MA 538 (Spring 2009, Yip) HW3 Solution

(B.1)

$$(\Omega, \mathcal{F}) \xrightarrow{T} (\Omega', \mathcal{F}')$$

$$\Omega = \bigcup_n A_n, \quad A_n \cap A_m = \phi, \ n \neq m$$

$$A_n \in \mathcal{F}.$$

Let $T_n = T\big|_{A_n}$

$$\mathcal{F}_n = \{A : \ , A \in \mathcal{F}, A \subseteq A_n\} = \mathcal{F} \cap A_n$$

<u>Claim</u> T is $\mathcal{F}/\mathcal{F}'$ meas. iff.

T_n is $\mathcal{F}_n/\mathcal{F}'$ meas. $\forall n$.

<u>Pf</u> "$\Longrightarrow$" $T_n^{-1}(B) = T^{-1}(B) \cap A_n \quad (B \in \mathcal{F}')$

$$\in \mathcal{F} \cap A_n = \mathcal{F}_n.$$

"$\Longleftarrow$" $T^{-1}(B) = \quad (B \in \mathcal{F}')$

$$= T^{-1}(B) \cap \underbrace{\left(\bigcup_n A_n\right)}_{\Omega} = \bigcup_n \left(A_n \cap T^{-1}(B)\right)$$

$$= \bigcup_n \underbrace{T_n^{-1}(B)}_{\in \mathcal{F}_n \subseteq \mathcal{F}} \in \mathcal{F}.$$

13.2 $(\Omega, \mathcal{F}) \xrightarrow{T} (\Omega', \mathcal{F}')$.

(a) $T^{-1}\mathcal{F}' = \{T^{-1}(A') : A' \in \mathcal{F}'\}$

$$T\mathcal{F} = \{A' : T^{-1}(A') \in \mathcal{F}\}$$

(i) $T^{-1}\mathcal{F}'$ & $T\mathcal{F}$ are both σ-fields.

(Proof is simple, making use of the fact that

$$T^{-1}\left(\bigcup_\alpha A'_\alpha\right) = \bigcup_\alpha T^{-1}(A'_\alpha)$$

$$T^{-1}\left((A')^c\right) = \left(T^{-1}(A')\right)^c \quad)$$

(ii) T is $\mathcal{F}/\mathcal{F}'$ meas (1)

iff. $T^{-1}\mathcal{F}' \subseteq \mathcal{F}$ (2)

iff. $\mathcal{F}' \subseteq T\mathcal{F}$. (3).

(They all follows from def. of meas. functions.)

(b) Let $X_j = \sum_i x_j^i \mathbb{1}_{A_j^i}$ $j = 1, 2, \cdots, n$.

Then

$\sigma(X_1, X_2, \cdots, X_n) = \sigma(\{A_j^i\}'s)$.

(c) $(\Omega, \mathcal{F}) \xrightarrow{T} (\Omega', \mathcal{F}')$

$$\sigma(T^{-1}\mathcal{A}') \qquad\qquad \sigma(\mathcal{A}')$$

$$\parallel$$

$$T^{-1}\sigma(\mathcal{A}')$$

this is a σ-field

(i) $\boxed{T^{-1}\mathcal{A}' \subseteq \overbrace{T^{-1}\sigma(\mathcal{A}')}}$ $(\because \mathcal{A}' \subseteq \sigma(\mathcal{A}'))$.

Hence $\sigma(T^{-1}\mathcal{A}') \subseteq T^{-1}\sigma(\mathcal{A}')$.

(ii) (Method 1) $T^{-1}\sigma(\mathcal{A}') \subseteq \sigma(T^{-1}(\mathcal{A}'))$?

$$\Updownarrow$$

$$\sigma(\mathcal{A}') \subseteq T[\sigma(T^{-1}(\mathcal{A}'))] \quad ?$$

Now $\mathcal{A}' \subseteq T(T^{-1}\mathcal{A}') \subseteq T[\sigma(T^{-1}\mathcal{A}')]$

$(\because T^{-1}\mathcal{A}' \subseteq \sigma(T^{-1}\mathcal{A}'))$

Hence $\sigma(\mathcal{A}') \subseteq T[\sigma(T^{-1}\mathcal{A}')]$

L as this is a σ-field.

$$\Downarrow$$

Thus $\boxed{T^{-1}\sigma(\mathcal{A}') \subseteq \sigma(T^{-1}\mathcal{A}')}$

(Method 2 of showing $T^{-1}(\sigma(\mathcal{A}')) \subseteq \sigma(T^{-1}(\mathcal{A}'))$)

Let $\mathcal{G} = \{ B \subseteq \Omega' : T^{-1}(B) \in \sigma(T^{-1}(\mathcal{A}')) \}$

Clearly $\mathcal{A}' \subseteq \mathcal{G}$

Now $\mathcal{G}$ is a σ-field!

