Note on $\inf$**,** $\sup$**,** $\liminf$**,** $\limsup$

(Reference: Rudin, *Principles of Mathematical Analysis*)

Definition: $\inf A$ – **biggest lower bound.**

Let A be a non-empty subset of R.

1. Suppose A is bounded from below, i.e. there exists an $M > -\infty$ such that $M < x$ for all $x \in A$.

   Then, $\inf A$ is THE number r with the following properties

   (a) for all $x \in A$, $r \leq x$;

   (b) for all s such that $r < s$, there exists an $x \in A$ such that $x < s$.

   (The uniqueness of r needs to be proved.)

2. Suppose A is not bounded from below, i.e. there exist x_n's from A, such that $x_n \longrightarrow -\infty$, then define $\inf A = -\infty$.

3. $\inf \emptyset = +\infty$. (This is in fact logically consistent.)

Definition: $\sup A$ – **smallest upper bound.**

Let A be a non-empty subset of R.

1. Suppose A is bounded from above, i.e. there exists an $M < \infty$ such that $x < M$ for all $x \in A$.

   Then, $\sup A$ is THE number r with the following properties

   (a) for all $x \in A$, $x \leq r$;

   (b) for all s such that $s < r$, there exists an $x \in A$ such that $s < x$.

   (The uniqueness of r needs to be proved.)

2. Suppose A is not bounded from above, i.e. there exist x_n's from A, such that $x_n \longrightarrow \infty$, then define $\sup A = \infty$.

3. $\sup \emptyset = -\infty$. (This is in fact logically consistent.)

Definition: $\liminf$**,** $\limsup$

Let A be an non-empty subset of R.

$$\begin{aligned} \liminf A &= \inf \{\text{limit points of } A\} \\ \limsup A &= \sup \{\text{limit points of } A\} \end{aligned}$$

where

$$\{\text{limit points of } A\} = \{r : \text{there exist } \{x_n\}_{n=1}^{\infty} \subseteq A \text{ such that } \lim_{n\to\infty} x_n = r.\}$$

Hence

$$\liminf A = \text{smallest limit points} \text{ and } \limsup A = \text{biggest limit points}$$

Usually $A = \{x_1, x_2, x_3, \ldots, x_n, \ldots\}$, i.e. A is a countably collection of points. Then

$$\{\text{limit points of } A\} = \left\{r : \text{there exist } \{x_{n_k}\}_{k=1}^{\infty} \subseteq A \text{ such that } \lim_{k\to\infty} x_{n_k} = r.\right\}$$

Practical Working Definition of $\liminf$ **and** $\limsup$

Let $A = \{x_1, x_2, x_3, \ldots\}$.

1. Assume $x_n > M > -\infty$ for all n. (Otherwise, $\liminf A = -\infty$.) Then $\liminf A$ is THE number r such that

   (a) for all $s < r$, there are only *finitely many points* x_i*'s* from A such that $x_i < s$.

   (b) for all $s > r$, there are *infinitely many points* x_i*'s* from A such that $x_i < s$.

   Another textbook definition of lim inf:

$$\liminf x_n = \lim_{N\to\infty} \left(\inf_{n\geq N} x_n \right)$$

2. Assume $x_n < M < \infty$ for all n. (Otherwise, $\limsup A = \infty$.) Then $\limsup A$ is THE number r such that

   (a) for all $s > r$, there are only *finitely many points* x_i*'s* from A such that $x_i > s$.

   (b) for all $s < r$, there are *infinitely many points* x_i*'s* from A such that $x_i > s$.

   Another textbook definition of lim sup:

$$\liminf x_n = \lim_{N\to\infty} \left(\sup_{n\geq N} x_n \right)$$

Remarks.

1. $\inf A \leq \liminf A \leq \limsup A \leq \sup A$

2. The limit $\lim_n x_n$ exists IF AND ONLY IF $\liminf_n x_n = \limsup_n x_n$. In this case,

$$\lim_n x_n = \liminf_n x_n = \limsup_n x_n.$$