MA/STAT 538 Spring 2009 (Yip) Solution of Final Exam.

1. Prove the following result (which is used in class several times):

If $\lim\limits_{n\longrightarrow\infty} a_n = L$, then $\lim\limits_{n\longrightarrow\infty} \frac{a_1+a_2+\cdots a_n}{n} = L$.

<u>Let ε be given</u>

Consider $\left|\frac{a_1+a_2+\cdots+a_n}{n} - L\right|$

$$= \left|\frac{(a_1-L)+(a_2-L)+\cdots+(a_n-L)}{n}\right|$$

$$\leq \left|\frac{(a_1-L)+\cdots+(a_{N_1}-L)}{n}\right| + \left|\frac{(a_{N_1+1}-L)+\cdots+(a_n-L)}{n}\right|$$

Now choose N_1 be such that

$|a_m - L| \leq \varepsilon/2 \quad \forall\, m \geq N_1 \quad (\text{as } \lim\limits_{n\to\infty} a_n = L)$.

& choose $n \geq N_2\ (\geq N_1)$ such that

$$\left|\frac{(a_1-L)+\cdots+(a_{N_1}-L)}{n}\right| \leq \frac{|a_1-L|+\cdots+|a_{N_1}-L|}{n} \leq \varepsilon/2.$$

(& automatically $\left|\frac{(a_{N_1+1}-L)+\cdots+(a_n-L)}{n}\right| \leq \frac{(n-N_1-1)\ \varepsilon/2}{n} \leq \varepsilon/2$)

Hence $\forall\, n \geq N_2$, $\left|\frac{a_1+\cdots+a_n}{n} - L\right| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$.

2. The Bochner Theorem says a continuous, complex valued function $\varphi(\cdot) : \mathbf{R} \longrightarrow \mathbf{C}$ with $\varphi(0) = 1$ is the characteristic function of some probability measure μ if and only if φ is semi-positive definite in the following sense: for any n real numbers $\lambda_1, \lambda_2, \dots \lambda_n$, and n complex numbers $c_1, c_2, \dots c_n$, it holds that

$$\sum_{i,j=1}^{n} \varphi(\lambda_i - \lambda_j) c_i \overline{c_j} \geq 0$$

where the over bar refers to complex conjugate.

Prove the "only if" part of the above statement.

$$\sum_{i,j} \varphi(\lambda_i - \lambda_j) c_i \overline{c_j}$$

$$= \sum_{i,j} \left(E e^{i(\lambda_i - \lambda_j)X} \right) c_i \overline{c_j}$$

$$= E\left[\sum_{i,j} e^{i(\lambda_i - \lambda_j)X} c_i \overline{c_j} \right]$$

$$= E\left[\sum_{i,j} e^{i\lambda_i X} c_i \, e^{-i\lambda_j X} \overline{c_j} \right]$$

$$= E\left[\sum_{i,j} e^{i\lambda_i X} c_i \, \overline{e^{i\lambda_j X} c_j} \right]$$

$(|z|^2 = z\overline{z})$

$$= E\left[\sum_i e^{i\lambda_i X} c_i \right] \overline{\left[\sum_j e^{i\lambda_j X} c_j \right]} = E\left| \sum_i e^{i\lambda_i X} c_i \right|^2$$

≥ 0

$(a_1 + \cdots + a_n)(b_1 + \cdots + b_n) = \sum_{i,j} a_i b_j$

3. Let X_λ denote a Poisson random variable with parameter λ, i.e. $\Pr(X_\lambda = k) = \frac{e^{-\lambda}\lambda^k}{k!}$ for $k = 0, 1, \ldots$. Prove that

$$\frac{X_\lambda - \lambda}{\sqrt{\lambda}} \Longrightarrow \mathcal{N}(0,1)$$

<u>M1</u>. Let $\lambda = n$ (an integer)

$$X_n \overset{\mathcal{D}}{\sim} X_1 + X_2 + \cdots + X_n$$

X_i – iid, Poisson, parameter 1.

$$\frac{X_n - n}{\sqrt{n}} \overset{\mathcal{D}}{\sim} \frac{X_1 + X_2 + \cdots + X_n - n}{\sqrt{n}}$$

$$= \frac{(X_1 - 1) + (X_2 - 1) + \cdots + (X_n - 1)}{\sqrt{n}}$$

$$\overset{\mathcal{D}}{\Longrightarrow} N(0,1) \qquad (\text{CLT})$$

In general, $\lambda = n + r$ where $0 \leq r < 1$.

$$\frac{X_\lambda - \lambda}{\sqrt{\lambda}} = \frac{X_n + X_r - n - r}{\sqrt{n+r}} = \left(\frac{X_n - n}{\sqrt{n+r}}\right) + \left(\frac{X_r - r}{\sqrt{n+r}}\right)$$

$$= \underset{\substack{\swarrow \\ 1}}{\frac{\sqrt{n}}{\sqrt{n+r}}} \underset{\substack{\Downarrow \mathcal{D} \\ N(0,1)}}{\left(\frac{X_n - n}{\sqrt{n}}\right)} + \underset{\substack{\swarrow \text{ in Prob.} \\ 0}}{\frac{X_r - r}{\sqrt{n+r}}} \overset{\mathcal{D}}{\Longrightarrow} N(0,1)$$

This is a scrap paper.

