MA 538 (Spring 2009, Yip) (Hw4 Solution) 6

20.14 (See back of Textbk.)

20.15

(a) $P\left(\frac{X}{Y} \leq a\right) = P(X \leq aY, Y > 0) + P(X \geq aY, Y < 0)$

$$= \int_0^\infty \left(\int_{-\infty}^{ay} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx\right)\frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy + \int_{-\infty}^0 \left(\int_{ay}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx\right)\frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy$$

Hence pdf of $\frac{X}{Y} = \frac{d}{da} P\left(\frac{X}{Y} \leq a\right)$

$$= \int_0^\infty \frac{y e^{-\frac{a^2y^2}{2}} e^{-\frac{y^2}{2}}}{2\pi}\,dy - \int_{-\infty}^0 \frac{y e^{-\frac{a^2y^2}{2}} e^{-\frac{y^2}{2}}}{2\pi}\,dy$$

$$= 2\int_0^\infty \frac{y e^{-\frac{(a^2+1)y^2}{2}}}{2\pi}\,dy = \boxed{\frac{1}{\pi}\frac{1}{a^2+1}}$$ Cauchy

1b) $X \sim$ unif dist. on $(-\pi/2, \pi/2)$

$$P\{\tan X \leq a\} = P\{X \leq \tan^{-1} a\}$$

$$= \frac{\tan^{-1} a + \pi/2}{\pi} \quad \text{— cdf}$$

$$\text{pdf} = \frac{d}{da}\left(\frac{\tan^{-1} a + \pi/2}{\pi}\right) = \frac{1}{\pi}\left(\frac{1}{a^2+1}\right) \quad \text{— Cauchy}$$

with $u=1$

(20.45).

(20.16) $X_1, \cdots, X_n$ — iid, $N(0,1)$.

$$\chi_n^2 = X_1^2 + X_2^2 + \cdots + X_n^2.$$

$$X_1 \sim \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

$$P(X_1^2 \leq a) = P(X_1 \leq \sqrt{a}) = \int_{-\infty}^{\sqrt{a}} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx$$

pdf of $X_1^2 = \dfrac{d}{da} \displaystyle\int_{-\infty}^{\sqrt{a}} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx$

$$= \frac{e^{-a/2}}{\sqrt{2\pi}} \frac{1}{2} a^{-1/2} = \boxed{C_1 a^{-1/2} e^{-a/2}}$$

$n=1$ ✓

Assume χ_n^2 has pdf given by $C_n x^{\frac{n}{2}-1} e^{-x/2}$

For $n+1$,

$\chi_{n+1}^2 = \chi_n^2 + \chi_1^2$, pdf given by convolution.

$$C_n C_1 \int_0^x y^{\frac{n}{2}-1} e^{-y/2} (x-y)^{-1/2} e^{-(x-y)/2}\, dy$$

$$= C_n C_1 e^{-x/2} \int_0^x y^{\frac{n}{2}-1} (x-y)^{-1/2}\, dy$$

Let $z = y/x$, $y = zx$

$$= C_n C_1 \, e^{-x/2} \int_0^1 (xz)^{\frac{n}{2}-1} (\cancel{x} - zx)^{-1/2} \, x \, dz$$

$$= C_n C_1 \, e^{-x/2} \, x^{\frac{n}{2}-1} \, x^{-1/2} \, x \underbrace{\int_0^1 z^{\frac{n}{2}-1} (1-\cancel{z})^{-1/2} \, dz}_{\text{a constant}}$$

$$= C_{n+1} \, x^{\frac{n+1}{2}-1} \, e^{-x/2} \qquad \text{(induction process done.)}$$

Hence pdf of χ_n^2 is given by $\boxed{C_n x^{\frac{n}{2}-1} e^{-x/2}}$

where C_n is given by : $C_n \int_0^\infty x^{n/2-1} e^{-x/2} \, dx = 1$

$$C_n = \frac{1}{\int_0^\infty x^{\frac{n}{2}-1} e^{-x/2} \, dx} \qquad {}^{//} \qquad \int_0^\infty 2^{\frac{n}{2}-1} x^{\frac{n}{2}-1} e^{-x} \, 2 \, dx$$

$$x \to x/2 \qquad = 2^{\frac{n}{2}} \int_0^\infty x^{\frac{n}{2}-1} e^{-x} \, dx$$

$$= 2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right) \quad (\text{Ex } 18.18 \text{ Pg } 240)$$

$$(t = n/2)$$

Hence $\boxed{C_n = \dfrac{1}{2^{\frac{n}{2}}\Gamma(n/2)}}$

(20.19) $A_{nm}(\varepsilon) = \{ |Z_k - Z| \leq \varepsilon, \; n \leq k \leq m \}$