(i) $T^{-1}(\Omega') = \Omega \in \sigma(T^{-1}(\mathcal{A}'))$

(ii) $T^{-1}(B) \in \sigma(T^{-1}(\mathcal{A}')) \Rightarrow T^{-1}(B^c) = [T^{-1}(B)]^c$

$\in \sigma(T^{-1}(\mathcal{A}'))$

(iii) $T^{-1}(B_1), T^{-1}(B_2), \ldots \in \sigma(T^{-1}(\mathcal{A}'))$

$\Rightarrow T^{-1}(B_1 \cap B_2 \cap \cdots) = T^{-1}(B_1) \cap T^{-1}(B_2) \cdots$

$\in \sigma(T^{-1}(\mathcal{A}'))$

Hence $\sigma(\mathcal{A}') \subseteq \mathcal{G}$

ie. $T^{-1}(\sigma(\mathcal{A}')) \subseteq \sigma(T^{-1}(\mathcal{A}'))$

13.6

$$(\mathbb{R}^1, \mathcal{F}) \xrightarrow{\ f\ } (\mathbb{R}^1, \mathcal{B})$$

$\mathcal{B} \subseteq \mathcal{F}$ iff. every continuous fct f is $\mathcal{F}/\mathcal{B}$ meas.

"$\Longrightarrow$" For any cont. fct $f: \mathbb{R}^1 \to \mathbb{R}^1$,
$f^{-1}(\mathcal{B}) \subseteq \mathcal{B} \subseteq \mathcal{F}$. So f is meas-$\mathcal{F}/\mathcal{B}$

"$\Longleftarrow$" Simply take f as the identity map:
$f(x) = x$

Then $f^{-1}(\mathcal{B}) = \mathcal{B} \underset{\substack{\uparrow \\ \text{by meas. of } f.}}{\subseteq} \mathcal{F}$

~~$\int g_N \le \lim_{n\to\infty} \int f_n.$~~

~~Then let $N \to \infty$ gives.~~

~~$\lim_{N\to\infty} \int g_N \le \lim_{n\to\infty} \int f_n.$~~

~~(Note. $\int g_1 \le \int g_2 \le \int g_3 \le \cdots$~~

~~Hence $\lim_{n\to\infty} \int g_n$ always exists, (might be infinity)~~

~~(2) is proved similarly (or is true by symmetry)~~

(16.8) Let f be integrable, i.e. $\int |f| \, d\mu < \infty$

By Thm 13.5 & Thm 15.1 (iii) (applied to f^+ & f^-),

$\exists\ g$ - simple fct, $g = \sum_{i=1}^{m<\infty} x_i \mathbb{1}_{A_i}$ s.t.

$$\int_\Omega |f - g| \, d\mu < \varepsilon/2.$$

Now consider

$$\left|\int_A f\,d\mu\right| = \left|\int_\Omega f\,\mathbb{1}_A\,d\mu\right|$$

$$= \left[\int_\Omega (f-g)\,\mathbb{1}_A\,d\mu + \int_\Omega g\,\mathbb{1}_A\,d\mu\right]$$

$$\leq \int_\Omega |f-g|\,d\mu + \int_\Omega |g\,\mathbb{1}_A|\,d\mu$$

$$\leq \frac{\varepsilon}{2} + \sum_{i=1}^{m} |x_i|\,\mu(A\cap A_i)$$

$$\leq \frac{\varepsilon}{2} + \sum_{i=1}^{m} |x_i|\,\mu(A).$$

If $\mu(A) \leq \dfrac{\varepsilon/2}{1+|x_1|+|x_2|+\cdots+|x_m|} = \delta$

then $\left|\int_A f\,d\mu\right| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$

(17.6) Given, $\int_0^\infty |f(x)|\,dx < \infty$.

