Homework 5
MA 538 Spring 2009 (Aaron N. K. Yip)
Due: 3pm, Mon, Apr. 20

From Textbook (Billingsley):
page 91 (section 6): #14
page 270 (section 20): #8, 9
page 294 (section 22): #5, 6, 7, 9
page 339 (section 25): #1, 13
page 353 (section 26): #5, 6, 9

Additional Problems

1. Consider the series $1 \pm \frac{1}{2} \pm \frac{1}{3} \pm \frac{1}{4} \pm \cdots$

   Prove that given any real number x, you can always choose the $\pm$ appropriately such that the series converges to x.

   Can you generalize this to other series?

2. Let $\{X_n\}$ be a sequence of independent random variables satisfying the following condition: there exists $\{b_n > 0\}$ such that $b_n \longrightarrow \infty$ and ($Y_k^n = X_k 1_{\{|X_k| \leq b_n\}}$):

   (a) $\lim_{n\to\infty} \sum_{k=1}^{n} P(|X_k| \geq b_n) = 0.$

   (b) $\lim_{n\to\infty} \frac{1}{b_n^2} \sum_{k=1}^{n} E\left[(Y_k^n)^2\right] = 0.$

   Let $a_n = \sum_{k=1}^{n} EY_k^n$ and $S_n = X_1 + X_2 + \ldots + X_n$. Prove that $\frac{S_n - a_n}{b_n} \longrightarrow 0$ in probability as $n \to \infty$.

3. Let $\{X_n\}$ be a sequence of iid random variables with the following common distribution:

$$\Pr(X_1 = i) = \Pr(X_1 = -i) = \frac{c}{i^2 \log i}, \quad \text{for } i = 3, 4, \ldots$$

   where c is the constant such that the condition $1 = \sum_{i=-\infty}^{\infty} \Pr(X_1 = i)$ holds.
   Prove that $\frac{S_n}{n} \longrightarrow 0$ in probability while P a.s. $\limsup_n \frac{S_n}{n} = +\infty$ and $\liminf_n \frac{S_n}{n} = -\infty$.
   (This problem thus gives an example in which the WLLN holds while the SLLN does not.)