MA/STAT 538 Midterm Solution (Spring 2006, Yip)

1. Prove the following statements:

   (a) Let $\{a_n\}$ be a sequence of numbers. Then

$$\liminf_n a_n = \sup_n \left\{ \inf_{k\geq n} a_k \right\} \qquad (*)$$

   (b) Let $(\Omega, \mathcal{F})$ be a sample space endowed with a σ-field and $\{A_n\}$ be a sequence of sets from $\mathcal{F}$. Then

$$\liminf_n 1_{A_n} = 1_{\liminf A_n}$$

   where $1_B(\omega) = 1$ for $\omega \in B$ and 0 otherwise.

(a) (Here $\underline{\lim}_n a_n \overset{\text{defined}}{=} \inf\{\text{limit pts of } a_n\}$)

① If $\{a_n\}$ is not bded from below, i.e. $\exists\, a_{n_i}$'s $\xrightarrow{n_i \to \infty} -\infty$

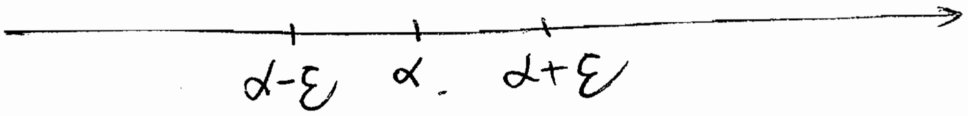
then L.H.S. of $(*) = -\infty$

For R.H.S. of $(*)$, $\forall n$, $\inf_{k\geq n} a_k = -\infty$

Hence $\sup_n \left\{ \inf_{k\geq n} a_k \right\} = -\infty =$ L.H.S. of $(*)$.

② So assume $\exists\, M > -\infty$ s.t. $a_n \geq -M$ $\forall n$.

Let $\alpha = \underline{\lim}_n a_n > -\infty$



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$\forall \varepsilon > 0, \ \exists N \ni \ \forall k > N \implies a_k \geq \alpha - \varepsilon$

Hence, $\forall n \geq N, \implies \inf_{k \geq n} a_k \geq \alpha - \varepsilon$

$$\implies \sup_n \left\{ \inf_{k \geq n} a_k \right\} \geq \alpha - \varepsilon$$

$\downarrow \varepsilon \to 0$

$$\implies \sup_n \left\{ \inf_{k \geq n} a_k \right\} \geq \alpha$$

$\forall \varepsilon > 0, \ \exists \{n_i\}$'s s.t. $n_i \to \infty$ & $a_{n_i} < \alpha + \varepsilon$

Hence, $\forall n, \quad \inf_{k \geq n} a_k \leq \alpha + \varepsilon$

$$\implies \sup_n \left\{ \inf_{k \geq n} a_k \right\} \leq \alpha + \varepsilon$$

$\downarrow \varepsilon \to 0$.

$$\to \sup_n \left\{ \inf_{k \geq n} a_k \right\} \leq \alpha$$

(b) $\underline{\lim}_n \mathbb{1}_{A_n} \overset{?}{=} \mathbb{1}_{\underline{\lim}_n A_n}$

As $\mathbb{1}_{A_n}$ takes on values of 0 or 1 only. So

$\boxed{\underline{\lim}_n \mathbb{1}_{A_n} = 0 \text{ or } 1.}$

if $x \in \underline{\lim}_n A_n$, then $\boxed{\mathbb{1}_{\underline{\lim}_n A_n}(x) = 1}$

But $x \in A_n$ $\forall$ n large enough.

Hence $\mathbb{1}_{A_n}(x) = 1$ $\forall$ n large enough

$$\Rightarrow \boxed{\underline{\lim}_n \mathbb{1}_{A_n}(x) = 1}$$

if $x \notin \underline{\lim}_n A_n$, then ~~$x \in \{A_n^c \text{ i.o}\}$~~ $\boxed{\mathbb{1}_{\underline{\lim}_n A_n}(x) = 0}$

But then $x \in \{A_n^c \text{ i.o}\}$

i.e. $\mathbb{1}_{A_n}(x) = 0$ for infinitely many n's.

$$\Rightarrow \boxed{\underline{\lim}_n \mathbb{1}_{A_n}(x) = 0}$$

2. Let f be a real valued measurable function of x (with respect to the usual Borel σ-field) which is integrable (with respect to the Lebesgue one-dimensional measure dx) in the sense that $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$. Show that:

$$\lim_{t\to 0} \int_{-\infty}^{\infty} |f(x+t) - f(x)|\, dx = 0$$

(Hint: Use approximation theorem or any other method of your choice.)

