Stochastic Curvature Driven Flows

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This article surveys some results concerning motion by mean curvature (MMC) for curves and surfaces. Emphasis will be put on methods of incorporating stochastic noise. In addition, we will demonstrate the idea of using noise to provide a selection principle for non-unique solutions.

We pursue the study of MMC for various reasons. The equations of MMC and related evolutions involve interesting and difficult nonlinear partial differential equations. Classical solutions to these equations do not exist globally in time due to the formation of singularities and topological changes of the surfaces. Various new mathematical devices are introduced to tackle these difficulties. Furthermore, MMC (and its anisotropic version) has wide range of applications in describing interfacial motions in materials science. Many solidification processes such as dendritic crystal growth are modeled by coupling the interfacial curvature flows with the diffusion of heat and other chemicals. These lead to various types of free boundary value problems.

Due to the physical origins of these evolutions, it appears that the consideration of noise is an important area of investigation. The noise can come from thermal fluctuations, impurities or the intrinsic instabilities of the deterministic evolutions. The mathematical formulations then involve stochastic partial differential equations. This subject is relatively new compared with its deterministic counterpart. One of the main difficulties in the study of these stochastic equations is how to combine the nonlinear effects of the geometric motions together with the stochastic perturbations.

In the following we will first introduce the definition of MMC and its methods of solutions. Then several results for the stochastic motions are described. Next we explain the selection principle brought out by noise when the deterministic equations have non-unique solutions. This gives an example that stochasticity can indeed change the behavior of the evolution. Some open problems will be discussed at the end.

1 Motion by Mean Curvature

In this section, we briefly describe MMC and its methods of solution. This motion has already been considered in some early works of materials science [49, 1]. We refer to [55, 56] for more detailed discussions.

A hyper-surface $M(t)$ of dimension $n-1$ in R^n is said to evolve by its mean curvature for $t \geq 0$ if the inward normal velocity V_n at every point of the surface equals the mean curvature κ:

$$V_n = \kappa. \tag{1}$$

In the classical setting, the mean curvature is defined as the sum of the principle curvatures. It can also be viewed as the first variation of the area functional (see (4)). The sign convention for κ is that a sphere will shrink. If the surface is parametrized by $\mathbf{F}(p,t) \in R^n$, the motion of $M(t)$ can be described as:

$$\frac{\partial \mathbf{F}(p,t)}{\partial t} = \kappa \mathbf{n}$$

where $\mathbf{n}$ is the unit inward normal. In this article, we will only consider the case when $M(t)$ is the boundary of some n-dimensional domain $D(t)$, i.e. $M(t) = \partial D(t)$. Then $M(t)$ is a curve or surface if $n = 2$ or 3 respectively. Heuristically if $M(t)$ is a curve, it will be smoothed by the evolution and become rounder and smaller. However, for general surfaces, singularities can develop.

For a circle or sphere, the motion can be completely described by the evolution of the radius $R(t)$ which satisfies

$$\frac{dR(t)}{dt} = -\frac{1}{R(t)}$$

The solution is given by $R(t) = \sqrt{R(0)^2 - 2t}$. This shows that the circle(sphere) will disappear in finite time. More generally, the motion of a graph $M(t) = \{(x, f(x,t)) : x \in R^{n-1}\}$ is described by the following quasi-linear parabolic equation:

$$\frac{\partial f}{\partial t} = \sqrt{1 + |\nabla f|^2}\,\mathrm{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) \tag{2}$$

The definition of MMC can be extended to some *anisotropic* version. This generalization has wide range of applications in modeling materials interfacial motions. For this purpose, we define the Φ-*surface energy* $\mathcal{E}_\Phi$ of a surface S to be:

$$\mathcal{E}_\Phi(S) = \int_S \Phi(\nu)\, d\sigma \tag{3}$$

where $\Phi : \{\nu \in R^n : |\nu| = 1\} \longrightarrow R_+$ is called the *surface energy integrand*; $\nu = -\mathbf{n}$ is the unit outward normal; σ is the surface measure of S. We further define the Φ-*mean curvature* κ_Φ to be the *variational derivative* of $\mathcal{E}_\Phi$. It has the property that for all smooth vector field $\mathbf{g} : R^n \longrightarrow R^n$,

$$\left.\frac{d}{ds}\mathcal{E}_\Phi(S_s)\right|_{s=0} = \int_S \kappa_\Phi \langle \mathbf{g},\, \nu\rangle\, d\sigma \tag{4}$$