M2 Use char. fct

$$E\left[e^{is\left(\frac{X_\lambda-\lambda}{\sqrt{\lambda}}\right)}\right] = e^{-is\sqrt{\lambda}}\, E e^{\frac{is X_\lambda}{\sqrt{\lambda}}}$$

$$= e^{-is\sqrt{\lambda}}\, E e^{i\left(\frac{s}{\sqrt{\lambda}}\right)X_\lambda}$$

$$\Big(\text{Note } E e^{itX_\lambda} = \sum_{k=0}^{\infty} e^{itk}\frac{e^{-\lambda}\lambda^k}{k!} = \sum_{k=0}^{\infty} e^{-\lambda}\frac{(e^{it}\lambda)^k}{k!}$$

$$= e^{-\lambda}e^{\lambda e^{it}} = e^{\lambda(e^{it}-1)}\Big)$$

$$= e^{-is\sqrt{\lambda}}\, e^{\lambda\left(e^{i\frac{s}{\sqrt{\lambda}}}-1\right)}$$

$$= e^{\lambda e^{i\frac{s}{\sqrt{\lambda}}}-\lambda-is\sqrt{\lambda}}$$

$$= e^{\left[\lambda\left(\cancel{1+}\cancel{\frac{is}{\sqrt{\lambda}}}-\frac{s^2}{2\lambda}-\frac{is^3}{\lambda\sqrt{\lambda}}+\cdots\right)\cancel{-\lambda}\cancel{-is\sqrt{\lambda}}\right]}$$

$$= e^{-\frac{s^2}{2}+o(1)} \xrightarrow{\lambda\to\infty} e^{-\frac{s^2}{2}} = \text{char. of } N(0,1)$$

4. Let $\{Y_i\}_{i\geq 1}$ be iid with $EY_i = 0$ and $\mathrm{Var}(Y_i) = 1$. Let $\{Z_i\}_{i\geq 1}$ be independent random variables which are also independent of the Y_i's with

$$\Pr(Z_i = i) = \frac{1}{i^2}, \quad \Pr(Z_i = -i) = \frac{1}{i^2} \quad \text{and} \quad \Pr(Z_i = 0) = 1 - \frac{2}{i^2}$$

Let $X_i = Y_i + Z_i$. Show that

$$\frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n}} \Longrightarrow \mathcal{N}(0,1)$$

$$\frac{X_1+X_2+\cdots+X_n}{\sqrt{n}}$$

$$= \underbrace{\frac{(Y_1+Y_2+\cdots+Y_n)}{\sqrt{n}}}_{\Big\Downarrow \mathcal{D} \atop N(0,1)} + \underbrace{\left(\frac{Z_1+Z_2+\cdots+Z_n}{\sqrt{n}}\right)}_{\Big\Downarrow}$$

Note that for: $P(Z_i \neq 0 \text{ i.o})$

Consider $\displaystyle\sum_{i=1}^{\infty} P(Z_i \neq 0)$

$$= \sum \frac{1}{i^2} < \infty$$

Hence $P(Z_i \neq 0 \text{ i.o.}) = 0$ (BCI)

so that $\dfrac{Z_1+\cdots+Z_n}{\sqrt{n}} \longrightarrow 0$ a.s.

5. (a) Let $\{X_n\}_{n\geq 0}$ be a sequence of iid random variables with mean zero and finite variance. Let a be any finite, positive number. Find

$$\lim_{n\longrightarrow\infty} \Pr\left(\max_{1\leq k\leq n} |X_k| \leq a\sqrt{n}\right).$$

What is the implication of your result in terms of the central limit theorem?