(17.4) Midterm #2 Spring 2006

(15) (15)

<u>Prove</u>: $\forall \varepsilon > 0$, $\lim_{\alpha \to \infty} \lambda\{x : x > \alpha : |f(x)| > \varepsilon\} =$

<u>Pf</u> By Markov-inequality:

$$\lambda\{x : x > \alpha, |f(x)| > \varepsilon\} \leq \frac{1}{\varepsilon} \int_\alpha^\infty |f(x)|\,dx.$$

$$= \frac{1}{\varepsilon} \int_0^\infty |f(x)| \mathbb{1}_{[\alpha,\infty)}(x)\,dx.$$

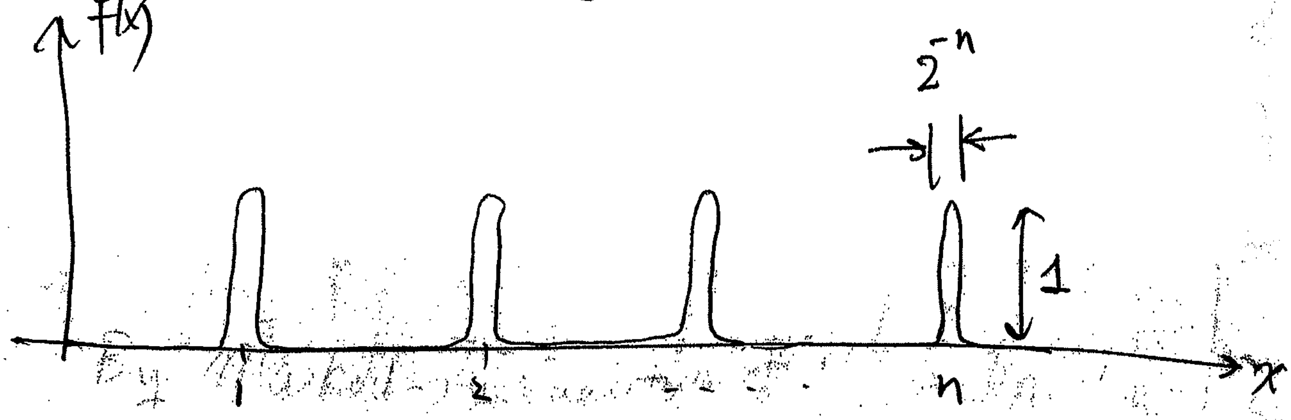
But ① $\left| |f(x)| \mathbb{1}_{[\alpha,\infty)}(x) \right| \leq |f(x)|$ — integrable.

② $\lim_{\alpha \to \infty} |f(x)| \mathbb{1}_{[\alpha,\infty)}(x) = 0 \quad \forall x.$

By LDCT, $\int_0^\infty |f(x)| \mathbb{1}_{[\alpha,\infty)}(x)\,dx \xrightarrow{\alpha \to 0} 0$

& hence the desired result.

eg of $f(x)$ need not go to 0 as $x \to \infty$



(17.7) $f_n(x) = x^{n-1} - 2x^{2n-1}$ $(n = 1, 2, \dots)$

① $\int_0^1 f_n(x)\,dx = 0 \Rightarrow \sum_{n=1}^{\infty} \int_0^1 f_n(x)\,dx = 0$

② Consider $\sum_{n=1}^{\infty} |f_n(x)|$

$$= \sum_{n=1}^{\infty} |x|^{n-1} \left[1 - 2x^n\right] \leq 3 \sum_{n=1}^{\infty} |x|^{n-1}$$

$$\leq \frac{3}{1 - |x|} < \infty$$

$$\forall\ x \in [0, 1)$$

Hence $\{f_n\}$ is absolutely convergent & the terms can be arbitrarily re-arranged & brackets arbitrarily removed

$$\int_0^1 \sum_{n=1}^{\infty} f_n(x)\, dx = \int_0^1 \sum_{n=1}^{\infty} (x^{n-1} - 2x^{2n-1})\, dx$$

$$= \int_0^1 \left(\sum_{n=1}^{\infty} x^{n-1}\right) - 2\left(\sum_{n=1}^{\infty} x^{2n-1}\right) dx$$