where for small s, S_s is the image of S under the diffeomorphism $\mathbf{G}_s(x) = x + s\mathbf{g}(x)$. In this sense, κ_Φ can be interpreted as the rate of change of $\mathcal{E}_\Phi$ per volume swept out by the deformation $\mathbf{G}_s$. We now define the motion by Φ-(weighted) mean curvature as

$$V_n = \kappa_\Phi \tag{5}$$

Note that the original definition of MMC (1) is the special case of (5) when Φ is *isotropic*, i.e. $\Phi(\nu) = 1$ for all ν. In order for equation (5) to be well-posed, it is usually required that the function $\tilde{\Phi}(\mathbf{v}) = |\mathbf{v}|\Phi(\frac{\mathbf{v}}{|\mathbf{v}|})$ is *uniformly convex* on R^n. However, this condition can be greatly relaxed (see [56]).

One special feature of MMC is that the surface energy is monotonically *decreasing* in time:

$$\frac{d}{dt}\mathcal{E}_\Phi(M(t)) = -\int_{M_t} \kappa_\Phi^2 \, d\sigma.$$

In fact, the motion can be viewed as the negative gradient flow for the surface energy with respect to some suitably chosen inner product on the space of velocity vector fields defined on $M(t)$ (see (8)).

1.1 Methods of Solution

One of the main difficulties in the investigation of MMC is that singularities and topological changes can happen. Existence of local in time classical solutions are proved in [25]. See also [6, 20]. Global in time solutions are proved to exist for embedded curves in the plane [25, 28]; convex surfaces [29] and surfaces given by a graph [16]. Long time behaviors of these solutions are also investigated in these papers. In [30], the self-similarity structures of the singularities are studied. We refer to [2] for a survey on this type of questions. On the other hand, various mathematical techniques have been introduced to construct solutions which can handle the singularity formations and topological changes. Here we will mention some of these approaches which are relevant to this article. The readers are referred to [55] for other methods of solution. The papers [50, 51] also contain comprehensive references.

The first global in time solutions were constructed by Brakke [10]. He considered the space of surfaces in the setting of *varifolds*. His construction also works in the case of higher co-dimensional flows such as those of curves in space. Unfortunately, the technique so far is only applicable in the isotropic version of the mean curvature flows (1). In addition, there is a high degree of non-uniqueness in the solutions constructed by his method. Nevertheless, his notion of solutions is very often regarded as the "definition" of MMC.

Another approach is using the level set formulation. This method constructs a time dependent function $u(x,t)$ such that its zero-level set (or any c-level set) $M(t)$ defined as

$$M(t) = \{x \in R^n : u(x,t) = 0\} \quad \text{or} \quad M(t) = \{x \in R^n : u(x,t) = c\}$$

evolves by MMC. This function would then be a solution to the following equation:

$$\frac{\partial u}{\partial t} = |\nabla u| \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) \tag{6}$$

The above is a degenerate parabolic equation. The solution is interpreted in the viscosity sense. In this setting, solution for φ exists globally in time and is unique [19, 12]. However, the level sets of u can *fatten*, i.e. $M(t) = \{x \in R^n : u(x,t) = 0 \text{ or } c\}$ can have *positive* Lebesgue n-dimensional measure. When this happens, the location of the actual sharp interface cannot be determined easily. (This issue will be the main topic of **Section 3**.)

Reaction diffusion equations and singular perturbation techniques are also used to study MMC. A function φ called the *phase-ordering parameter* is introduced. It evolves according to the following *Allen-Cahn* equation [1]:

$$\frac{\partial\varphi}{\partial t} = \triangle\varphi - \frac{1}{\epsilon^2}F'(\varphi) \tag{7}$$

The function F is the double-well potential: $F(\varphi) = (1-\varphi^2)^2$ and ϵ is a small positive number. It can be shown that given reasonable initial data, φ will quickly develop a transition region of thickness ϵ which separates the domains $\{x \in R^n : \varphi(x,t) \approx 1\}$ and $\{x \in R^n : \varphi(x,t) \approx -1\}$. As $\epsilon \to 0$, it is proved in [13, 15] that the *zero-level set* of φ evolves according to MMC when the curvature flow has smooth solution. This result has been extended to the setting of viscosity and varifold solutions in [18, 50, 51] and [31]. The paper [8] describe a general technique to tackle the problem of front propagations using viscosity and singular perturbation methods.

