Spring 2008 (Yip MA/STAT 538) Midterm

1. Consider an i.i.d. sequence $\{X_n\}_{n\geq 1}$ of Cauchy random variables, i.e. the p.d.f. of each of X_n's is given by:

$$f(x) = \frac{1}{\pi(x^2+1)}$$

Let $\{a_n\}_{n\geq 1}$ be a sequence of positive, deterministic numbers. Show that $P(|X_n| \geq a_n \text{ i.o.})$ can only take the values of 0 or 1 and it happens if and only if $\sum_{n=1}^{\infty} \frac{1}{a_n} < \infty$ or $= \infty$.

As $\{X_n\}$'s are independent, by Kolmogorov-0-1-Law,

$$P\{|X_n| \geq a_n \text{ i.o}\} = 0 \text{ or } 1$$

(since $\{|X_n| \geq a_n \text{ i.o}\}$ is a tail event.)

By Borel-Cantelli-Lemma I & II,

$$P\{|X_n| \geq a_n \text{ i.o}\} = 0 \text{ or } 1 \iff \sum_{n=1}^{\infty} P(|X_n| \geq a_n) \begin{array}{c} < \infty \\ \text{or} \\ = \infty \end{array}$$

Now, $$P(|X_n| \geq a_n) = 2\int_{a_n}^{\infty} \frac{dx}{\pi(x^2+1)}$$

Hence $$\sum_{n=1}^{\infty} P(|X_n| \geq a_n) \begin{array}{c} < \infty \\ \text{or} \\ = \infty \end{array} \quad \text{iff} \quad \sum_{n=1}^{\infty} \int_{a_n}^{\infty} \frac{dx}{x^2+1} \begin{array}{c} < \infty \\ \text{or} \\ = \infty \end{array}$$

This is a scrap paper.

<u>if</u> $\{a_n\}$ <u>does not</u> converge to infinity as $n \to \infty$,

then $\sum_{n=1}^{\infty} \int_{a_n}^{\infty} \frac{dx}{x^2+1} = \infty$ which gives also $\sum \frac{1}{a_n} = \infty$

& thus $P\{|X_n| \geq a_n \text{ i.o.}\} = 1$.

if $\{a_n\}$ does converge to infinity as $n \to \infty$

then $\frac{1}{2}\int_{a_n}^{\infty} \frac{dx}{x^2} \leq \int_{a_n}^{\infty} \frac{dx}{x^2+1} \leq \int_{a_n}^{\infty} \frac{dx}{x^2}$

$x^2+1 \leq 2x^2$

i.e. $\frac{1}{2}\frac{1}{a_n} \leq \int_{a_n}^{\infty} \frac{dx}{x^2+1} \leq \frac{1}{a_n}$.

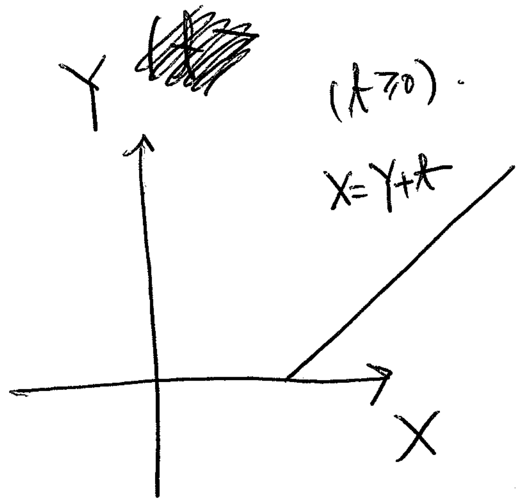
So $\sum_{n=1}^{\infty} \int_{a_n}^{\infty} \frac{dx}{x^2+1} \; \begin{matrix} < \infty \\ = \infty \end{matrix} \iff \sum_{n=1}^{\infty} \frac{1}{a_n} \; \begin{matrix} < \infty \\ = \infty \end{matrix}$

2. Let X and Y be two independent random variables having exponential distribution with parameter λ, i.e. the p.d.f.'s of X and Y are given by $\lambda e^{-\lambda t}$ for $t \geq 0$.

   Let $Z = X - Y$. Find the p.d.f. of Z.

pdf of Z $= f(t) = P(Z=t) = P(X-Y=t)$.

$P(X-Y=t) = P(X=Y+t)$.