$\forall \varepsilon, \ \exists \ g = \sum_{i=1}^{N<\infty} x_i \mathbb{1}_{A_i}$ s.t. $A_i = (a_i, b_i)$

&. $\int |f(x) - g(x)|\, dx < \varepsilon/3$

Thm 13.5,
15.1 (iii)
<u>11.4 (ii)</u>.

Now $\int |f(x+t) - f(x)|\, dx$

$$\leq \int |f(x+t) - g(x+t)| + \int |g(x+t) - g(x)| + \int |g(x) - f(x)|$$

$$\leq \varepsilon/3 + \int |g(x+t) - g(x)| + \varepsilon/3 \quad (*)$$

As g is given by simple intervals, note that

$$\lambda\{(a_i+t, b_i+t) \,\Delta\, (a_i, b_i)\} = 2t.$$

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Hence if t is choosen small enough, say

$$t < \frac{\varepsilon/3}{2N[1+|x_1|+|x_2|+\cdots+|x_N|]}$$

then $\int |g(x+t) - g(x)|\, dx < \varepsilon/3$

So that $\boxed{(*) < \varepsilon}$

⑥

3. Let $(\Omega, \mathcal{F}_0, P)$ be a sample space endowed ~~with a field~~ $\mathcal{F}_0$ and a countably-additive measure P on $\mathcal{F}_0$. Assume P is finite, i.e. $P(\Omega) < \infty$. Consider the following definitions:

$$\text{For all } A \subseteq \Omega, \quad P^*(A) = \inf \left\{ \sum_n P(A_n) : A \subseteq \bigcup_n A_n, A_n \in \mathcal{F}_0 \right\}$$
$$\mathcal{M} = \{ A \subseteq \Omega : P^*(A \cap E) + P^*(A^c \cap E) = P^*(E), \text{ for all } E \subseteq \Omega \}$$

(For this problem, you can assume the facts that P^* is a countably additive measure on the σ-field $\mathcal{M}$, $\mathcal{F}_0 \subseteq \mathcal{M}$ and $P^*|_{\mathcal{F}_0} = P$, i.e. P^* extends P from $\mathcal{F}_0$ to $\mathcal{M}$.) Prove the following statements.

(a) $\mathcal{M}$ is a complete σ-field in the sense that if $P^*(N) = 0$, then $N \in \mathcal{M}$.

(b) For all $A \in \mathcal{M}$, there exists $B, C \in \sigma(\mathcal{F}_0)$ such that $B \subseteq A \subseteq C$ and $P^*(C \backslash B) = 0$.

(c) For all $A \in \mathcal{M}$ and $\epsilon > 0$, there exists a $D \in \mathcal{F}_0$ such that $P^*(A \Delta D) < \epsilon$, where $A \Delta D = (A \backslash D) \cup (D \backslash A)$.

(a). We need to show that if $P^*(N) = 0$, then

$$P^*(N \cap E) + P^*(N^c \cap E) = P^*(E) \quad \forall E \subseteq \Omega$$

$\geq$ always true by monotonicity of P^*

$\leq$? ← OK.

Consider

$$P^*(N \cap E) + P^*(N^c \cap E) \leq P^*(N) + P^*(E)$$

(by monotonicity)

$$= P^*(E)$$

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(b) From the definition of $P^*(A)$,

$\forall\, 2^{-i}$, $\exists\, A_n^i$ s.t. $\sum_n P(A_n^i) \le P^*(A) + 2^{-i}$

without loss of generality, assume $\{A_n^i\}_{n=1}^{\infty}$ are disjoint.

(Otherwise, take $\tilde{A}_n^i = A_n^i \setminus \{A_1^i \cup A_2^i \cup \cdots A_{n-1}^i\}$).