In [6, 48], variational minimization and time discretization procedures are used to construct solutions for MMC. This method can also handle a model for dendritic crystal growth [5, 47]. The main idea is to write the evolution $\{X(t), t \geq 0\}$ as a *negative gradient flow* with respect to some energy functional $\mathcal{E}$:

$$\frac{d}{dt}X(t) = -\nabla\mathcal{E}(X(t)), \quad t \geq 0 \tag{8}$$

where $X(t)$ is a generic symbol for an element of some state space $\mathcal{X}$. The solution for (8) is constructed by time stepping: given $X(i\triangle t)$, $X((i+1)\triangle t)$ is chosen to be a *minimizer* of the following functional:

$$\mathcal{F}(X) = \mathcal{E}(X) + \frac{1}{2\triangle t}\left\|X - X(i\triangle t)\right\|^2, \quad X \in \mathcal{X} \tag{9}$$

where $\|\cdot\|$ is the norm for $\mathcal{X}$ which is compatible with the gradient ∇ in (8). Energy estimates and compactness for the function space are then proved in order to extract a convergent subsequence as $\triangle t \to 0$. The mathematical framework to handle the above minimization problem is geometric measure theory [23]. In **Sections 2.1** and **2.2**, this approach will be combined with stochastic perturbations to construct random evolutions of subsets of R^n.

2 Stochastic Curvature Driven Flows

In this section, we mention some mathematical results which can incorporate stochastic noise into curvature driven flows. This type of flows includes the previous MMC ((1) or (5)) and also some mathematical models for dendritic crystal growth.

As mentioned earlier, due to the underlying physical motivations of our equations, it is natural to consider noise which can come from the background fluctuations or the intrinsic instabilities of the deterministic evolutions. One excellent example is the dendritic pattern formation in crystal growth. There exists many models to explain the side-branching events, the shape of the tips and their speed of propagation. Noise is found to be an important factor in the process. However, its consequences have not been completely understood yet [34, 44]. Noise can also come from some microscopic models for these growth processes. It would be interesting to investigate in what form the noise survives (if at all) by going to a larger spatial scale descriptions.

A natural mathematical framework of studying stochastic evolution is by means of stochastic partial differential equations (SPDEs). There are several existence results for linear and semi-linear equations [58, 24, 14, 41]. The regularity of the solutions depends on the competition between the smoothing and roughening effects of the deterministic equations and the stochastic driving forces. However, for highly nonlinear equations such as those used in MMC and other geometric motions, not much is known.

A simple stochastic version of MMC (1) or (5) is given by

$$V_n = \kappa + \dot{W}(x,t). \tag{10}$$

Here $\dot{W}(x,t)$ is a white noise which is required to be sufficiently correlated in the spatial variables in order to give solutions in reasonable function spaces. A simpler version of (10) is the following evolution equation for a graph:

$$\frac{\partial f}{\partial t} = \sqrt{1+|\nabla f|^2}\,\mathrm{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) + \sqrt{1+|\nabla f|^2}\circ\frac{dW(x,t)}{dt} \tag{11}$$

where $\circ$ refers to the Stratonovich interpretation of stochastic calculus. In [37, 38], equations similar to the above are derived from a stochastic version of the Allen-Cahn equation (7). Equation (11) can be solved using the techniques developed by Lions and Souganidis [45, 46] (see **Section 2.3**). However, even when $W(x,t)$ is spatially constant, it seems not clear that the solution will remain smooth for all time.

There are many other phenomenological equations used to describe similar growth processes. These include the KPZ equation [33] and some higher order equations to model surface diffusion:

$$\frac{\partial f}{\partial t} = A_1\triangle f + A_2|\nabla f|^2 + \eta_1;$$

$$\frac{\partial f}{\partial t} = -A_1\triangle^2 f + A_2\triangle|\nabla f|^2 + \eta_2$$

where the surface is described by a height function f; A_1 and A_2 are positive numbers; η_1 and η_2 are some stochastic driving forces. The work [57] formally derives these equations from microscopic models. Questions of particular interests include the dynamical scaling behaviors of the interfacial structures. We refer to [7, 40] for more detail discussions and references.

Even though stochastic effects are known to exist for a long time, there are still many outstanding open problems concerning the existence and statistical properties of the solutions. In the following, we will describe some results that can give global in time solutions for some stochastic versions of MMC.

