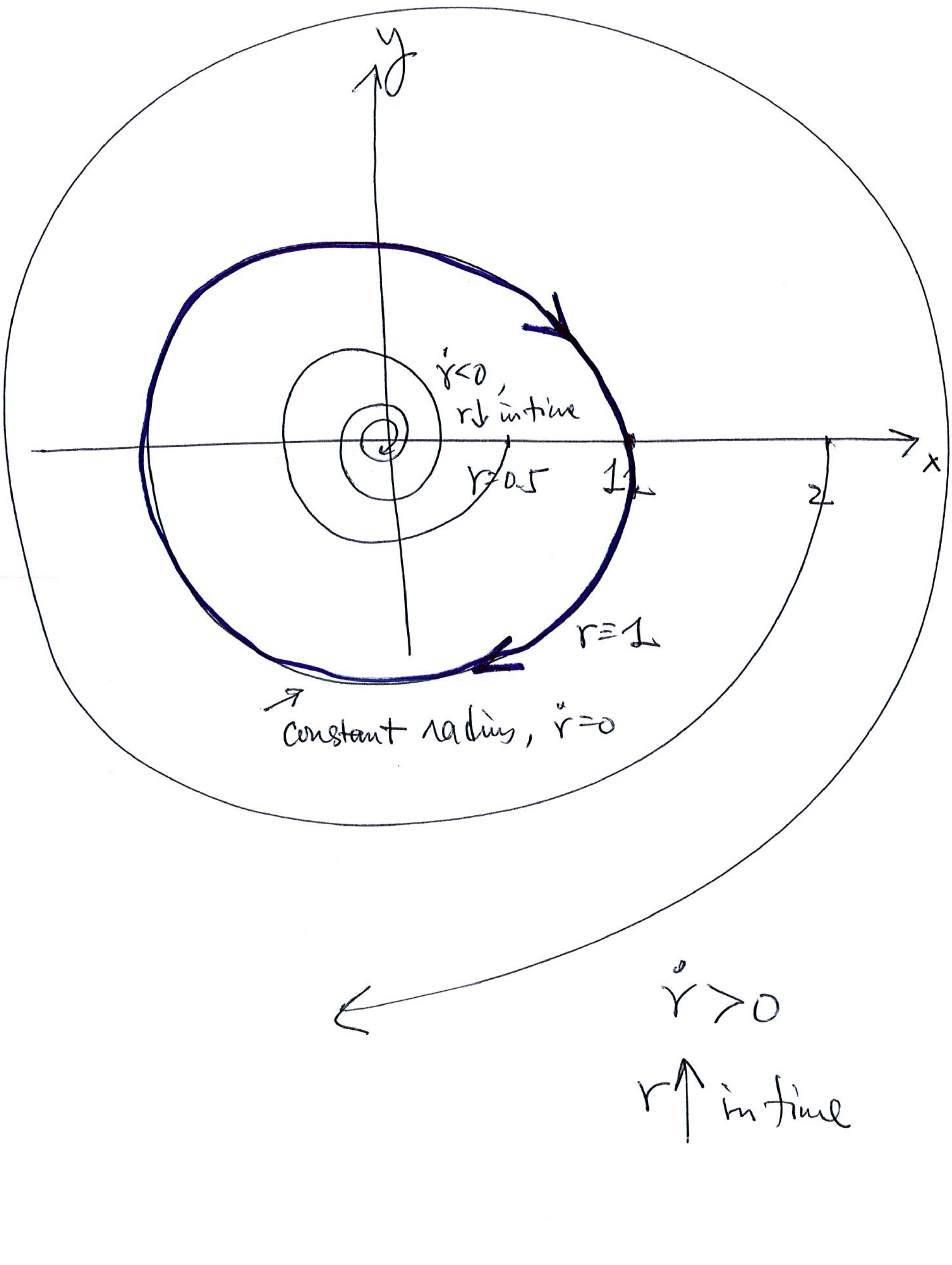
(c) $\dot{\theta} = -1 \overset{<0}{} \Rightarrow$ angle rotate counter-clockwise with constant speed

$$\boxed{\dot{r} = -r + r^3 = r(r^2-1)} = r(r-1)(r+1)$$

$r > 1 \Rightarrow \dot{r} > 0 \Rightarrow r \uparrow$ in time

$0 < r < 1 \Rightarrow \dot{r} < 0 \Rightarrow r \downarrow$ in time

$r = 1 \Rightarrow \dot{r} = 0 \Rightarrow r$ is constant!


y
x
$\dot{r}<0$,
$r\downarrow$ in time
$r=0.5$
1
2
$r=1$
constant radius, $\dot{r}=0$
$\dot{r}>0$
$r\uparrow$ in time