Set $C^i = \bigcup_n A_n^i \in \sigma(\mathcal{F}_0)$ $\quad (A \subseteq C^i)$

& $C = \bigcap_i C^i \in \sigma(\mathcal{F}_0)$. $\quad (A \subseteq \bigcap C^i = C)$

then $P^*(C) = P^*(A)$ & $P^*(C \setminus A) = 0$

(Note that $A \subseteq C$)

For B, follow the <u>same</u> procedure for A^c & find $\tilde{B}$

So that $P^*(\tilde{B}) = P^*(A^c)$, $\quad A^c \subseteq \tilde{B} \in \mathcal{F} = \sigma(\mathcal{F}_0)$

Now set $B = \tilde{B}^c \subseteq A$, we will similarly have

$P^*(B) = P^*(A)$ (Note: $P(\Omega) < \infty$, by assumption)

(c) $\forall \varepsilon$, $\exists$ A_n's s.t. $A \subseteq \bigcup_n A_n$

$$\sum_n P(A_n) \leq P^*(A) + \varepsilon/2$$

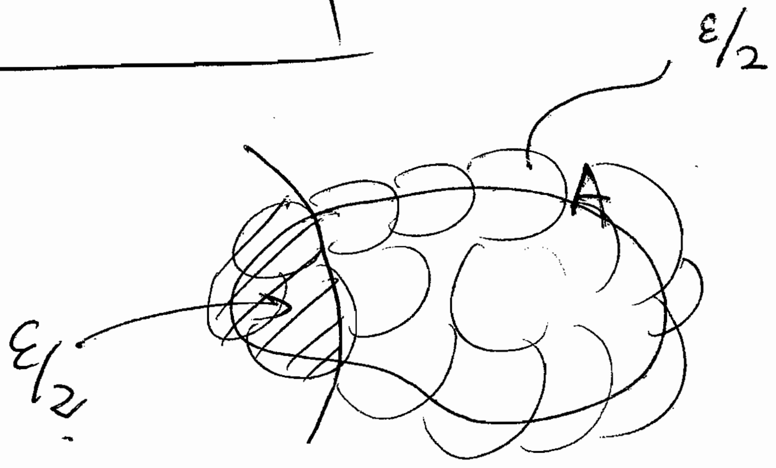
Without loss of generality, assume again A_n's are disjoint.

Consider $\tilde{D} = \bigcup_{n=1}^{\infty} A_n$

As $P^*(\tilde{D}) = \sum_{n=1}^{\infty} P(A_n) < \infty$

$\exists N$ s.t. $\sum_{n=N}^{\infty} P(A_n) < \varepsilon/2$

Now set $\boxed{D = \bigcup_{n=1}^{N-1} A_n}$.



4. Let $\{X_1, X_2, \ldots X_n\}$ be n mutually independent discrete random variables, each having the *uniform distribution* on $\{1, 2, \ldots N\}$, i.e. $P(X_i = k) = N^{-1}$ for $k = 1, 2, \ldots N$. Let U_n and V_n be the smallest and largest values of the X_i's. Find the distribution of U_n and V_n, i.e. find $P(U_n = k)$ and $P(V_n = k)$.