(b) Let $\{X_n\}_{n\geq 0}$ be a sequence of iid random variables with a common pdf $p(x) = |x|^{-3}$ for $|x| \geq 1$ and $p(x) = 0$ otherwise. Let a be any finite, positive number. Find

$$\lim_{n\longrightarrow\infty} \Pr\left(\max_{1\leq k\leq n} |X_k| \leq a\sqrt{n\log n}\right).$$

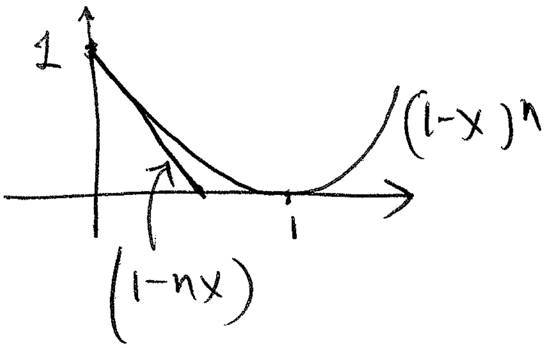
What is the implication of your result in terms of the central limit theorem?

(a) $$P\left(\max_{1\leq k\leq n} |X_k| \leq a\sqrt{n}\right)$$

$$= P(|X_1| \leq a\sqrt{n})^n$$

$$= \left[1 - P(|X_1| \geq a\sqrt{n})\right]^n$$

Note: $(1-x)^n \geq 1-nx$ for $(0 \leq x \leq 1)$



$$\geq 1 - nP(|X_1| \geq a\sqrt{n}).$$

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Now: $$nP(|X_1| \geq a\sqrt{n}) \leq \frac{\cancel{n}}{a^2\cancel{n}} \underbrace{\int_{\{|X_1| \geq a\sqrt{n}\}} |X_1|^2 \, dP}_{\downarrow\ n\to\infty \atop 0}$$

$$\to 0$$

Hence $$P\left(\max_{1\leq k\leq n} |X_k| \leq a\sqrt{n}\right) \xrightarrow{n\to\infty} 1$$

In terms of CLT, $$\frac{X_1+X_2+\cdots+X_n}{\sqrt{n}} \xRightarrow{\mathcal{D}} N(0,1)$$

i.e. roughly $X_1+\cdots+X_n \sim Z\sqrt{n}$, $Z\sim N(0,1)$

each term ~~contributes~~ makes only very small contribution to the overall sum.

This is a scrap paper.

(b) $$P\left(\max_{1 \le k \le n} |X_k| \le a\sqrt{n \log n} \right)$$

(Similar to (a)) $$= \left[1 - P\left(|X_1| \ge a\sqrt{n \log n}\right)\right]^n$$

$$= \left(1 - 2\int_{a\sqrt{n \log n}}^{\infty} \frac{dx}{x^3}\right)^n$$

$$= \left(1 - \frac{1}{a n \log n}\right)^n \xrightarrow{n \to \infty} 1$$

Same conclusion as (a)

(Note: CLT in (b) takes the form:

$$\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n \log n}} \xrightarrow{\mathcal{D}} N(0,1) \Big).$$

6. Let $\{X_n\}_{n\geq 1}$ be iid nonnegative random variables. Let c be some number in the range $(0,1)$. Show that

$$\sum_{n=1}^{\infty} e^{X_n} c^n \text{ is } \begin{cases} < \infty \text{ a.s} & \text{if } EX_1 < \infty \\ = \infty \text{ a.s} & \text{if } EX_1 = \infty \end{cases}$$

(Hint: make use of midterm of this semester, problem #4: $\limsup_n \frac{X_n}{n}$ equals zero or infinity depending on whether EX_1 is finite or infinity.)

For $EX_1 < \infty$: $\overline{\lim}_n \frac{X_n}{n} = 0$ P a.s.

i.e. $\forall n$ large, $\frac{X_n}{n} \leq \varepsilon$, i.e. $X_n \leq \varepsilon n$

Hence $\sum_{n=1}^{\infty} e^{X_n} c^n \leq \sum_{n=1}^{\infty} e^{\varepsilon n} c^n$

$= \sum_{n=1}^{\infty} (e^{\varepsilon} c)^n$ choose ε small enough s.t. $e^{\varepsilon} c < 1$ (since $0 < c < 1$)

$< \infty$. P a.s.

This is a scrap paper.

For $EX_1 = \infty$, $\overline{\lim} \frac{X_n}{n} = \infty$ P a.s.

i.e. there are $\underbrace{\text{infinitely many } n\text{'s}}_{}$ s.t.

$$\frac{X_n}{n} \geq M \gg 1 \quad (M \text{ to be chosen}).$$

so $e^{X_n} c^n \geq e^{Mn} c^n = (e^M c)^n$

Choose $M \gg 1$ s.t. $e^M c > 1$

Then $e^{X_n} c^n \geq 1$ i.o.

