MA/STAT 538, Spring 2009, Yip)

1. Recall the definition that a random variable X is called *integrable* if $E|X| < \infty$.

   (a) Prove that if X and Y are two integrable and *independent* random variables, then XY is also integrable and that $E(XY) = (EX)(EY)$.

   (b) Give an example of two integrable random variables X and Y such that XY is not integrable.

   (c) Give an example of two random variables X and Y such that $X + Y$ is integrable and yet X and Y are not integrable.

(a) $$E|XY| \overset{(LMCT)}{=} \lim_{N\to\infty} E|X|1_{\{|X|\le N\}}|Y|1_{\{|Y|\le N\}}$$

$$\overset{X,Y\text{-ind.}}{=} \lim_{N\to\infty}\left(E|X|1_{\{|X|\le N\}}\right)\left(E|Y|1_{\{|Y|\le N\}}\right)$$

$$\overset{LMCT}{=} E|X|\,E|Y| < \infty$$

The same proof gives

$$E(XY) = EX\,EY$$

(b) $(\Omega = [0,1],\ \mathcal{B},\ P = x)$

Let $X = Y$. $X(x) = \frac{1}{\sqrt{x}}$ — integrable

$Y(x) = \frac{1}{\sqrt{x}}$ — integrable

but $XY = \frac{1}{x}$ — not integrable.

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$$\left(\Omega = [0,1],\ \mathcal{B},\ \lambda\right)$$

(c) $X = \frac{1}{x}$ (not integrable)

$Y = -\frac{1}{x}$ (not integrable)

but $X + Y \equiv 0$ (integrable).

2. Let $\{B_n\}_{n\geq 1}$ be a sequence of events. Suppose there is a $\delta > 0$ such that $P(B_n) \geq \delta$ for all n. Prove that $P(B_n \text{ i.o.}) \geq \delta$.

M1

$$P(\overline{\lim} B_n) \geq \overline{\lim} P(B_n) \geq \delta > 0$$

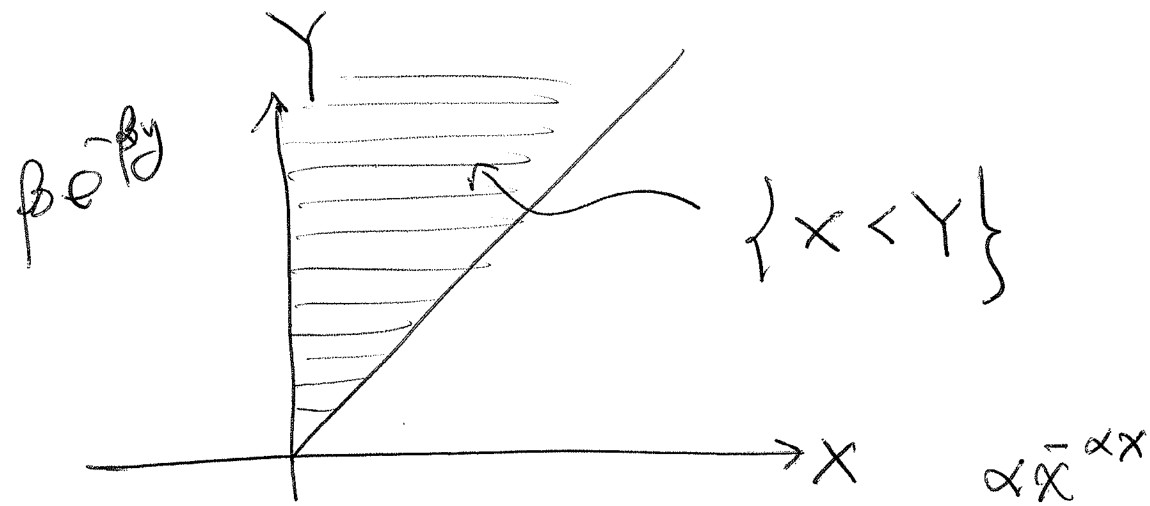
M2

$$P(\overline{\lim} B_n) = P\left(\bigcap_N \bigcup_{n=N}^{\infty} B_n\right)$$

$$= \lim_{N\to\infty} P\left(\bigcup_{n=N}^{\infty} B_n\right)$$

$$\geq \lim_{N\to\infty} P(B_n) \geq \delta > 0$$

3. Let X and Y be two independent, exponentially distributed random variables with parameter α and β, i.e. the pdfs of X and Y are given by $\alpha e^{-\alpha x}$ and $\beta e^{-\beta y}$ for $x, y \geq 0$. Find $P(X < Y)$.



