

MA/STAT 538 Final Exam Solu (Tip)

Spring 2006

1. Let X_1, X_2, \dots be i.i.d. random variables with finite second moment. Prove the following statements:

(a) For all $\epsilon > 0$, $nP(|X_1| \geq \epsilon\sqrt{n}) \rightarrow 0$ as $n \rightarrow \infty$.

(b) $\frac{1}{\sqrt{n}} \max_{1 \leq k \leq n} |X_k| \rightarrow 0$ in probability as $n \rightarrow \infty$.

$$(a) \quad nP(|X_1| \geq \epsilon\sqrt{n}) = n \int_{\{|X_1| \geq \epsilon\sqrt{n}\}} dP$$

$$\leq n \frac{1}{(\epsilon\sqrt{n})^2} \int_{\{|X_1| \geq \epsilon\sqrt{n}\}} |X_1|^2 dP$$

$$= \frac{1}{\epsilon^2} \int_{\Omega} |X_1|^2 \mathbb{1}_{\{|X_1| \geq \epsilon\sqrt{n}\}} dP$$

$$\xrightarrow{n \rightarrow \infty} 0 \quad (\text{LDCT}) \quad \left(\underbrace{|X_1|^2 \mathbb{1}_{\{|X_1| \geq \epsilon\sqrt{n}\}}}_{\downarrow n \rightarrow \infty} \leq |X_1|^2 \right)$$

0 ↑ integrable

This is a blank page.

(b) Method 1.

$$P\left(\frac{\max_{1 \leq k \leq n} |X_k|}{\sqrt{n}} \geq \varepsilon\right) \xrightarrow{?} 0$$

$$P\left(\max_k |X_k| \geq \varepsilon\sqrt{n}\right) \xrightarrow{?} 0$$

Note: $\left\{\max_k |X_k| \geq \varepsilon\sqrt{n}\right\} = \bigcup_{k=1}^n \left\{|X_k| \geq \varepsilon\sqrt{n}\right\}$

Hence $P\left(\max_k |X_k| \geq \varepsilon\sqrt{n}\right) = P\left(\bigcup_{k=1}^n \left\{|X_k| \geq \varepsilon\sqrt{n}\right\}\right)$

$$\stackrel{(X_k \text{ - i.i.d.})}{\leq} \underbrace{n P\left(\left\{|X_1| \geq \varepsilon\sqrt{n}\right\}\right)}_{\substack{\downarrow n \rightarrow \infty \\ 0 \quad \text{Part (a)}}}$$

Method 2

$$P\left(\frac{\max_k |X_k|}{\sqrt{n}} \leq \varepsilon\right) \xrightarrow{?} 1$$

$$P\left(\max_k |X_k| \leq \varepsilon\sqrt{n}\right) \xrightarrow{?} 1$$

$$= P(|X_k| \leq \varepsilon\sqrt{n})^n \xrightarrow{?} 1$$

$$\Leftrightarrow n \log P(|X_k| \leq \varepsilon\sqrt{n}) \xrightarrow{?} 0$$

"

$$n \log [1 - P(|X_k| \geq \varepsilon\sqrt{n})]$$

$$\cong n P(|X_k| \geq \varepsilon\sqrt{n}) \xrightarrow{\text{Part (a)}} 0$$

Note: $\log(1+x) \approx x + o(x^2)$

2. Let X_1, X_2, \dots be i.i.d. random variables. Let also N be a Poisson random variable with parameter λ which is independent of all the X_n 's. Consider the random sum variable:

$$S_N = X_1 + X_2 + \dots + X_N \quad (S_0 = 0).$$

(A Poisson random variable N with parameter $\lambda > 0$ has distribution given by: $P(N = k) = e^{-\lambda} \frac{\lambda^k}{k!}$ for $k = 0, 1, 2, \dots$)

- (a) Find the characteristic function of S_N . Express and simplify your answer in terms of the characteristic function of X_1 .
- (b) Prove that S_N is infinitely divisible.

$$(a) \quad E e^{it S_N} = E e^{it(X_1 + X_2 + \dots + X_N)}$$

$$= \sum_{k=0}^{\infty} E e^{it(X_1 + \dots + X_k)} P(N=k)$$

$$= \sum_{k=0}^{\infty} (E e^{it X_1})^k P(N=k)$$

$$= \sum_{k=0}^{\infty} [\varphi_{X_1}(t)]^k \frac{e^{-\lambda} \lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\varphi_{X_1}(t) \lambda)^k}{k!}$$

$$= e^{-\lambda} e^{\lambda \varphi_{X_1}(t)} = \boxed{e^{\lambda(\varphi_{X_1}(t) - 1)}}$$

④

This is a blank page.

$$(b) \quad \varphi_{S_N}(t) = e^{\lambda(P_X(t)-1)} \\ = \left[e^{\frac{\lambda}{n}(P_X(t)-1)} \right]^n$$

Method 2

Note $e^{\frac{\lambda}{n}(P_X(t)-1)}$ is the char. fun. of S_N
but with N -Poisson, parameter $\frac{\lambda}{n}$.