2.1 Stochastic Motion by Mean Curvature

In [59], the author constructed a random evolution of a subset $K(t)$ of R^n such that its boundary $\partial K(t)$ evolves continuously in time formally according to the following stochastic version of (5):

$$V_n(p) = \kappa_\Phi(p) + \left\langle \dot{\mathbf{W}}(p,t),\, \mathbf{n} \right\rangle, \quad p \in \partial K(t) \tag{12}$$

where $\{\dot{\mathbf{W}}(\cdot,t) : R^n \longrightarrow R^n, t \geq 0\}$ is a *white noise* vector field (to be defined later).

We consider the space of subsets of R^n in the setting of *sets of finite perimeter*:

$$\mathcal{S} = \left\{ K \subset R^n : \left| \int_K \operatorname{div}(\mathbf{G})\, d^n x \right| \leq C \sup_x \|\mathbf{G}(x)\| \right\} \tag{13}$$

where $\mathbf{G} : R^n \longrightarrow R^n$ is any C^1 vector field. The space $\mathcal{S}$ is endowed with the L^1-metric: $d(K_1, K_2) = \mathcal{L}^n(K_1 \triangle K_2)$ where $\mathcal{L}^n$ is the Lebesgue n-dimensional measure and $K_1 \triangle K_2 = (K_1 \backslash K_2) \cup (K_2 \backslash K_1)$ is the symmetric difference between two sets. For any such set K, we can define a natural notion of boundary ∂K (called the *reduced boundary*) which coincides with the classical definition when K is bounded by a smooth surface. We refer to [27, 17] for the proofs of these statements.

The solution for (12) is obtained first by performing two procedures during each discrete time interval: $[i\triangle t, (i+1)\triangle t)$ $(i = 0, 1, 2 \ldots)$ and then let $\triangle t \to 0$. At the beginning of each interval, we choose $K(i\triangle t^+)$ to be a *minimizer* of the following functional:

$$\mathcal{F}(K) = \mathcal{E}_\Phi(\partial K) + \frac{1}{2\triangle t} \int_{K \triangle K(i\triangle t^-)} \operatorname{dist}(x, \partial K(i\triangle t^-))\, d\mathcal{L}^n, \quad K \in \mathcal{S} \tag{14}$$

where $\mathcal{E}_\Phi$ is the Φ-surface energy (3) and $\operatorname{dist}(x, A) = \inf\{\|x - p\| : p \in A\}$ is the distance function from x to $A \subset R^n$. This change from $K(i\triangle t^-)$ to $K(i\triangle t^+)$ can be shown to be an approximation to the deterministic motion law $V_n = \kappa_\Phi$.

During $(i\triangle t, (i+1)\triangle t)$, we use a *white noise* vector field $\dot{\mathbf{W}}(x,t)$ to deform the set $K(i\triangle t^+)$. This vector field is *smooth* in space but *white* in time. It can be constructed by using the following representation:

$$\dot{\mathbf{W}}(x,t) = \sum_i \mathbf{F}_i(x)\, \frac{dW_i(t)}{dt} \tag{15}$$

where the $\mathbf{F}_i(x)$'s and $W_i(t)$'s are smooth vector fields and independent Wiener processes. The summation can be over a finite or countably infinite collections. From this $\dot{\mathbf{W}}(x,t)$, we can define a *two-parameters family of diffeomorphisms* of R^n: $\varphi_{s,t}(x)$ which are the solutions to the following stochastic differential equation:

$$\varphi_{s,t}(x) = x + \int_s^t \dot{\mathbf{W}}(\varphi_{s,r}(x), r)\, dr, \quad x \in R^n,\, t \geq s \geq 0. \tag{16}$$

The machinery for this construction can be found in [42]. Using this $\varphi_{s,t}(\cdot)$, we define:

$$K(t) = \varphi_{i\triangle t, t}(K(i\triangle t^+)), \quad t \in (i\triangle t, (i+1)\triangle t). \tag{17}$$

The two procedures (14) and (17) are then alternated to produce a stochastic evolution of a subset $K^{\triangle t}(t)$ of R^n for $t \geq 0$. It is shown in [59] that the probability measures $P^{\triangle t}$ induced by $\{K^{\triangle t}(t) \in \mathcal{S} : t \geq 0\}$ on some appropriate function space has a weak limit (upon taking a subsequence) as $\triangle t \to 0$. Furthermore, any limiting measure concentrates on the space $C(R^+, \mathcal{S})$ of $K(t)$ which evolves continuously in time ($t \geq 0$).