$$P(X<Y) = \int_0^{\infty} \left[\int_0^{y} \alpha e^{-\alpha x} \, dx \right] \beta e^{-\beta y} \, dy$$

$$= \int_0^{\infty} \left(1 - e^{-\alpha y}\right) \beta e^{-\beta y} \, dy$$

$$= \int_0^{\infty} \beta e^{-\beta y} \, dy - \beta \int_0^{\infty} e^{-(\alpha+\beta) y} \, dy$$

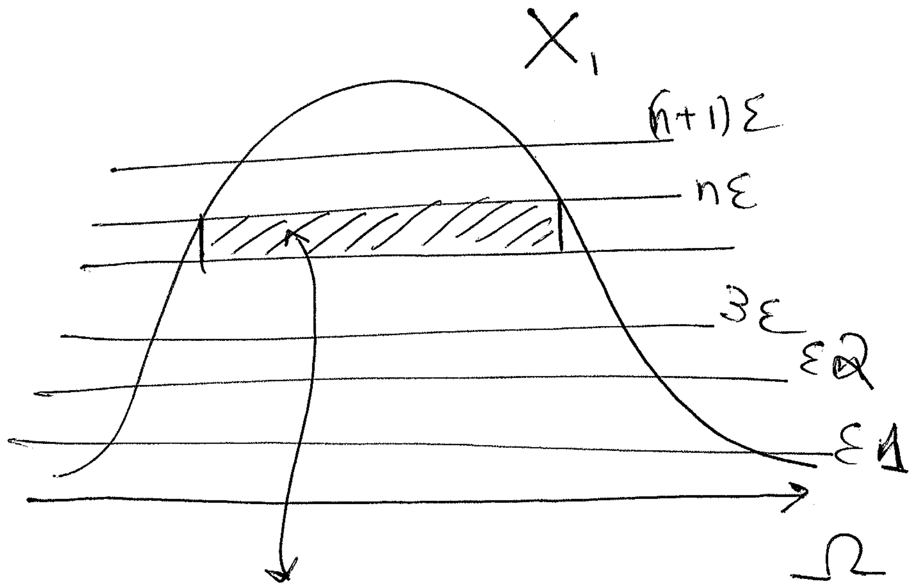
$$= 1 - \frac{\beta}{\alpha+\beta} = \boxed{\frac{\alpha}{\alpha+\beta}}$$

4. Let $X_1, X_2, \ldots$ be i.i.d. non-negative random variables. Prove that $\limsup_n \frac{X_n}{n}$ equals zero or infinity depending on whether EX_1 is finite or infinity.

若 $EX_1 < \infty$

then $\forall \varepsilon$, consider $\sum_{n=1}^{\infty} P(X_n \geq \varepsilon n)$

$= \sum_{n=1}^{\infty} P(X_1 \geq \varepsilon n)$



Area $= \varepsilon P(X_1 \geq n\varepsilon)$

$= \frac{1}{\varepsilon} \sum_{n=1}^{\infty} P(X_1 \geq n\varepsilon)\,\varepsilon \leq \frac{1}{\varepsilon} EX_1 < \infty$

B-C I $\Rightarrow$ $P(X_n \geq \varepsilon n \text{ i.o}) = 0$

i.e. $\forall \varepsilon$, $P\left(\frac{X_n}{n} \geq \varepsilon \text{ i.o}\right) = 0 \Rightarrow$ $\overline{\lim} \frac{X_n}{n} \leq \varepsilon$ P a.s $(\varepsilon \to 0)$

