Homework 1

MA 538 Spring 2009 (Aaron N. K. Yip)

Due: 3pm, Monday, Feb. 9

From Textbook (Billingsley):

page 33: #2.3, 2.4, 2.13, 2.15

page 46: #3.1, 3.2, 3.3, 3.5

Additional Problems

1. Consider the iid coin tossing experiment such that $P(\text{Head}) = p$ (and $P(\text{Tail}) = q = 1 - p$). Recall the definition $P_n(k) = {}_nC_k p^k q^{n-k}$ for $0 \leq k \leq n$.

   (a) Prove that $P_n(k) \sim \frac{1}{\sqrt{2\pi npq}} e^{-\frac{z^2}{2}}$ in the sense that $\lim \frac{P_n(k)}{\frac{1}{\sqrt{2\pi npq}} e^{-\frac{z^2}{2}}} = 1$ where the limit is taken such that $n, k \to \infty$ and $\frac{k - np}{\sqrt{npq}}$ converge to a finite value z,
   (Hint: use Stirling's Formula and express k in terms of z.)

   (b) From the above, prove the Central Limit Theorem for the coin tossing experiment, i.e. for any finite values a and b,

   $$\lim_{n\to\infty} \sum_{a\sqrt{npq} \leq k - np \leq b\sqrt{npq}} P_n(k) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz$$

   (Hint: consider Riemann sum.)

2. Consider the iid coin tossing experiment with n tossing. Suppose $p = P(\text{Head}) = \frac{\lambda}{n}$ where λ is some fixed number. Find the following limit:

   $$\lim_{n\to\infty} P_n(k)$$

   What is the common name for the limiting object (distribution)?

3. Let Ω be a set – to avoid logical tautology, assume Ω has infinitely many elements. Let

   $$\begin{aligned} \mathcal{F} &= \{A \subset \Omega : \#(A) < \infty, \text{ or } \#(A^c) < \infty\} \\ \mathcal{G} &= \{A \subset \Omega : A \text{ is countable or } A^c \text{ is countable.}\} \end{aligned}$$

   Show that

(a) $\mathcal{F}$ is a field.

(b) $\mathcal{G}$ is a σ-field.

(c) $\mathcal{G} = \sigma(\mathcal{F})$, i.e. $\mathcal{G}$ is the σ-field generated by $\mathcal{F}$.

4. Let P be a *finitely additive* probability measure on a field $\mathcal{F}$. Prove that the following four statements are equivalent, i.e. one statement is true *if and only if* any other is true:

   (a) P is *countably-additive*, i.e. for any $A_n \in \mathcal{F}$, disjoint and $\bigcup_n A_n \in \mathcal{F}$, then $P(\bigcup_n A_n) = \sum_n P(A_n)$.

   (b) P is continuity from above, i.e. for any $A_n \in \mathcal{F}$, $A_1 \supset A_2 \supset A_3 \cdots$ and $\bigcap_n A_n \in \mathcal{F}$, then $P(\bigcap_n A_n) = \lim_n P(A_n)$.

   (c) P is continuity from below, i.e. for any $A_n \in \mathcal{F}$, $A_1 \subset A_2 \subset A_3 \cdots$ and $\bigcup A_n \in \mathcal{F}$, then $P(\bigcup_n A_n) = \lim_n P(A_n)$.

   (d) P is continuity from above at the empty set, i.e. for any $A_n \in \mathcal{F}$, $A_1 \supset A_2 \supset A_3 \cdots$ and $\bigcap_n A_n = \emptyset$, then $\lim_n P(A_n) = 0$.

5. With the definition of the outer-measure P^* and measurable set $\mathcal{M}$ as defined in class (and the textbook) show that for all $A, B \in \mathcal{M}$ and disjoint, then

$$P^*((A \cup B) \cap E) = P^*(A \cap E) + P^*(B \cap E)$$

   (This is the part I omited in class, which will then complete the proof of Theorem 3.1.)

6. (This problem shows that the idea of using outer-measure is a very natural procedure.)

   Let Ω be a sample space and you are given the following items – *measuring devices*:

   - $\mathcal{F}$: a collection of some subsets of Ω.
     Note that $\mathcal{F}$ is NOT required to be a field. But for simplicity, assume that $\emptyset$ and Ω are both in $\mathcal{F}$.
   - S: a map from $\mathcal{F}$ to *non-negative* real number.
     Note that S is NOT required to be additive or even monotone, i.e. it is not necessarily true that $S(A) \leq S(B)$ if $A \subset B$. But for simplicity, assume $P(\emptyset) = 0$.

   Then consider the following definitions:

- Outer-measure for any subset A of Ω:

$$S^*(A) = \inf\left\{\sum_n S(A_n): \;\; A \subset \bigcup_n A_n, \;\; A_n \in \mathcal{F}, \;\; n = 1, 2, 3, \ldots\right\}$$

- Measurable subsets of Ω:

$$\mathcal{M} = \{A: \;\; S^*(A \cap E) + S^*(A^c \cap E) = S^*(E) \;\text{ for all } E \subset \Omega\}$$

Using just the above information, prove or disprove that S^* is a countably-additive measure on the σ-field $\mathcal{M}$. If not true, indicate which statement(s) is not true and where the proof breaks down. Note that we cannot in general assert that $S^*(\Omega) = 1$ and hence S^* might not be a probability measure.