$$P(U_n \geq k) = P(X_1 \geq k, X_2 \geq k, \ldots, X_n \geq k)$$

$$= P(X_1 \geq k)^n = \left(\frac{N-k+1}{N}\right)^n$$

So

$$P(U_n = k) = P(U_n \geq k) - P(U_n \geq k+1)$$

$$= \boxed{\left(\frac{N-k+1}{N}\right)^n - \left(\frac{N-k}{N}\right)^n}$$

$$P(V_n \leq k) = P(X_1 \leq k, \ldots, X_n \leq k)$$

$$= P(X_1 \leq k)^n = \left(\frac{k}{N}\right)^n$$

So $P(V_n = k) = P(V_n \leq k) - P(V_n \leq k-1)$

$$= \left(\frac{k}{N}\right)^n - \left(\frac{k-1}{N}\right)^n$$

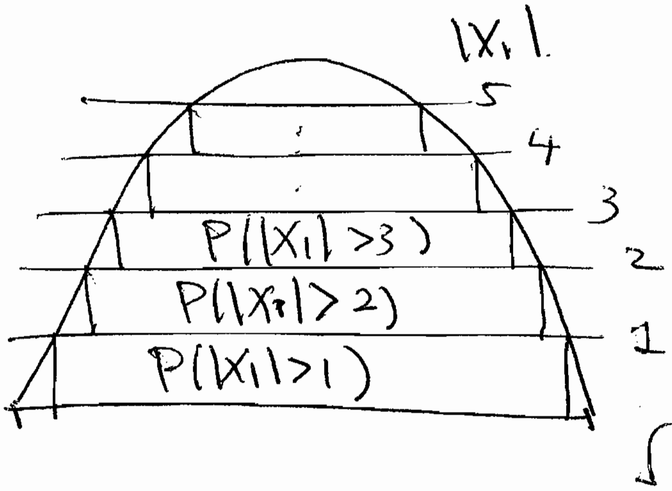
5. Let $\{X_n\}$ be a sequence of independent identically distributed random variables. Prove that

$$E|X_1| < \infty \text{ if and only if } P(|X_k| > k \text{ i.o.}) = 0.$$

As the $\{X_n\}$'s are independent,

$$P(|X_k|>k \text{ i.o.}) = 0 \overset{\overset{BC\text{-}1}{\longleftarrow}}{\underset{\underset{BC\text{-}2}{\Longrightarrow}}{\Longleftrightarrow}} \sum_{k=1}^{\infty} \underset{\underset{\displaystyle P(|X_1|>k)}{\parallel}}{P(|X_k|>k)} < \infty$$

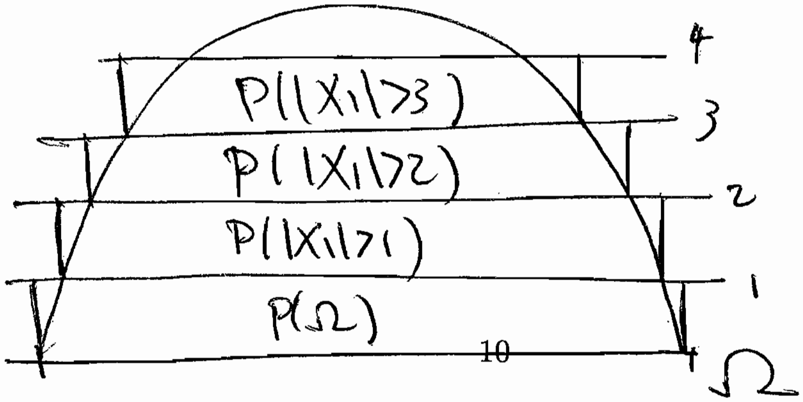
Now:



(Fig 1)

Hence $$\boxed{\sum_{k=1}^{\infty} P(|X_1|>k) < E|X_1|} \quad (*)$$

Also



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Hence

$$\boxed{E|X_1| < 1 + \sum_{k=1}^{\infty} P(|X_1| > k)} \quad (**)$$

So (*) & (**) $\Longrightarrow$

$$E|X_1| < \infty \iff \sum_{k} P(|X_1| > k) < \infty$$

Note
(*) & (**) are special cases of (21.9) Pg 275 of Billingsley.

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⑫

6. Let $\{X_n\}$ be a sequence of independent identically distributed random variables. Suppose that the X_n assume only positive values and that $E(X_n) = A < \infty$ and $E(X_n^{-1}) = B < \infty$. Let $S_N = X_1 + X_2 + \cdots X_N$. Prove the following statements:

   (a) $E(S_N^{-1}) < \infty$.

   (b) $E(X_k S_N^{-1}) = N^{-1}$ for $k = 1, 2, \ldots n$.