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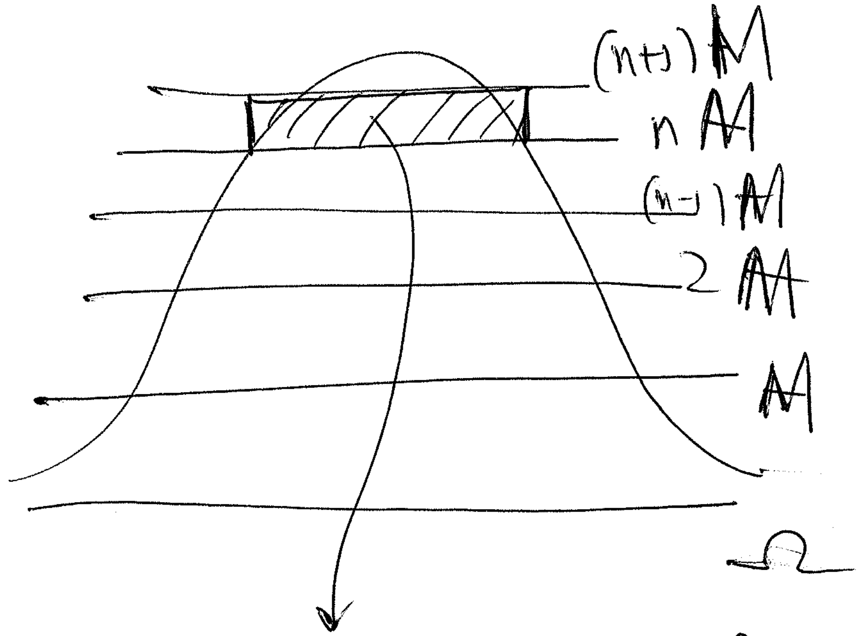
X_n – iid

Similarly, if $EX_1 = \infty$

then $\forall M$, Consider $\sum_{n=1}^{\infty} P(X_n \geq Mn)$

$= \sum_{n=1}^{\infty} P(X_1 \geq Mn)$



Area = $MP(X \geq nM)$

$= \frac{1}{M} \sum_{n=1}^{\infty} P(X_1 \geq Mn) M$

$\geq \frac{1}{M} EX_1 = \infty$

B-C-II (ind) X_n's $\Rightarrow$ $P(X_n \geq Mn \text{ i.o}) = 1$

i.e. $\forall M$, $P(\frac{X_n}{n} \geq M \text{ i.o}) = 1 \Rightarrow$ $\overline{\lim} \frac{X_n}{n} \geq M$, $M \to \infty$

5. Let $X_1, X_2, \ldots$ be a sequence of i.i.d., integrable random variables. Let T be another random variable which takes value in positive integers and is independent of all of the X_i's. Define

$$S_T = \sum_{i=1}^{T} X_i$$

Prove the following:

(a) $ES_T = E(T)E(X_1)$.

(b) $\mathrm{Var}(S_T) = [E(X_1)]^2 \,\mathrm{Var}(T) + E(T)\mathrm{Var}(X_1)$.