Show (1) $Z_n \to Z$ P a.s. $\iff \forall \varepsilon \; \lim_n \lim_m P(A_{nm}(\varepsilon)) = 1$

(2) $Z_n \to Z$ in Prob $\iff \forall \varepsilon \; \lim_n P(A_{nn}(\varepsilon)) = 1$

(2) is by def:

$Z_n \to Z$ in Prob if $\forall \varepsilon$, $P(|Z_n - Z| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0$

i.e. $\lim_n P(A_{nn}(\varepsilon)) = 1$

(1) $Z_n \to Z$ P a.s

$\overline{\lim_n}(|Z_n - Z| \geq \varepsilon)$

$\iff \forall \varepsilon, \; P\{ |Z_n - Z| \geq \varepsilon \text{ i.o} \} = 0$

$\iff \forall \varepsilon, \; P\{ \underline{\lim_n}(|Z_n - Z| \leq \varepsilon) \} = 1$

$$\Longleftrightarrow \forall \varepsilon \quad P\left\{ \bigcup_{N} \bigcap_{n \geq N} \{ |Z_n - Z| \leq \varepsilon \} \right\} = 1$$

$$\Longleftrightarrow \forall \varepsilon \quad \lim_{N \to \infty} P\left\{ \underbrace{\bigcap_{n \geq N}^{\infty} \{ |Z_n - Z| \leq \varepsilon \}}_{\lim\limits_{M \to \infty} \bigcap\limits_{N \leq n \leq M} \{ |Z_n - Z| \leq \varepsilon \}} \right\} = 1$$

$$= \lim_{N \to \infty} \left[\lim_{M \to \infty} P\left\{ \underbrace{\bigcap_{N \leq n \leq M} \{ |Z_n - Z| \leq \varepsilon \}}_{A_{NM}(\varepsilon)} \right\} \right]$$

$$\Longleftrightarrow \lim_{N \to \infty} \left[\lim_{M \to \infty} P(A_{NM}(\varepsilon)) \right] = 1$$

(20.21)

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables which is *Cauchy in probability*, i.e. for all $\epsilon > 0$, there exists an $N > 0$ such that $P(|X_m - X_n| > \epsilon) < \epsilon$ for all $m, n \geq N$. Show that there exists a random variable X such that $X_n \longrightarrow X$ in probability as $n \longrightarrow \infty$.

Let $\varepsilon = 2^{-k}$. There exist $\{n_k\}_{k=1}^{\infty}$ s.t. $n_1 \leq n_2 \leq \cdots \to \infty$

& $P(|X_{n_k} - X_{n_{k+1}}| \geq 2^{-k}) \leq 2^{-k}$.

For these choices,

$$\sum_{k=1}^{\infty} P(|X_{n_k} - X_{n_k}| \geq 2^{-k}) < \infty$$

$$\xrightarrow{\text{BC-1}} P\{|X_{n_k} - X_{n_{k+1}}| \geq 2^{-k} \text{ i.o}\} = 0$$

<u>Note</u> $\sum_{k=1}^{\infty} 2^{-k} < \infty \Rightarrow \{X_{n_k}\}_{k=1}^{\infty}$ Cauchy.

$\Big(\forall \varepsilon, \exists K > 0$ s.t. $\forall i, j \geq K$

$$|X_{n_i} - X_{n_j}| \leq \sum_{k=i}^{j-1} |X_{n_k} - X_{n_{k+1}}| \leq \sum_{k=i \geq K}^{j-1} 2^{-k} \leq \varepsilon \Big)$$

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i.e. $\exists X$ s.t. $X_{n_k} \xrightarrow[k\to\infty]{a.s} X$

<u>Claim</u> $X_n \xrightarrow{n\to\infty} X$ in prob. (for the <u>whole</u> sequence)

<u>Pf</u>: $P(|X_n - X| \geq \varepsilon)$

$= P(|X_n - X_{n_k} + X_{n_k} - X| \geq \varepsilon)$

$\leq P(\{|X_n - X_{n_k}| \geq \varepsilon/2\} \cup \{|X_{n_k} - X| \geq \varepsilon/2\})$

$\leq P\{|X_n - X_{n_k}| \geq \varepsilon/2\} + P\{|X_{n_k} - X| \geq \varepsilon/2\}$

$\downarrow$ n, n_k large enough $\qquad \downarrow$ $k\to\infty$

0 (by assumption of this problem) $\qquad 0$ (as. $X_{n_k} \xrightarrow{a.s} X$)

20.22

(a) <u>Method 1</u> $\quad X_n \longrightarrow X$ in Prob

$\exists\, X_{n_k}$ s.t. $X_{n_k} \xrightarrow{k\to\infty} X$ P a.s.

i.e. P a.s, ω $\quad X_{n_k}(\omega) \uparrow X(\omega)$.