(1) $t \geqslant 0$.

$$P(X=Y+t) = \int_0^\infty P(X=s+t)\,P(Y=s)\,ds$$

$$= \int_0^\infty \lambda e^{-\lambda(s+t)}\,\lambda e^{-\lambda s}\,ds \qquad \left(\int_0^\infty e^{-2\lambda s}\,ds = \frac{1}{2}\right)$$

$$= \lambda^2 e^{-\lambda t}\int_0^\infty e^{-2\lambda s}\,ds = \frac{\lambda e^{-\lambda t}}{2}.$$

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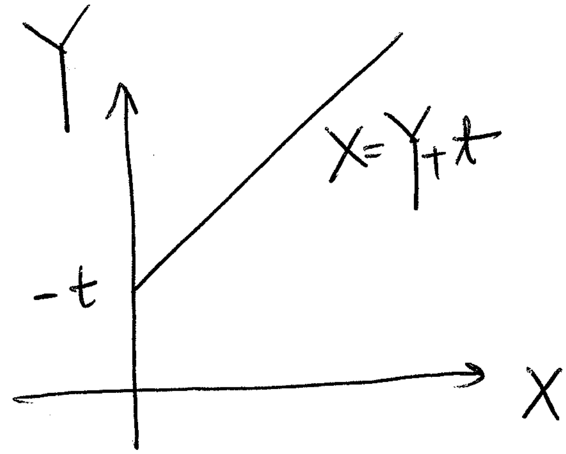
(2) $t \leq 0$



$$P(X = Y+t) = \int_{-t}^{\infty} P(X = s+t)\, P(Y = s)\, ds$$

$$= \int_{-t}^{\infty} \lambda e^{-\lambda(s+t)} \lambda e^{-\lambda s}\, ds$$

$$= \lambda^2 e^{-\lambda t} \int_{-t}^{\infty} e^{-2\lambda s}\, ds = \lambda e^{-\lambda t} \frac{e^{2\lambda t}}{2}$$

$$= \frac{\lambda}{2} e^{\lambda t}.$$

Hence pdf of Z

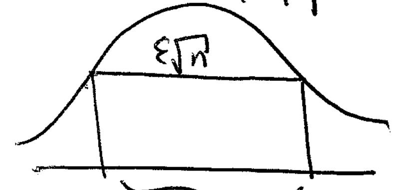
$$\boxed{f(t) = \frac{\lambda}{2} e^{-\lambda |t|}}$$

3. Let $X_1, X_2, \ldots$ be i.i.d. random variables with finite second moment, i.e. $E|X|^2 < \infty$. Prove the following statements:

(a) For all $\epsilon > 0$, $\lim_{n\to\infty} nP\left(|X_1| \geq \epsilon\sqrt{n}\right) = 0$.

(b) For all $\epsilon > 0$, $\lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}} \max_{1\leq k\leq n} |X_k| \geq \epsilon\right) = 0$

(Hint: make use of some inequalities.)



「a). $\cancel{P(|X_1| \geq \varepsilon\sqrt{n}) \leq \frac{E|X_1|^2}{(\varepsilon\sqrt{n})^2}}$

$$\varepsilon^2 n P\{|X_1| \geq \varepsilon\sqrt{n}\} \leq \int_{\{|X_1| \geq \varepsilon\sqrt{n}\}} |X_1|^2 \, dP$$

$$= \int_\Omega \underbrace{|X_1|^2 1_{\{|X_1| \geq \varepsilon\sqrt{n}\}}} \, dP$$

$\downarrow$ as $n \to \infty$

0

$\xrightarrow{n\to\infty} 0$. (LDCT.

$|X_1|^2 1_{\{|X_1| \geq \varepsilon\sqrt{n}\}} \leq |X_1|^2$.

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(b) M1 $$P\left\{\frac{\max_{1\le k\le n}|X_k|}{\sqrt{n}}\geq \varepsilon\right\} = P\left\{\max_{1\le k\le n}|X_k|\geq \varepsilon\sqrt{n}\right\}$$

$$P\left\{\max_{1\le k\le n}|X_k|\leq \varepsilon\sqrt{n}\right\} = \left[P\left\{|X_1|\leq \varepsilon\sqrt{n}\right\}\right]^n.$$

$(\because \{X_k\}\text{-ind.})$

$$= \left[1 - P(|X_1|\geq \varepsilon\sqrt{n})\right]^n.$$

$$= \left[1 - \frac{nP(|X_1|\geq \varepsilon\sqrt{n})}{n}\right]^n \xrightarrow{\text{as } n\to\infty} 1$$

Take the log: $$n\log\left(1-P(|X_1|\geq \varepsilon\sqrt{n})\right)$$

$$\cong nP(|X_1|\geq \varepsilon\sqrt{n}) \xrightarrow{n\to\infty} 0 \text{ by (a)}$$