   (c)

$$E\left(\frac{S_M}{S_N}\right) = \begin{cases} \frac{M}{N} & \text{if } M \le N \\ 1 + (M-N)AE(S_N^{-1}) & \text{if } M > N \end{cases}$$

(a) $$\frac{1}{S_N} = \frac{1}{X_1+X_2+\cdots+X_N} < \frac{1}{X_1}$$

Hence $$E\left(\frac{1}{S_N}\right) < E\left(\frac{1}{X_1}\right) = B < \infty$$

(b) As the X_n's are iid, by symmetry, we have:

$$E\left(\frac{X_k}{X_1+X_2+\cdots+X_N}\right) = E\left(\frac{X_{k'}}{X_1+X_2+\cdots+X_N}\right) \quad (k \neq k').$$

But

$$1 = E\left[\frac{X_1+X_2+\cdots+X_N}{X_1+X_2+\cdots+X_N}\right] = \sum_{i=1}^{N} E\left[\frac{X_i}{X_1+\cdots+X_N}\right] = N\,E\left[\frac{X_1}{X_1+\cdots+X_N}\right]$$

So $$\boxed{E\left[\frac{X_k}{X_1+\cdots+X_N}\right] = \frac{1}{N}}$$

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(c). $$E\left(\frac{S_M}{S_N}\right) \overset{(a)}{=} \frac{M}{N} \qquad (M \le N)$$

for $\underline{M > N}$

$$E\left(\frac{X_1 + \cdots + X_N + \cdots + X_M}{X_1 + \cdots + X_N}\right)$$

$$= E\left[1 + \frac{X_{N+1} + \cdots + X_M}{X_1 + X_2 + \cdots + X_N}\right]$$

X_n's indep of $X_1, \ldots N$ for $n > N$

$$= 1 + (M-N)\, E X_1\, E\left[\frac{1}{\cancel{X}_1 + \cdots + X_N}\right]$$

$$= 1 + (M-N)\, A\, E(S_N^{-1}).$$

7. Consider the infinitely many identical, independent coin tossing experiments, with outcomes denoted by $X_n = -1$ or 1. Assume $0 < P(X_1 = 1) < 1$.

   (a) Let **S** be a given, fixed pattern of K-long sequence of -1 and 1's. Let also $A_n = \{(X_n, X_{n+1}, \ldots X_{n+K-1}) = \mathbf{S}\}$. Show that $P(A_n \text{ i.o.}) = 1$. i.e. almost surely, the pattern **S** will appear infinitely often.

   (b) Let $Z_n = X_1 + X_2 + \cdots X_n$. Suppose $P(X_1 = 1) \neq \frac{1}{2}$. Show that $P(Z_n = 0 \text{ i.o.}) = 0$, i.e. if the coins are biased, then almost surely, the event of *equalization* — $Z_n = 0$ — cannot occur infinitely often.

(a). Let $B_n = \{(X_{nK}, X_{nK+1}, \cdots, X_{nK+K-1}) = S\}$

Then $\{B_n\}$'s indep.

Note $\{B_n \text{ i.o}\} \subseteq \{A_n \text{ i.o}\}$

Hence BC-2 gives that

if $\sum P(B_n) = \infty \Rightarrow P(B_n \text{ i.o}) = 1$.

But $P(B_n) = P(B_1) > 0$, a fixed no.

$\left(= P(X=1)^{\#\text{ of } 1\text{'s in } S} \; P(X=-1)^{\#\text{ of } -1\text{'s in } S}\right.$

So $\boxed{\sum_n P(B_n) = \infty}$

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(b). By B-C-1, it suffices to show that

$$\sum_n P(Z_n) < \infty$$

Clearly, we just need to consider

$$\sum_{n=1}^{\infty} P(Z_{2n}=0) < \infty$$ (can only equalize at 2n's).

But $P(Z_{2n}=0) = {}_{2n}C_n \, p^n (1-p)^n$

$$\boxed{\begin{array}{l} p = P(X=1) \\ 1-p = P(X=-1) \end{array}}$$

$$= \frac{(2n)!}{(n!)^2} [p(1-p)]^n$$

(by Stirling's Formula: $\lim_{n \to \infty} \frac{n!}{n^n e^{-n} \sqrt{2\pi n}} = 1$, Hw1 Add prob. #1)

$$\approx \frac{4^n}{\sqrt{n}} [p(1-p)]^n = \frac{[4p(1-p)]^n}{\sqrt{n}}.$$

Now if $p \neq \frac{1}{2}$, $\propto 4p(1-p) < 1$, Hence

$$\boxed{\sum_n \frac{1}{\sqrt{n}} [4p(1-p)]^n < \infty}$$ QED