MA/STAT 538 (Spring 2009, Yip)
HW 6 Solution

#27.1 $P(Z_{nk}=1)=p_{nk}$, $P(Z_{nk}=0)=1-p_{nk}$

(23.14) $\sum_{k=1}^{r_n} p_{nk} \xrightarrow{n\to\infty} \lambda \geq 0$, $\max_k p_{nk} \xrightarrow{n\to\infty} 0$

Note $\varphi_{nk}(t) = E e^{itZ_{nk}} = e^{it}p_{nk} + 1 - p_{nk}$

$= 1 + (e^{it}-1)p_{nk}$

Need: $\prod_{k=1}^{r_n} \varphi_{nk}(t) \xrightarrow{n\to\infty} e^{\lambda(e^{it}-1)}$

$\qquad = \varphi_{\text{Poisson},\lambda}(t)$

$$\prod_{k=1}^{r_n}\left[1+(e^{it}-1)p_{nk}\right]$$

$$= e^{\sum_{k=1}^{r_n}\log\left(1+(e^{it}-1)p_{nk}\right)}.$$

<u>Note:</u> $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots \quad (|x|<1)$

$$= x - \frac{x^2}{2}\underbrace{\left(1 - \frac{x}{3} + \frac{x^2}{4} - \frac{x^3}{5}\cdots\right)}_{\text{finite}}$$

$$= x - O(x^2).$$

Thus $\sum_{k=1}^{r_n}\log\left(1+(e^{it}-1)p_{nk}\right)$

$$= \sum_{k=1}^{r_n}\left[(e^{it}-1)p_{nk} + C\,p_{nk}^2\right]$$

$$= \sum_{k=1}^{r_n}(e^{it}-1)p_{nk} + C\sum_{k=1}^{r_n}p_{nk}^2$$

$$\xrightarrow{n\to\infty} (e^{it}-1)\lambda \quad + \text{error.}$$

$$|\text{error}| \le C \underbrace{\left(\max_k p_{nk}\right)}_{\substack{n\to\infty \\ \downarrow \\ 0}} \underbrace{\sum_k p_{nk}}_{\lambda}$$

#27.2 $\varphi_{\frac{S_n}{n}}(t) = E e^{it\frac{X_1+\cdots+X_n}{n}}$

$$= \left[\varphi\left(\frac{t}{n}\right)\right]^n, \quad \varphi = \varphi_{X_1}(t)$$

So need to show that

$$\varphi\left(\frac{t}{n}\right)^n \xrightarrow{n\to\infty} e^{itm} \qquad m = EX_1.$$

By (27.5) $\left\| \varphi\left(\frac{t}{n}\right)^n - \left(e^{i\frac{t}{n}m}\right)^n \right\|$

$$\leq n \left\| \varphi\left(\frac{t}{n}\right) - e^{i\frac{t}{n}m} \right\|.$$

$$= t \underbrace{\frac{\left\| \varphi\left(\frac{t}{n}\right) - e^{i\frac{t}{n}m} \right\|}{\left(\frac{t}{n}\right)}}$$

$$\downarrow n\to\infty \ ?$$

$$0$$

So need to show:

$$\frac{\varphi(\varepsilon) - e^{i\varepsilon m}}{\varepsilon} \overset{?}{\xrightarrow{\varepsilon \to 0}} 0 \qquad \left(\varepsilon = \frac{t}{n}\right)$$

$$= \frac{1}{\varepsilon}\left[E e^{i\varepsilon X} - e^{i\varepsilon m}\right] \qquad (|m| < \infty)$$

$$= \frac{1}{\varepsilon}\left[E\left(1 + i\varepsilon X + \text{error}_1\right) - 1 - i\varepsilon m + \text{error}_2\right]$$

$$= \frac{1}{\varepsilon}\left[\cancel{1 + i\varepsilon m} + E\,\text{error}_1 \cancel{- 1 - i\varepsilon m} + \text{error}_2\right]$$

$$= \frac{1}{\varepsilon}\left[E(\text{error}_1) + \text{error}_2\right]$$

$(264)_1 \Rightarrow \quad |\text{error}_2| \leq \min\{\varepsilon^2 m^2, 2\varepsilon m\}$

$$\leq \varepsilon^2 m^2$$

So $\dfrac{\text{error}_2}{\varepsilon} \xrightarrow{\varepsilon \to 0} 0$

(26.4), $\Rightarrow |\text{error}_1| \leq \min\{\varepsilon^2|X_1|^2, 2\varepsilon|X_1|\}$