But $\quad X_{n_k}(\omega) \leq \lim_n X_n(\omega) \leq \lim_k X_{n_k}(\omega) = X(\omega)$

also., as $k \to \infty$

Hence $\boxed{\lim_n X_n(\omega) = X(\omega)}$

<u>Method 2</u> $\quad X_n \longrightarrow X$ P a.s.

iff $\forall \varepsilon$, $\quad P\{\, |X_n - X| \geq \varepsilon \text{ i.o} \,\} = 0$

$P\{\, X - X_n \geq \varepsilon \text{ i.o} \,\} = 0$

i.e. $P\left\{ \bigcap_N \bigcup_{n \geqslant N} \{X - X_n \geqslant \varepsilon\} \right\} = 0$

i.e. $\lim_{N \to \infty} P\left\{ \bigcup_{n \geqslant N} \{X - X_n \geqslant \varepsilon\} \right\} = 0$

But as $X_1 \leq X_2 \leq \cdots$

Hence $\{X - X_n \geqslant \varepsilon\} \subset \{X - X_N \geqslant \varepsilon\}$
$(n \geqslant N)$

Hence $P\left\{ \bigcup_{n \geqslant N} \{X - X_n \geqslant \varepsilon\} \right\} \leq P\{X - X_N \geqslant \varepsilon\} \xrightarrow{N \to \infty} 0$!!!

by assumption: $X_n \to X$ in Prob.

(21.14) Let μ be the dist. mean for X] indep (14)

ν - - - - - - - Y

Then the dist mean for $X+Y$ is given by convolution: $\overset{\shortparallel}{Z}$

$$m(H) = \int_{-\infty}^{\infty} \nu(H-x)\, d\mu(x)$$

Now suppose $E|Z| < \infty$

i.e. $$\int_{-\infty}^{\infty} |z|\, m(dz)$$

$$= \int_{z=-\infty}^{\infty} |z| \int_{x=-\infty}^{\infty} \nu(dz-x)\, d\mu(x)$$

$$= \int_{x=-\infty}^{\infty} \underbrace{\left[\int_{z=-\infty}^{\infty} |z| \, \nu(dz - x) \right]} d\mu(x) \quad < \infty$$

<u>Hence</u> $\displaystyle\int_{z=-\infty}^{\infty} |z| \, \nu(dz - x) \quad < \infty$ for some x.

<u>Note</u>:

$$E|Y| = \int_{-\infty}^{\infty} |z| \, \nu(dz)$$

$$= \int_{-\infty}^{\infty} |z - x| \, \nu(dz - x)$$

$$\leq \int_{-\infty}^{\infty} \left(|z| + |x| \right) \nu(dz - x)$$

$$= \underbrace{\int_{-\infty}^{\infty} |z| \, \nu(dz - x)}_{< \infty} + \underbrace{\int_{-\infty}^{\infty} |x| \, \nu(dz - x)}_{= |x| < \infty}$$

Hence $E|Y| < \infty$

& $$E|X| = E|X+Y-Y| \le \underbrace{E|X+Y|}_{<\infty} + \underbrace{E|Y|}_{<\infty}$$

4.20

$$f(x;\alpha,u) = \frac{\alpha^u}{\Gamma(u)} x^{u-1} e^{-\alpha x}$$

Moment generating fct

$$M(s) = \int_0^\infty e^{sx} f(x;\alpha,u)\,dx$$

$$= \int_0^\infty \frac{\alpha^u}{\Gamma(u)} x^{u-1} e^{-\alpha x} e^{sx}\,dx$$

$$= \frac{\alpha^u}{\Gamma(u)} \int_0^\infty x^{u-1} e^{-(\alpha-s)x}\,dx \qquad \alpha-s>0$$

let $y = (\alpha-s)x$

$$= \frac{\alpha^u}{\Gamma(u)} \int_0^\infty \left(\frac{y}{\alpha-s}\right)^{u-1} e^{-y} \frac{dy}{(\alpha-s)}$$

$$= \frac{\alpha^u}{\Gamma(u)} \left(\alpha - s\right)^{-u+1} (\alpha - s)^{-1} \int_0^\infty y^{u-1} e^{-y} \, du$$

$$= \frac{\alpha^u (\alpha - s)^{-u}}{\cancel{\Gamma(u)}} \underbrace{\cancel{\int_0^\infty y^{u-1} e^{-y} \, du}}_{= \Gamma(u)}$$