Hence $\sum_n e^{X_n} c^n = \infty$. P a.s

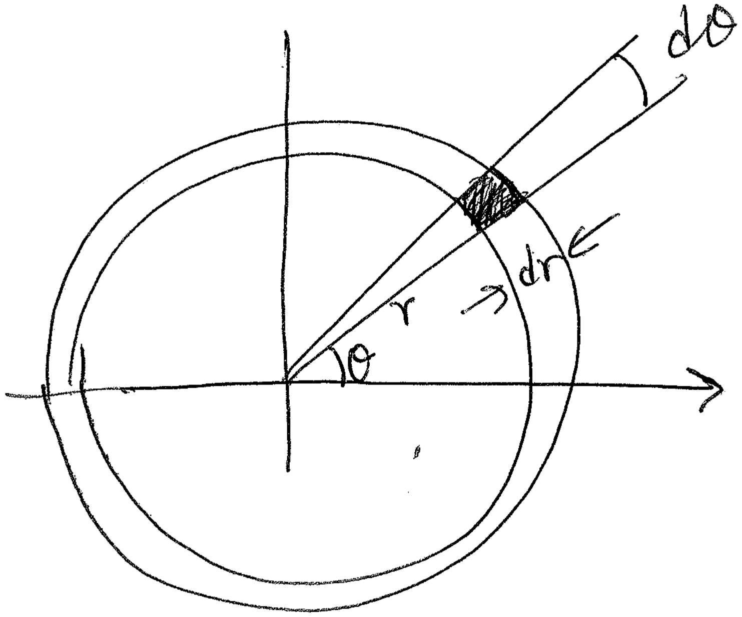
7. Let U and V be independent random variables that are uniformly distributed in $[0,1]$. Define:

$$X = \sqrt{-2\log(U)}\cos(2\pi V) \text{ and } Y = \sqrt{-2\log(U)}\sin(2\pi V).$$

Show that X and Y are independent and $\mathcal{N}(0,1)$ distributed.

Use Polar Coordinates : $\sqrt{x^2+y^2} = -2\log U \; (= r)$

$\tan^{-1}\left(\frac{Y}{X}\right) = 2\pi V \; (= \theta)$



Let $p(x,y)$ be the joint pdf of X & Y in cartesian coord. & $q(r,\theta)$ be the joint pdf of X & Y in polar coord.

Then $p(x,y)\,dx\,dy = q(r,\theta)\,r\,dr\,d\theta$

Compute $q(r,\theta)$:

Prob. of shaded area $= P\left(\sqrt{-2\log U} \in (r, r+dr), d\theta\right)$:

This is a scrap paper. as U & V are independent, so are

$$\underset{\left(-2\log u\right)}{\underset{\shortparallel}{r}} \quad \& \quad \underset{\left(2\pi V\right)}{\underset{\shortparallel}{\theta}}$$

Hence $P\left(\sqrt{-2\log u} \in (r, r+dr),\ d\theta\right)$

$$= P\left(\sqrt{-2\log u} \in (r, r+dr)\right) P(d\theta)$$

<u>Note</u> $P(d\theta) = \dfrac{d\theta}{2\pi}$

For $P\left(\sqrt{-2\log u} \in (r, r+dr)\right)$ $\simeq r^2 + 2r\,dr$

$$= P\left(-2\log u \in \left(r^2, (r+dr)^2\right)\right)$$

$$= P\left(\log u \in \left(-\frac{r^2}{2} - r\,dr,\ -r^2/2\right)\right)$$

$$= P\left(u \in \left(e^{-r^2/2} e^{-r\,dr},\ e^{-r^2/2}\right)\right)$$

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$$= \left(e^{-r^2/2} - e^{-r^2/2} e^{-r\,dr}\right)$$

$$= e^{-r^2/2}\left(1 - e^{-r\,dr}\right)$$

$$= e^{-r^2/2}\left(1 - 1 + r\,dr + \cdots\right)$$

$$= r e^{-r^2/2}\,dr.$$

Hence Prob. of shaded area $= \dfrac{r e^{-r^2/2}\,dr\,d\theta}{(2\pi)}$.

But the area of the shaded area $= \underline{r\,dr\,d\theta}$.

Hence $g(r,\theta) = \dfrac{e^{-r^2/2}}{2\pi} = \dfrac{e^{-\left(\frac{x^2+y^2}{2}\right)}}{2\pi} = \dfrac{e^{-x^2/2}}{\sqrt{2\pi}}\,\dfrac{e^{-y^2/2}}{\sqrt{2\pi}}$

ie. $p(x,y) = \dfrac{e^{-x^2/2}}{\sqrt{2\pi}} \cdot \dfrac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}$.

ie. X, Y are iid $N(0,1)$.