Hence $E|\text{error}_1| \leq E\min\{\varepsilon^2|X_1|^2, 2\varepsilon|X_1|\}$

$$= \int_{\{|X_1|\geq M\}} \min\{\varepsilon^2|X_1|^2, 2\varepsilon|X_1|\}\, dP + \int_{\{|X_1|\leq M\}} \min\{\varepsilon^2|X_1|^2, 2\varepsilon|X_1|\}\, dP$$

$$\leq \int_{\{|X_1|\geq M\}} 2\varepsilon|X_1|\, dP + \int_{\{|X_1|\leq M\}} \varepsilon^2 \cancelto{M}{|X_1|}^2\, dP$$

$$\leq 2\varepsilon \int_{\{|X_1|\geq M\}} |X_1|\, dP + \varepsilon^2 M^2.$$

Hence $\frac{1}{\varepsilon} E\,|\text{error}_1|$

$$\leq \underbrace{2 \int_{\{|X_1| \geq M\}} |X_1|\, dP}_{\text{But term can be made as small as possible by choosing } M \text{ large}} + \varepsilon M^2$$

$\varepsilon M^2 \xrightarrow{\varepsilon \to 0} 0$

27.4 $\quad |X_{nk}| \le M_n, \qquad \dfrac{M_n}{\sigma_n} \xrightarrow{n \to \infty} 0$

Lyapounov Condition

$$\frac{\sum_{k=1}^{r_n} E|X_{nk}|^{2+\delta}}{\left(\sum_{k=1}^{r_n} E|X_{nk}|^2\right)^{\frac{2+\delta}{2}}} = \frac{\sum_{k=1}^{r_n} E\left(|X_{nk}|^{\delta}|X_{nk}|^2\right)}{\left(\sum_{k=1}^{r_n} E|X_{nk}|^2\right)^{2+\delta/2}}$$

$$\le \frac{M_n^{\delta}\left(\sum_{k=1}^{r_n} \cancel{E}|X_{nk}|^2\right)}{\left(\sum_{k=1}^{r_n} E|X_{nk}|^2\right)^{2+\delta/2}} = \frac{M_n^{\delta}\,\cancel{\sigma_n^2}}{(\sigma_n^2)^{\cancel{2}+\delta/2}}$$

$$= \frac{M_n^{\delta}}{\sigma_n^{\delta}} \longrightarrow 0$$

Lyapounov $\Longrightarrow$ Lindeberg.

27.8 <u>CLT</u> for:

$Z_1 + Z_2 + \cdots + Z_n$

$\{Z_k\}_{k \geq 1}$ — indep, $P(Z_k = 1) = 1/k$

$P(Z_k = 0) = 1 - 1/k$

Let $Z_k' = Z_k - \frac{1}{k}$,

$EZ_k' = 0$

$Var(Z_k') = Var(Z_k) = \frac{1}{k} - \frac{1}{k^2} \sim \frac{1}{k}$

$\sigma_{nk}^2 = \frac{1}{k}$

$\sigma_n^2 = \sum_{k=1}^{n} \sigma_{nk}^2 = \sum_{k=1}^{n} \frac{1}{k} \sim \log n.$

Apply Lindeberg Condition:

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2} \sum_{k=1}^{n} \left(\int_{\{Z_k \geq \varepsilon \sigma_n\}} Z_k^2 \, dP \right)$$

$$= \lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \left(\int_{\{Z_k \geq \varepsilon \sqrt{\log n}\}} Z_k^2 \, dP \right)$$

Note $|Z_k| \leq 1$, so when n is so large that $\varepsilon\sqrt{\log n} > 1$ (ε - fixed)

then $\{Z_k \geq \varepsilon\sqrt{\log n}\} = \phi$.

$$\frac{Z_1 + \cdots + Z_n - \log n}{\sqrt{\log n}} \xrightarrow[n\to\infty]{\mathcal{D}} N(0,1)$$

Hence the $\underline{\underline{\lim_{n\to\infty} = 0}}$

So
$$\frac{Z_1 + Z_2 + \cdots + Z_n - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)}{\sqrt{\log n}} \xrightarrow{\mathcal{D}} N(0,1)$$