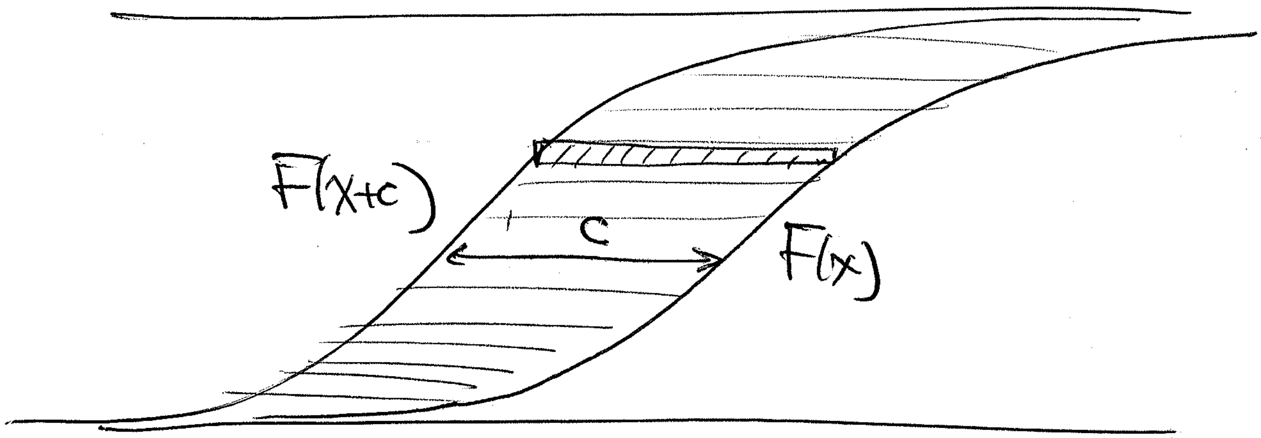
$$= \int_0^1 \frac{1}{1-x} - \frac{2x}{1-x^2}\, dx$$

$$= \int_0^1 \frac{1-x}{1-x^2}\, dx = \int_0^1 \frac{1}{1+x}\, dx = \ln 2$$

<u>Hence</u>, $\int_0^1 \sum_n f_n\, dx \neq \sum_n \int_0^1 f_n(x)\, dx$

(This shows the hypothesis for Thm 16.6, 16.7, Cor. are important.)

18.13



Use Fubini, cut the area horizontally.

18.14 $\boxed{\text{Thm } 18.4}$

Additional Problem

#1 [Bill Thm 16.1 (ii) Pg 206]

Additional Problem

#2 $\{A_n\}$ — indep. events.

(a) $P\{\underline{\lim} A_n\} = 1$

$\Longleftrightarrow P\{\overline{\lim} A_n^c\} = 0$

BC I, II

$\Longleftrightarrow \boxed{\sum_n P(A_n^c) < \infty}$

(b) $P(N \leq k) = P\{w \in A_k, A_{k+1}, \cdots\}$

$= P\{w \in A_k \cap A_{k+1} \cap \cdots\}$

$= \prod_{i=k}^{\infty} P(A_i)$

(c) $P(N=k) = P(N \le k) - P(N \le k-1)$

$$= \prod_{i=k}^{\infty} P(A_i) - \prod_{i=k-1}^{\infty} P(A_i)$$

$$= P(A_{k-1}^c) \prod_{i=k}^{\infty} P(A_i)$$

Hence $EN = \sum_{k=1}^{\infty} k P(N=k)$

$$= \sum_{k=1}^{\infty} k P(A_{k-1}^c) \left[\prod_{i=k}^{\infty} P(A_i) \right] < \infty$$

(The above can be simplified further!)

Assume $\sum_n P(A_n^c) < \infty$

Otherwise, by (a), $P(\underline{\lim} A_n) = 0 \Rightarrow N = \infty$ P a.s.

(of course) $\Rightarrow EN = \infty$

$$\sum_n P(A_n^c) = \sum_n \left(1 - P(A_n)\right) < \infty$$

<u>Claim</u> $\boxed{\lim_{k\to\infty} \prod_{i=k}^{\infty} P(A_i) = 1}$

<u>Pf</u> $\log \prod_{i=k}^{\infty} P(A_i)$

$$= \sum_{i=k}^{\infty} \log P(A_i) = \sum_{i=k}^{\infty} \log\Big[1 + \underbrace{P(A_i) - 1}_{\to 0}\Big]$$

$$\simeq \sum_{i=k}^{\infty} \left[P(A_i) - 1\right] \quad (< \infty)$$

Hence $\lim_{k\to\infty} \log\left[\prod_{i=k}^{\infty} P(A_i)\right] = 0$

i.e. $\boxed{\lim_{k\to\infty} \prod_{i=k}^{\infty} P(A_i) = 1}$

In this case,

$$\sum_{k=1}^{\infty} k P(A_{k-1}^c) \left[\prod_{i=k}^{\infty} P(A_i) \right] < \infty$$

$$\Big\Updownarrow$$

$$\sum_{k=1}^{\infty} k P(A_{k-1}^c) < \infty$$

i.e. $\boxed{\sum_{k=1}^{\infty} k \left(1 - P(A_{k-1})\right) < \infty}$