Ex 18.18 Pg 240

$$\boxed{M(s) = \left(1 - \frac{s}{\alpha}\right)^{-u}}$$

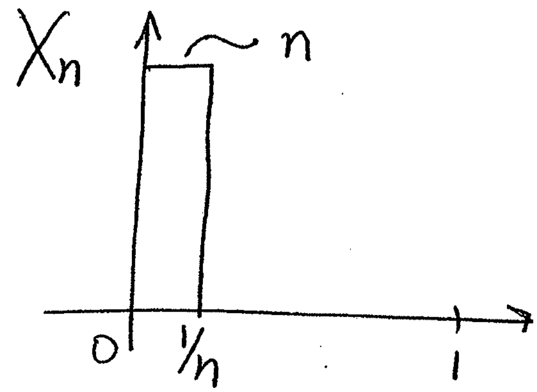
$$k^{th} \text{ moment} = \left. \frac{d^k M(s)}{ds^k} \right|_{s=0}$$

$$= \frac{(-u)}{(-\alpha)} \frac{(-u-1)}{(-\alpha)} \frac{(-u-2)}{(-\alpha)} \cdots \frac{(-u-k+1)}{(-\alpha)} = \frac{u(u+1)(u+2)\cdots(u+k-1)}{\alpha^k}$$

Additional Problem

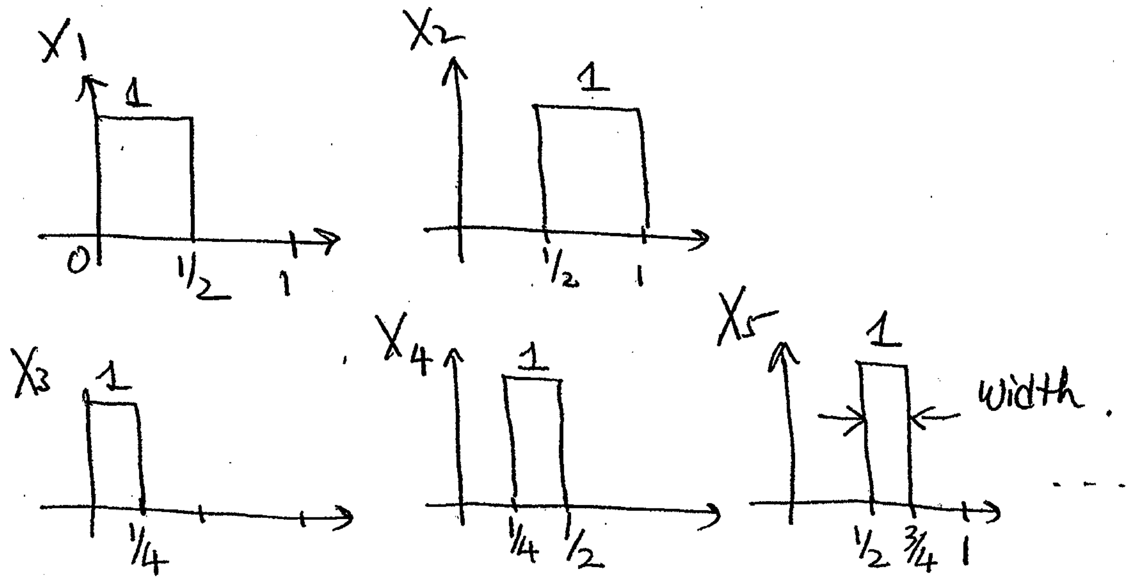
#1 Let $(\Omega, \mathcal{F}, P) = ((0,1), \mathcal{B}, \text{Leb-meas})$.
(standard example)

(a)



Then $X_n \to 0$ a.s.
But $E(X_n^2) \to \infty$.

(b)



then $EX_n^2 \to 0$ but not a.s.

(c) Same as (b) but change height 1 to $\left(\frac{1}{\text{width}}\right)$. Then $X_n \to 0$ in prob.
But $E(X_n^2) \to \infty$

#2

$$E\left[\frac{1}{n}\left(S_n - \sum_{i}^{n} m_i\right)\right]^2$$

$$= \frac{1}{n^2} E\left(X_1 + X_2 + \cdots + X_n - m_1 - m_2 - \cdots - m_n\right)^2$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} E(X_i - m_i)^2$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \sigma_i^2 = \frac{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2}{n^2}$$

$$= \frac{\sigma_1^2 + 2\frac{\sigma_2^2}{2} + 3\frac{\sigma_3^2}{3} + \cdots + n\frac{\sigma_n^2}{n}}{n^2}$$

$$\leq \frac{\sigma_1^2 + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{3} + \cdots + \frac{\sigma_n^2}{n}}{n}. \xrightarrow{\text{as } n\to\infty} 0$$

(<u>Simple 504 Lemma:</u>
If $m_n \to 0$, then $\frac{m_1 + m_2 + \cdots + m_n}{n} \xrightarrow{n\to\infty} 0$)