27.16

$$\frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-u^2/2}\,du$$

$$= \frac{1}{\sqrt{2\pi}}\int_x^\infty \frac{u\,e^{-u^2/2}}{u}\,du$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_x^\infty \frac{-d\,e^{-u^2/2}}{u}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[-\frac{e^{-u^2/2}}{u}\Bigg|_x^\infty - \int_x^\infty \frac{e^{-u^2/2}}{u^2}\,du\right]$$

$$= \frac{e^{-x^2/2}}{\sqrt{2\pi}\,x} - \frac{1}{\sqrt{2\pi}}\int_x^\infty \frac{e^{-u^2/2}}{u^2}\,du$$

<u>Note</u>

$$\int_x^\infty \frac{e^{-u^2/2}}{u^2}\,du$$

$$= \int_x^\infty \frac{u e^{-u^2/2}}{u^3}\,du$$

$$\le \frac{1}{x^3}\int_x^\infty u e^{-u^2/2}\,du$$

$$= \frac{1}{x^3} e^{-x^2/2} \underset{x\to\infty}{\ll} \frac{1}{x} e^{-x^2/2}$$

27.17 $P[S_n \geq a_n \sqrt{n}]$

$$= \int_{a_n}^\infty \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz \underset{(27.16)}{\sim} \frac{1}{\sqrt{2\pi}} \frac{1}{a_n} e^{-a_n^2/2}$$

Write

$$\frac{1}{\sqrt{2\pi}}\,\frac{1}{a_n}\,\cancel{e^{-a_n^2/2}} = e^{-\frac{a_n^2}{2}(\cancel{1}+\zeta_n)}$$

Then $$a_n\sqrt{2\pi} = e^{\frac{a_n^2}{2}\zeta_n}$$

$$\ln a_n + \ln\sqrt{2\pi} = \frac{a_n^2}{2}\zeta_n$$

Hence $$\zeta_n = 2\left(\frac{\ln a_n + \ln\sqrt{2\pi}}{a_n^2}\right) \xrightarrow{a_n\to\infty} 0$$

~~$\therefore\ S_n^2 \Rightarrow n\log n$~~

~~Thus, CLT $\Rightarrow$~~

~~$$\frac{X_1+X_2+\cdots+X_n}{\sqrt{n\log n}} \Rightarrow N(0,1)$$~~

27.18 (a) $E\left[\left(\dfrac{S_n-n}{\sqrt{n}}\right)^-\right]$ $\left(\begin{aligned} S_n &= \text{Poisson},\ n \\ &= \sum_{k=1}^{n} \text{Poisson}, 1\end{aligned}\right)$

$$= \sum_{k=0}^{n}\left(\frac{n-k}{\sqrt{n}}\right)\frac{e^{-n}n^k}{k!}$$

$$= e^{-n}n^{-1/2}\underbrace{\sum_{k=0}^{n}(n-k)\frac{n^k}{k!}}$$

$$= \sum_{k=0}^{n}\frac{n^{k+1}}{k!} - \sum_{k=0}^{n}\frac{n^k}{(k-1)!}$$

$$= \frac{n^{n+1}}{n!}$$

$$= \frac{e^{-n}n^{n+1/2}}{n!}$$

(b) CLT $\Longrightarrow$ $\dfrac{S_n - n}{\sqrt{n}} \xrightarrow{n\to\infty} N$

By Cor 1, Pg 334 $\Longrightarrow$ $\left(\dfrac{S_n - n}{\sqrt{n}}\right)^- \longrightarrow N^-$

(Take $f(x) = x^- = \begin{cases} 0 & x \geq 0 \\ -x & x \leq 0 \end{cases}$).

(c). In order to show that

$$E\left(\frac{S_n - n}{\sqrt{n}}\right)^- \longrightarrow EN^- \qquad (*)$$

Apply Thm 25.12, Cor Pg 338 : $r=1$, $\varepsilon = 1$

$$E\left(\frac{S_n - n}{\sqrt{n}}\right)^2 = \frac{Var(S_n)}{n} = \frac{n}{n} = 1 < \infty \; ! \quad \forall n.$$

Thus $(*)$ is true.

(d) $EN^- = \dfrac{1}{2\pi}\displaystyle\int_0^\infty x e^{-x^2/2}\, dx = \dfrac{1}{\sqrt{2\pi}}$

Then (a), (c) $\Longrightarrow$ $\dfrac{e^{-n} n^{n+1/2}}{n!} \longrightarrow \dfrac{1}{\sqrt{2\pi}}$. $\Rightarrow$ Stirling's