2.2 Stochastic Dendritic Crystal Growth

The previous approach of combining variational minimization and stochastic perturbation has also been used in a similar fashion to prove the existence of a solution to a model of stochastic dendritic crystal growth. The evolution describes the motion of a *crystal interface* $\partial K(t)$ coupled with an underlying stochastic heat equation. This is an example of a *free boundary value* problem as the location of the interface is one of the unknown variables.

For many solidification processes, both the curvature of the crystal interface and the diffusion of heat or chemicals are important factors in determining the overall patterns of the crystal shape. We refer to [43] for a description of various physical systems and the associated mathematical models.

In [60], the author constructed stochastic processes $\{K(t) \in \mathcal{S}, T(t) \in L^2(\mathcal{D}) : t \geq 0\}$ which describe the evolution of a crystal shape and temperature field in a domain $\mathcal{D}$. These processes satisfy the following statements:

- **Stochastic Heat Equation**:

$$d_t(c_{K(t)}T(x,t)) = \mathrm{div}(\Sigma_{K(t)} \nabla T(x,t))\, dt + c_{K(t)} f(T(x,t)) d_t W(x,t) \tag{18}$$

  where $c_{K(t)}$ and $\Sigma_{K(t)}$ are the specific heat capacity and conductivity matrix respectively. Both of them can take on different values in $K(t)$ (the solid phase) and $\mathcal{D} \backslash K(t)$ (the liquid phase). $W(x,t)$ is a *spatially correlated* Wiener process in $L^2(\mathcal{D})$. f is some function which controls the noise in regions of extreme temperature values.

- **Gibbs-Thomson Relation**:

$$\kappa_\Phi(p) = H(T(p,t)), \quad p \in \partial K(t) \tag{19}$$

  which says that the curvature at any point of the crystal interface balances with a function H of the temperature value T. For practical purposes, $H(T)$ is approximatedly equal to $-T$.

Furthermore, the heat distribution defined as $Q = c_K T$ evolves continuously in time in the *Monge-Kantorovich* norm which is a natural notion of distance for many heat transfer problems.

The Gibbs-Thomson relation (19) describes the *equilibrium* situation for the following more general *kinetic undercooling* condition:

$$V_n(p) = \mu(\kappa_\Phi(p) - H(T(p,t))), \quad p \in \partial K(t) \tag{20}$$

when μ (which is called the mobility function) tends to infinity. Not many mathematical results are known for the above motion law. In the classical setting, [11] proves the convergence of phase-field models (which are similar to the Allen-Cahn equation (7)) to various sharp interfacial limits related to (20). The work [52] has constructed a weak solution.

2.3 Stochastic Viscosity Solution

Recently, Lions and Souganidis in [45, 46] have extended the viscosity method to solve the following fully nonlinear SPDE:

$$d_t u = F(D^2 u, Du, x, t)\, dt + \sum_i H_i(Du, x, t) \circ dW_i(x, t) \tag{21}$$

where F is degenerate elliptic; H_i's are functions satisfying some smoothness assumption; $W_i(x,t)$'s are independent Wiener processes. The above equation can be used to formulate a wide range of front propagation problems in random environments. In particular, it can handle the following stochastic version of the level set equation (6):

$$d_t u = |\nabla u| \mathrm{div} \left(\frac{\nabla u}{|\nabla u|} \right) + \alpha(x) |\nabla u| \circ d_t W(t) \tag{22}$$

where $\alpha(x)$ is some space dependent function.

One of the main results in [45, 46] states that under *any reasonable smoothing* of the white noise term in (21), the corresponding approximated solutions will converge to the *same* limit. A notion of weak solution for (21) and some uniqueness results are also given.

The methods outlined in the previous sections give the first type of results which can incorporate stochastic noise into these geometric motions. However, many questions concerning the properties of the stochastic evolutions remain open. In the next section, we will tackle the issue of non-uniqueness of the deterministic solutions.

3 Noise and Uniqueness of Motion by Mean Curvature

In this section, we demonstrate the possibility of using stochastic noise to provide a *selection principle* when the deterministic equations have *non-unique* solutions. We illustrate this idea in the case of MMC. We first give some examples of non-uniqueness and then motivate the need to have a selection method.