Since N_λ is inf. div. (Poisson λ)

Method 2

$$N_\lambda = N_{\frac{\lambda}{n}}^{(1)} + N_{\frac{\lambda}{n}}^{(2)} + \dots + N_{\frac{\lambda}{n}}^{(n)}$$

Hence $S_{N_\lambda} = \underbrace{S_{N_{\frac{\lambda}{n}}} + S_{N_{\frac{\lambda}{n}}} + \dots + S_{N_{\frac{\lambda}{n}}}}_{n \text{ of them.}}$

3. Let X_1, X_2, \dots and Y_1, Y_2, \dots be two independent sequences of i.i.d. random variables such that each variable has exponential distribution with parameter λ , i.e. the p.d.f.'s of X_1 and Y_1 equal $\lambda e^{-\lambda t}$ for $t \geq 0$. Or in terms of c.d.f.:

$$P(X_1 \leq t) = P(Y_1 \leq t) = \begin{cases} 1 - e^{-\lambda t} & \text{for } t \geq 0 \\ 0 & \text{for } t \leq 0 \end{cases}$$

Let $Z_n = X_n - Y_n$.

- (a) Find the p.d.f. of Z_n . (Hint: Use convolution or characteristic functions or whatever.)

- (b) Show that $\sum_{n=1}^{\infty} \frac{Z_n}{n}$ converges almost surely.

$$(a) \quad P(Z = t) = P(X - Y = t)$$

$$(t \geq 0) \quad = P(X = Y + t)$$

$$= \int_0^{\infty} P(X = t + s, Y = s) ds$$

$$= \int_0^{\infty} P(X = t + s) P(Y = s) ds$$

$$= \int_0^{\infty} \lambda e^{-\lambda(t+s)} \lambda e^{-\lambda s} ds$$

$$= \lambda^2 e^{-\lambda t} \int_0^{\infty} e^{-2\lambda s} ds$$

$$= \frac{\lambda}{2} e^{-\lambda t}$$

This is a blank page.

$$(t \leq 0) : P(Z=t) = P(Y = X-t)$$

$$= \int_0^{\infty} P(Y = s-t) P(X=s) ds$$

$$= \int_0^{\infty} \lambda e^{-\lambda(s-t)} \lambda e^{-\lambda s} ds$$

$$= \lambda^2 e^{\lambda t} \int_0^{\infty} e^{-2\lambda s} ds$$

$$= \frac{\lambda e^{\lambda t}}{2}$$

$$\text{So pdf of } Z = \frac{\lambda e^{-\lambda|t|}}{2}$$

constant &
finite!

(b) (Apply Kolmogorov Criterion Thm 2.6)

$$\sum_{n=1}^{\infty} \frac{\text{Var}(Z_n)}{n} = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{Var}(Z_n) = \sum_{n=1}^{\infty} \frac{1}{n^2} \underbrace{(\text{Var} X_n + \text{Var} Y_n)}_{\text{finite}}$$

$< \infty$.

4. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables which is *Cauchy in probability*, i.e. for all $\epsilon > 0$, there exists an $N > 0$ such that $P(|X_m - X_n| > \epsilon) < \epsilon$ for all $m, n \geq N$. Show that there exists a random variable X such that $X_n \rightarrow X$ in probability as $n \rightarrow \infty$.

Let $\epsilon = 2^{-k}$. There exist $\{n_k\}_{k=1}^{\infty}$ s.t. $n_1 < n_2 < \dots \rightarrow \infty$

$$\& P(|X_{n_k} - X_{n_{k+1}}| \geq 2^{-k}) \leq 2^{-k}$$

For these choices,

$$\sum_{k=1}^{\infty} P(|X_{n_k} - X_{n_{k+1}}| \geq 2^{-k}) < \infty$$

BC-1

$$\Rightarrow P\left\{ |X_{n_k} - X_{n_{k+1}}| \geq 2^{-k} \text{ i.o.} \right\} = 0$$

Note $\sum_{k=1}^{\infty} 2^{-k} < \infty \Rightarrow \{X_{n_k}\}_{k=1}^{\infty}$ Cauchy.