M2

$$P\left\{ \frac{\max_{1 \le k \le n} |X_k|}{\sqrt{n}} \geq \varepsilon \right\}$$

$$= P\left\{ \max_{1 \le k \le n} |X_k| \geq \varepsilon\sqrt{n} \right\}$$

$$= P\left\{ \exists\, 1 \le k \le n \ni |X_k| \geq \varepsilon\sqrt{n} \right\}$$

$$= P\left\{ \bigcup_{k=1}^{n} \left\{ |X_k| \geq \varepsilon\sqrt{n} \right\} \right\}$$

$$\leq \sum_{k=1}^{n} P\left\{ |X_k| \geq \varepsilon\sqrt{n} \right\} \qquad \text{sub-additivity} \quad (\cancel{X_k\text{-iid}})$$

$$= n\,P\left\{ |X_1| \geq \varepsilon\sqrt{n} \right\} \qquad (X_k\text{-iid})$$

$$\xrightarrow[n\to\infty]{} 0 \quad (\text{by (a)})$$

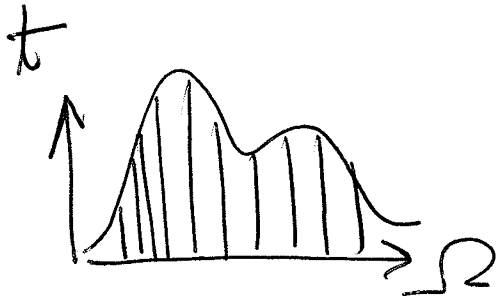
4. Prove that for any integrable random variable X,

$$E(X) = \int_0^\infty P(X > t)\,dt - \int_{-\infty}^0 P(X < t)\,dt$$

(Assume $X \geqslant 0$)

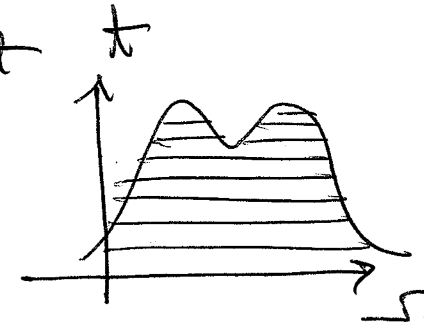
M1 $$EX = \int_\Omega X(\omega)\, dP(\omega)$$

$$= \int_\Omega \int_0^{X(\omega)} dt\, dP(\omega)$$



$\boxed{\text{Fubini}}$

$$= \int_0^\infty \int_{\{X \geqslant t\}} dP(\omega)\, dt$$



$$= \int_0^\infty P(X > t)\, dt$$

(Note: $\int_0^\infty P(X \geqslant t)\, dt = \int_0^\infty P(X > t)\, dt$

as $P(X \geqslant t)$ & $P(X > t)$ only differs at countably many t. !)

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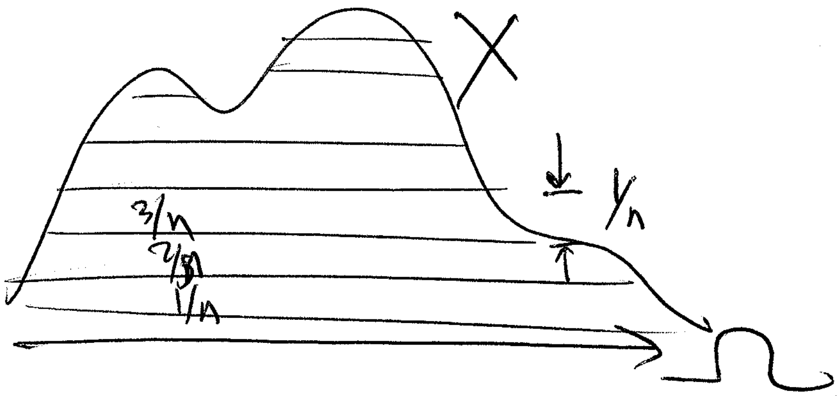
M2 (Use approximation)



$$\int_{\{X>0\}} X\,dP = \lim_{n\to\infty} \sum_{i=1}^{\infty} P\left(X > \frac{i}{n}\right)\frac{1}{n}.$$

$$= \int_0^{\infty} P(X>t)\,dt$$

(Can simply use Rieman integral as $P(X>t)$ is a decreasing fct.).

5. Given two random variables X and Y. Prove that X and Y are independent if and only if

$$E[f(X)g(Y)] = E[f(X)]E[g(Y)]$$

for any two bounded, continuous functions f and g.