Two Touching Circles. Given the pair of circles as in Figure 1, they can evolve according to different solutions such as (A) or (B) if the intersection point I is broken *vertically* or *horizontally*. If I is kept fixed, the solution (C) can be produced. All three evolutions satisfy the definition of MMC (1) in the varifold and viscosity formulations.

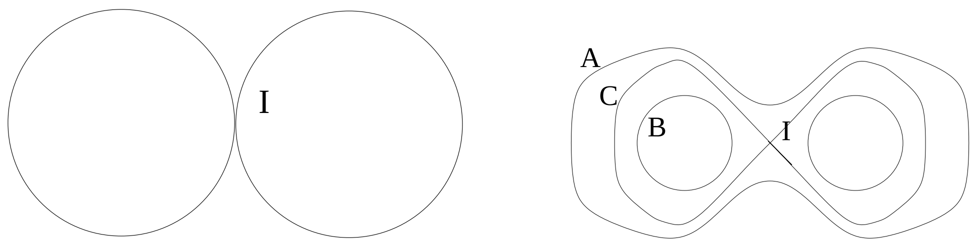

Figure 1.

We remark that both (A) and (B) are *stable* solutions while (C) is *unstable*.

Two Separating Circles [9]**.** Consider a pair of circles which are initially apart from each other (Figure 2). They evolve according to the following MMC with a driving force:

$$V_n = \kappa - g(t). \tag{23}$$

The $g(t)$ is a time dependent function which *decreases from positive to negative values*. Its exact form is chosen in such a way that the two circles will enlarge at the beginning and then touch at some time t_*. It is the time when $g(t)$ becomes zero and then negative. At this moment, the two circles will try to separate again. Multiple solutions such as (D) or (E) can now develop. Presumably it is also possible to have an unstable solution similar to (C) in Figure 1.

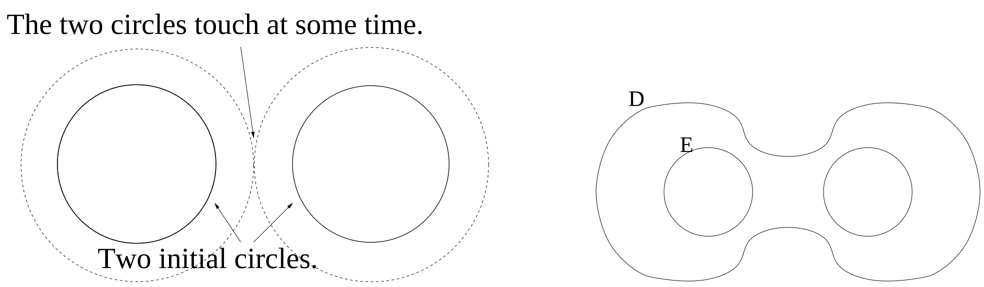

Figure 2.

The works [3, 39, 4] also contain some non-uniqueness examples in higher dimensions.

The above non-uniqueness phenomenon corresponds to the fact that the solutions of MMC *do not* depend *continuously* on the initial data (in the L^1 topology). If we approximate the initial data from the *inside* and *outside* using *smooth* curves, the approximating solutions will not stay close to each other. In the language of maximum principle, it means that there is a gap between the sub- and super-solutions. In the level set formulation, if the curves are the zero-level set of an initial data for $u(x,t)$ which solves (6), then $M(t) = \{x \in R^2 : u(x,t) = 0\}$ will instantaneously have *positive* Lebesgue two-dimensional measure for $t > 0$. It can be shown that any solution to MMC (for example, the varifold solutions in [10]) is trapped in this fattened region. However, its exact location cannot be easily determined.

This non-uniqueness issue hinders many of the mathematical statements concerning MMC. Brakke in [10] proved that his varifold solution almost everywhere is a smooth manifold. But he needs to assume that his solutions satisfy a unit density hypothesis which is not known to be true by his construction. This appears to be related to the non-uniqueness phenomenon. The works [21, 22, 32] explain this in more detail. In addition, even though the convergence of the Allen-Cahn equation (7) to MMC has been established in the viscosity and varifold sense ([18, 50, 51] and [31]), again it is not clear which sharp interfacial limit it converges to when fattening occurs. From the gradient flow point of view of MMC (8), it seems that non-uniqueness happens exactly at the critical points of the energy functional (3). It would be interesting to investigate the behavior of the solutions near these critical points, especially in such a geometric setting. Furthermore, since we have in mind the application of MMC to some physical processes such as solidifications, non-uniqueness could mean that the model is incomplete. Hence the formulation of a selection principle would have both mathematical and physical interests.