($\forall \epsilon, \exists K > 0$ s.t. $\forall i, j \geq K$)

$$|X_{n_i} - X_{n_j}| \leq \sum_{k=i}^{j-1} |X_{n_k} - X_{n_{k+1}}| \leq \sum_{k=i}^{j-1} 2^{-k} \leq \epsilon$$

⑧

This is a blank page.

$$\text{i.e. } \exists X \text{ s.t. } X_{n_k} \xrightarrow[k \rightarrow \infty]{\text{a.s.}} X$$

Claim $X_n \xrightarrow{n \rightarrow \infty} X$ in prob. (for the whole sequence)

Pf: $P(|X_n - X| \geq \varepsilon)$

$$= P(|X_n - X_{n_k} + X_{n_k} - X| \geq \varepsilon)$$

$$\leq P(\{|X_n - X_{n_k}| \geq \frac{\varepsilon}{2} \} \cup \{|X_{n_k} - X| \geq \frac{\varepsilon}{2}\})$$

$$\leq P\{|X_n - X_{n_k}| \geq \frac{\varepsilon}{2}\} + P\{|X_{n_k} - X| \geq \frac{\varepsilon}{2}\}$$

\downarrow n, n_k
large enough

0

(by assumption of this problem) (9)

\downarrow $k \rightarrow \infty$
0

(as $X_{n_k} \xrightarrow{\text{a.s.}} X$)

5. Let T_1, T_2, \dots be a sequence of i.i.d. exponentially distributed random variables with parameter λ , i.e. the p.d.f. of T_1 equals $\lambda e^{-\lambda t}$ for $t \geq 0$. Or in terms of c.d.f.:

$$P(T_1 \leq t) = \begin{cases} 1 - e^{-\lambda t} & \text{for } t \geq 0 \\ 0 & \text{for } t \leq 0 \end{cases}$$

Let $M_n = \max_{1 \leq k \leq n} T_k$. Compute the following limits as a function of x :

- (a) $\lim_{n \rightarrow \infty} P(M_n \leq x)$
 (b) $\lim_{n \rightarrow \infty} P\left(\frac{M_n}{\log n} \leq x\right)$
 (c) $\lim_{n \rightarrow \infty} P\left(M_n - \frac{1}{\lambda} \log n \leq x\right)$

Explain in words (in four or five sentences) what the above results ((a), (b), and (c)) reveal about the convergence in distribution of M_n and state also the relationship *between* the results (a), (b), and (c).

(a) (Obviously just need to consider $x \geq 0$)

$$\begin{aligned} P(M_n \leq x) &= P\left(\max_k T_k \leq x\right) = P(T_1 \leq x)^n \\ &= (1 - e^{-\lambda x})^n \\ &\xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

(b) (Obviously just need to consider $x \geq 0$)

$$\begin{aligned} P(M_n \leq x \log n) &= P(T_1 \leq x \log n)^n \\ &= (1 - e^{-\lambda x \log n})^n \end{aligned}$$

$$= \left[1 - \frac{1}{n^{\lambda x}} \right]^n$$

Lemma from Calculus:

$$\left[1 - \frac{1}{n^{\lambda x}} \right]^n \xrightarrow{n \rightarrow \infty} \begin{cases} 1 & \lambda x > 1 \\ e^{-1} & \lambda x = 1 \\ 0 & 0 \leq \lambda x < 1. \end{cases}$$

CPF: $n \log \left(1 - \frac{1}{n^{\lambda x}} \right)$

$$\approx -\frac{n}{n^{\lambda x}} = -\frac{1}{n^{\lambda x - 1}} \xrightarrow{n \rightarrow \infty} \begin{cases} 0 & \lambda x > 1 \\ -1 & \lambda x = 1 \\ -\infty & 0 \leq \lambda x < 1 \end{cases}$$

(c) $\mathbb{P}(M_n - \frac{1}{\lambda} \log n \leq x) = \mathbb{P}(M_n \leq x + \frac{1}{\lambda} \log n)$

$$= \mathbb{P}(\tau_1 \leq x + \frac{1}{\lambda} \log n)^n = \left[1 - e^{-\lambda \left(x + \frac{1}{\lambda} \log n \right)} \right]^n$$

$$= \left[1 - \frac{e^{-\lambda x}}{n} \right]^n \xrightarrow{n \rightarrow \infty} e^{-e^{-\lambda x}}$$

(11)

Physical interpretation of results

(a) $M_n \xrightarrow{D} \infty$ ($P(M_n \leq x) \rightarrow 0$
 $\forall x < \infty$)

(b) $\frac{M_n}{\log n} \xrightarrow{D} \frac{1}{\lambda}$ (a constant!)

(c) $M_n - \frac{1}{\lambda} \log n \xrightarrow{D} \text{dist. } \bar{e} e^{-\lambda x}$

Note: this is a valid cdf

as $e^{-e^{-\lambda x}} \rightarrow 0$ ($x \rightarrow -\infty$)

$e^{-e^{-\lambda x}} \rightarrow 1$ ($x \rightarrow +\infty$)

In words, M_n goes to infinity (a) with asymptotic rate $\frac{1}{\lambda} \log n$ (b). Statement (c) gives the statistics of fluctuation around $\frac{1}{\lambda} \log n$