(a) $ES_T = E\left[X_1 + X_2 + \cdots + X_T\right]$

$$= \sum_{n=1}^{\infty} E\left(X_1 + X_2 + \cdots + X_n\right) 1_{\{T=n\}}$$

$$= \sum_{n=1}^{\infty} E\left(X_1 + \cdots + X_n\right) E\, 1_{\{T=n\}}$$

$$= \sum_{n=1}^{\infty} n(EX_1) P(T=n)$$

$$= EX_1 \sum_{n=1}^{\infty} nP(T=n) = EX\,ET.$$

$$\mathrm{Var}(S_T) = E(S_T^2) - (ES_T)^2$$

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(b). $E(S_T^2)$

$$= \sum_{n=1}^{\infty} E(X_1 + \cdots + X_n)^2 \, P(T=n)$$

$(a \quad b \quad c)^2$

$a^2 + b^2 + c^2$

$2ab + 2bc + 2bc$

$$= \sum_{n=1}^{\infty} \left(nE(X_i^2) + n(n-1)\, EX_i EX_j \right) P(T=n)$$

6

$3 \cdot 2$

$a \quad b \quad c \quad d$

$2ab \quad 2bc \quad 2cd$

$2ac \quad 2ad$

$2bd$

4×3

$$= \sum_{n=1}^{\infty} \left[n E(X_1^2) + n(n-1)(EX_1)^2 \right] P(T=n).$$

$$= \sum_{n=1}^{\infty} nP(T=n)\, E(X_1^2) + (E(X_1))^2 \sum_{n=1}^{\infty} (n^2 - n) P(T=n)$$

$$= E(X_1^2)\, E(T) + (EX_1)^2 \left[E(T^2) - (ET) \right]$$

$$= \left(E(X_1^2) - (EX_1)^2 \right) ET + (EX_1)^2 E(T^2)$$

6. In this problem, $\{N_n\}_{n\geq 1}$ denotes some sequence of positive-integer-valued random variables such that $N_n \longrightarrow \infty$ with *probability one (i.e. P a.s.).*

   (a) Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables converging to 0 with *probabilty one.* Prove that X_{N_n} converges to zero with *probability one.*

   (b) Give an example of a $\{N_n\}$ (satisfying the stated assumption at the beginning of this problem) and a sequence of random variables $\{X_n\}$ which converges to zero *in probability* but X_{N_n} *does not* converge to zero in probability (and hence of course, cannot converge to zero with probability one).

   (c) Give an example of a sequence of random variables $\{X_n\}$ which converges to zero *in probability* but $\dfrac{X_1 + X_2 + \cdots X_n}{n}$ *does not* converge to zero in probability.

(a) $X_n \longrightarrow 0$ P a.s

i.e. $\exists \Omega_1$ s.t. $P(\Omega_1) = 1$ + $\forall \omega \in \Omega_1$ (fixed), $X_n(\omega) \xrightarrow{n\to\infty} 0$

$N_n \longrightarrow 0$ P a.s

i.e. $\exists \Omega_2$ s.t. $P(\Omega_2) = 1$ + $\forall \omega \in \Omega_2$ (fixed), $N_n(\omega) \xrightarrow{n\to\infty} \infty$

Hence $\forall \omega \in \Omega_1 \cap \Omega_2$ (fixed) Note $P(\Omega_1 \cap \Omega_2) = 1$

$X_{N_n(\omega)}(\omega) \xrightarrow{n\to\infty} 0$

$N_n(\omega) \to \infty$, ω fixed

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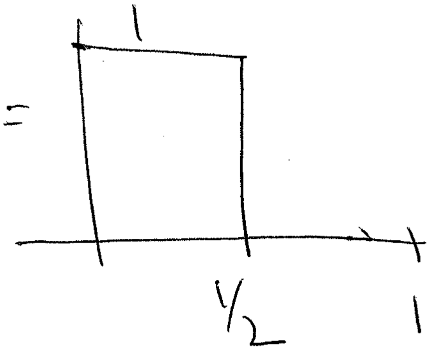
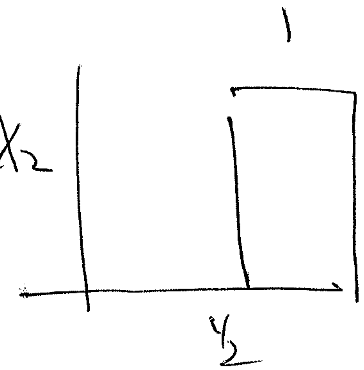
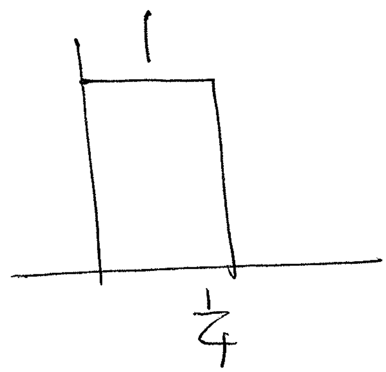
(b) Consider $X_1 =$  X_2 

X_3 

$\forall \omega \in (0,1)$, let $N_n(\omega)$ = the n^{th} index k s.t. $X_k(\omega) = 1$

then $N_n(\omega) \xrightarrow{n \to \infty} \infty$ $\quad \forall \omega$ (fixed)

but $X_{N_n(\omega)}(\omega) \equiv 1 \quad \forall n, \omega$!