We now give two examples illustrating that the addition of stochastic noise can provide a selection principle. The results are joint work with Souganidis [53].

Consider the following stochastic version of MMC (1) and (23):

$$V_n = \kappa + \epsilon \dot{W}(t) \ \text{ and } \ V_n = \kappa - g(t) + \epsilon \dot{W}(t) \tag{24}$$

where $W(t)$ is a Wiener process and ϵ is a small parameter. We apply these motion laws to the curves in Figures 1 and 2. They lead to some random evolutions of curves $\{C_\epsilon(t) : t > 0\}$ for each $\epsilon > 0$. When we let $\epsilon \to 0$, the following selection principles are obtained.

Almost Sure Uniqueness. Starting from the initial curve in Figure 1, *almost surely* (with respect to the Wiener measure), the curve $C_\epsilon(t)$ converges to the solution (A) of the deterministic MMC (1).

Uniqueness in Probability Law. Starting from the initial curve in Figure 2, the curve $C_\epsilon(t)$ converges with probability 0.5 to solution (D) and probability 0.5 to solution (E).

The first result seems to be the strongest possible in terms of the convergence of stochastic processes. The second statement in effect has constructed a *unique probability measure on the space of solutions*. We feel that this latter formulation has wider applications.

4 Open Problems

In this section, we will discuss some open problems.

The results in the previous section on the selection principle highly rely on the specific geometries involved, especially the tangency condition of the two circles when they meet or come together. How can this be generalized to other types of fattening? In addition, as already seen in **Section 1.1**, there are several methods of constructing solutions for MMC. These solutions satisfy the following hierarchy:

$$\begin{pmatrix} \text{Smooth} \\ \text{flows} \end{pmatrix} \subseteq \begin{pmatrix} \text{Limits of} \\ \text{Allen-Cahn (7)} \end{pmatrix} \subseteq \begin{pmatrix} \text{Brakke} \\ \text{flows [10]} \end{pmatrix} \subseteq \begin{pmatrix} \text{Levet-set} \\ \text{flows (6)} \end{pmatrix}$$

The flows constructed by the variational method [6, 48] seem to lie in between the smooth flows and the Allen-Cahn limits as they deal with sets of finite perimeter. It would be interesting to understand this hierarchy more quantitatively. For example, can any Brakke flow be a limit of the Allen-Cahn equation? Can the space of all Brakke flows fill the whole fattening region in the level set formulation? The answers to these questions can further our understanding of the differences between the methods of constructing solutions for MMC.

Another question is to understand the curvature flows from a more microscopic point of view. This falls into the regime of bridging the descriptions from different length scales. Such question appears frequently in physics and in particular materials science. MMC is proved to be the hydrodynamic limit of interactive particle systems under Glauber-Kawasaki dynamics [35] and Glauber dynamics with Kac potential [36]. Usually these approaches produce deterministic evolution equations. Stochastic equations come from by considering the fluctuations. However, there is still not yet a derivation of a fully nonlinear stochastic MMC. We refer to [54, 26] for more discussions and references. In addition, what role do these microscopic descriptions play in the selection of non-unique solutions?

Furthermore, the asymptotic limits of the following stochastic Allen-Cahn equation is not completely understood yet:

$$\frac{\partial \varphi}{\partial t} = \triangle \varphi - \frac{1}{\epsilon^2}\left(F'(\varphi) + \epsilon^{\gamma}\dot{W}(x,t)\right) \tag{25}$$

where γ is some postive number. It seems that this can be handled by the method developed in [45, 46]. However, the regularity and statistical properties of the surfaces are not known. So far, all the theories require some spatial correlation for $\dot{W}(x,t)$. What would happen if it is replaced by *space-time* white noise (or some regularized version $\dot{W}^{\eta}(x,t)$ where the parameter η can depend on ϵ)? Such noise is considered in many works (see for example [37, 38]). The main difficulty in tackling this problem is that the deterministic planar standing or traveling front used in approximating curve fronts can be destroyed by the roughness of the noise. How to define the structures and locations of the stochastic fronts can be challenging. The answers to these questions can help us relate the randomness in the interfacial motions to the noise in the background environment.

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