Recall def: X & Y are independent iff $\forall A, B \in \mathcal{B}^1$ (Borel 1-dim set),

$$P(X\in A, Y\in B) = P(X\in A)P(Y\in B)$$

i.e. $E[1_A(X)\, 1_B(Y)] = E[1_A(X)]\, E[1_B(Y)]$

" $\Longleftarrow$ " (Assume $E[f(X)g(Y)] = E[f(X)]E[g(Y)]$ $\forall f, g$ — bd cont. fct)

By $\boxed{\text{(Thm) 17.1}}$ Can approximate any meas. fct by continuous fct: $1_A(x) \cong f(x)$, $1_B(y) \cong g(y)$

$$1_A(x) 1_B(y) \cong f(x) g(y).$$

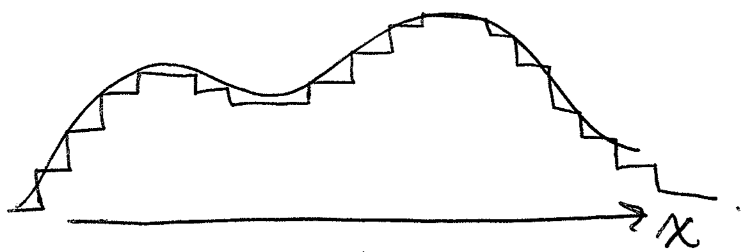
Hence $E[f(X)g(Y)] = E[f(X)]E[g(Y)]$

$$\downarrow \qquad\qquad \downarrow$$

$E[1_A(X)1_B(Y)]$ $\qquad$ $E[1_A(X)]E[1_B(Y)]$

This is a scrap paper.

"$\Longrightarrow$" Approximate any <u>Continuous</u> fct by <u>step functions</u> / or simple function). Thm 13.5 & 17.1



Or simply use Fubini Thm on product measures:

$$E f(X) g(Y) = \iint_{\mathbb{R}\times\mathbb{R}} f(x) g(y) (\mu\times\nu)(dx, dy)$$

$$= \int_{\mathbb{R}} f(x) \mu(dx) \int_{\mathbb{R}} g(y) \nu(dy)$$

$$= E f(X) E g(Y).$$

6. The p.d.f. of the Gamma distribution with parameter (α, k) $(\alpha, k > 0)$ is given by

$$f_{\alpha,k}(x) = \frac{\alpha(\alpha x)^{k-1}e^{-\alpha x}}{(k-1)!} \quad \text{(for } x \geq 0\text{)}$$

(In this problem, for simplicity, k is assumed to be a positive integer.)

(a) What is the common name for $f_{\alpha,1}$?

(b) Let X and Y be two independent Gamma random variables with parameters (α, m) and $(\alpha, 1)$. Show that $X+Y$ is has Gamma distribution with parameter $(\alpha, m+1)$?

(c) Let X and Y be two independent Gamma random variables with parameters (α, m) and (α, n). Show that $X+Y$ is has Gamma distribution with parameter $(\alpha, m+n)$?

(d) Given $p, q > 0$ and $p+q=1$. Let $N, X_1, X_2, \ldots$ be independent random variables such that $P(N=n) = q^{n-1}p$ for $n \geq 1$ and each X_k's has the exponential distribution with parameter α, i.e. its p.d.f. is given by: $\alpha e^{-\alpha x}$ for $x \geq 0$.
Show that $X_1 + X_2 + \cdots X_N$ has density $f_{\alpha p,1}$.

(a) $f_{\alpha,1}(x) = \alpha e^{-\alpha x}$ $(x \geq 0)$ exp, parameter α.

(b). $P(X+Y=t) = \int_0^t P(X=t-s)P(Y=s)\,ds$

$$= \int_0^t \frac{\cancel{\alpha}(\cancel{\alpha}(t-s))^{k-1}e^{-\alpha(t-s)}}{(k-1)!}\,\cancel{\alpha} e^{-\alpha s}\,ds$$

$$= \frac{\alpha^{k+1}e^{-\alpha t}}{(k-1)!}\int_0^t (t-s)^{k-1}\,ds = \frac{\alpha^{k+1}e^{-\alpha t}}{k!}t^k$$

$$= \frac{\alpha(\alpha t)^k e^{-\alpha t}}{k!} = f_{\alpha,k+1}$$

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(c) $f_{\alpha,k} * f_{\alpha,1} = f_{\alpha,k+1}$.

By induction, $f_{\alpha,k} * f_{\alpha,l} = f_{\alpha,k+l}$.

(d) $P(X_1 + \cdots + X_N = t)$

$$= \sum_{n=1}^{\infty} P(X_1 + \cdots + X_n = t,\ N = n)$$

independent

$$= \sum_{n=1}^{\infty} \underbrace{P(X_1 + \cdots + X_n = t)}_{\text{Gamma},\ \alpha,\ n.} P(N = n)$$

$$= \sum_{n=1}^{\infty} \frac{\alpha(\alpha t)^{n-1} e^{-\alpha t}}{(n-1)!} q^{n-1} p = \alpha e^{-\alpha t} p \sum_{n=1}^{\infty} \frac{(\alpha t)^{n-1}}{(n-1)!} q^{n-1}.$$

This is a scrap paper.

$$= \alpha e^{-\alpha t} p \; e^{\alpha q t}$$

$$= \alpha p \; e^{-(1-q)\alpha t}$$

$$= \alpha p \; e^{-p\alpha t} = \boxed{f_{\alpha p, 1}}$$