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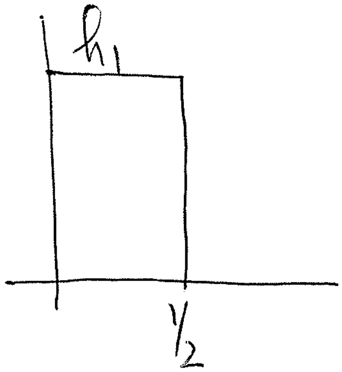
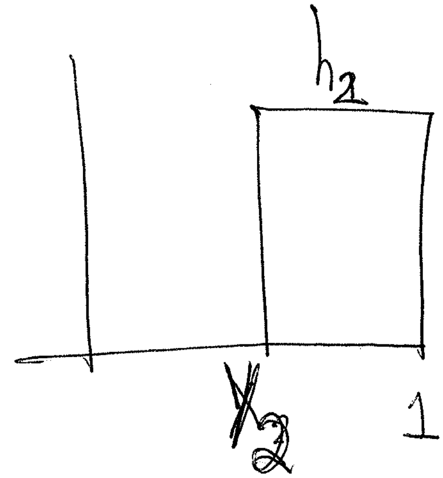
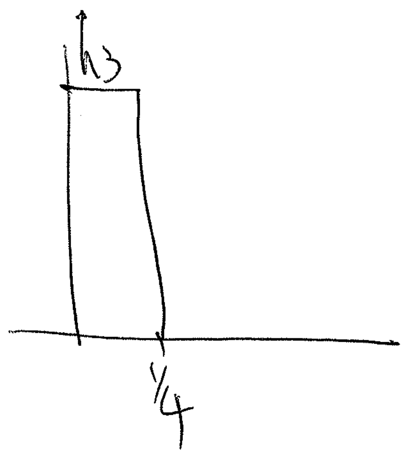
(c) $X_1 =$



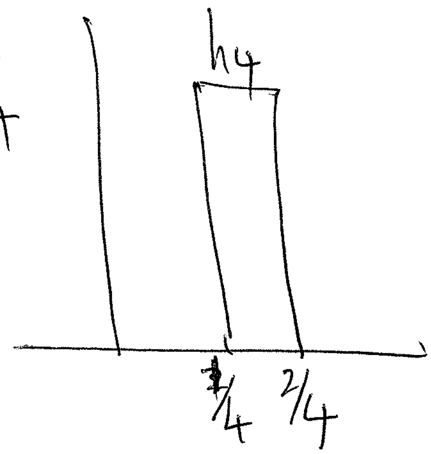
X_2



X_3



X_4



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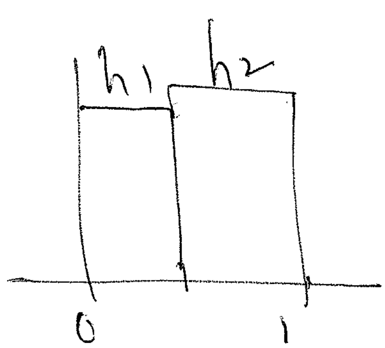
Choose $h_1, h_2 \dots$ SO LARGE that (eg $h_n = 2^{n+2}$)

$$\frac{X_1(\omega)+X_2(\omega)+\cdots+X_n(\omega)}{n} \geq \frac{1}{2} \text{ (for example)}$$

$\not\to 0.$ $\forall n$

eg. $\frac{X_1+X_2}{2} =$



and $\frac{X_1+X_2+X_3+X_4+X_5+X_6}{6}$