FINITE TIME BLOW-UP OF PARABOLIC SYSTEMS WITH NONLOCAL TERMS

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Abstract. In this paper, we study the blow-up phenomena for a class of parabolic systems with nonlocal terms, called *shadow systems* which are often used to approximate reaction-diffusion systems when one of the diffusion rates is large. Existence of finite blow-up solutions are characterized based on the parameters in the shadow systems. Two different approaches are employed to overcome the difficulties caused by the appearance of nonlocal terms and the lack of comparison principles. One is based on integral estimates, while the other relies on the Schauder Fixed Point Theorem. This is a continuation of the work [13]. In particular, we improve the earlier results concerning blow-up solutions to the optimal case.

1. Introduction

Reaction-diffusion systems of the following form have been used extensively in modeling various phenomena in many branches of science

$$(1.1)\qquad \begin{cases} u_t = d_1\Delta u + f(u,v) & \text{in } \Omega\times(0,T),\\ \tau v_t = d_2\Delta v + g(u,v) & \text{in } \Omega\times(0,T),\\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega\times(0,T),\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x) & \text{in } \Omega, \end{cases}$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the usual Laplace operator, Ω is a bounded smooth domain in $\mathbb{R}^n$ with unit outward normal vector ν on its boundary $\partial\Omega$; d_1, d_2 are two positive constants representing the diffusion rates of the two substances u, v respectively, the number $\tau > 0$ is related to the response rate of v versus the change in u, and f and g are two smooth functions generally referred to as the reaction terms.

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When one of the the diffusion rates, say d_2, is very large, it seems natural to analyze (1.1) by first letting $d_2 \to \infty$ in (1.1) which formally causes $v(x,t)$ to tend to a spatially constant but time dependent function $\xi(t)$. Then the overall model can be replaced by the following which is often called the *shadow system* of the original model:

$$\begin{cases} u_t = d_1 \Delta u + f(u,\xi) & \text{in } \Omega \times (0,T), \\ \tau \xi_t = \frac{1}{|\Omega|}\int_\Omega g(u,\xi)dx & \text{in } (0,T), \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x),\ \xi(0) = \xi_0 & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $|\Omega|$ is the measure of Ω. This idea is due to Keener [10] and is typical in studying activator-inhibitor models.

The purpose of this paper is to study the finite time blow-up phenomena occurring in the shadow system of the following well-known Gierer-Meinhardt system [5, 16]:

$$\begin{cases} u_t = d_1 \Delta u - u + \frac{u^p}{v^q} & \text{in } \Omega \times (0,T), \\ \tau v_t = d_2 \Delta v - v + \frac{u^r}{v^s} & \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \geq 0,\ v(x,0) = v_0(x) > 0 & \text{in } \Omega, \end{cases} \tag{1.3}$$

where the exponents p, q, r are positive and s is nonnegative, and satisfy

$$0 < \frac{p-1}{r} < \frac{q}{s+1}. \tag{1.4}$$

The above condition is imposed so that the constant equilibrium solution $(1,1)$, which is stable in the corresponding ODE system, becomes unstable due to the introduction of the diffusion terms in (1.3) with d_1 small and d_2 large. See [16] and the references therein. This phenomenon is generally referred to as Turing's "diffusion-driven instability" [20], which is a remarkable idea in modeling pattern formation in various branches of science (e.g., biology and chemistry).

According to the formal derivation mentioned above, its shadow system is as follows:

$$\begin{cases} u_t = d_1 \Delta u - u + \frac{u^p}{\xi^q} & \text{in } \Omega \times (0,T), \\ \tau \xi_t = -\xi + \frac{1}{|\Omega|}\int_\Omega \frac{u^r}{\xi^s}dx & \text{in } (0,T), \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \geq 0,\ \xi(0) = \xi_0 > 0 & \text{in } \Omega. \end{cases} \tag{1.5}$$

There are quite a few works dedicated to the blow-up phenomena in parabolic equations and systems. See [1, 12, 19] and references therein. In most cases, comparison principles are employed to guarantee the existence of blow-up solutions, even in systems. Unfortunately, they do not hold for the Gierer-Meinhardt system (1.3) and its shadow version (1.5).

In [13], the authors compare the dynamics of shadow system (1.2) with their original reaction-diffusion systems (1.1) for the Gierer-Meinhardt model (1.3). In this case, global existence was proved in [9] for the range $\frac{p-1}{r} < 1$ and it has already been known that in the case $\frac{p-1}{r} > 1$ even for the corresponding ordinary differential equations, i.e. when u_0 and v_0 are suitable constants in (1.3), blow-up happens at finite time. (See [17] for a complete description of all solutions to the corresponding kinetic version of (1.3).) This only leaves the critical case $\frac{p-1}{r} = 1$ still open. However for the corresponding shadow system (1.5), the situation is remarkably different. To be more specific, the following results were established in [13].

Theorem A. *If $0 < \frac{p-1}{r} < \frac{2}{n+2}$, then every solution of the shadow system (1.5) exists for all time $t > 0$.*

Theorem B. *Suppose that Ω is the unit ball $B_1(0)$, and that $p = r$, $\tau = s+1-q$ and $0 < \frac{p-1}{r} < \frac{q}{s+1} < 1$. If $\frac{p-1}{r} > \frac{2}{n}$, $n \geq 3$, then (1.5) has finite time blow-up solutions for suitable choices of initial values u_0 and ξ_0.*

Recently, [18] proves that every solution of the shadow system (1.5) exists for all time $t > 0$ if $\frac{p-1}{r} = \frac{2}{n+2}$.

It is worth discussing the motivations for the current work. First, note that there is a big difference between the global existence results for the original Gierer-Meinhardt system (1.3) and its shadow version (1.5). The former holds for $\frac{p-1}{r} < 1$ while the latter blows up for $\frac{p-1}{r} > \frac{2}{n}$. The corresponding ODE (kinetic) system also has global existence for $\frac{p-1}{r} \leq 1$ [17]. This clearly reveals an intricate discrepancy between $1 \ll d_2 < \infty$ and $d_2 = \infty$. It also shows that the formal replacement of $1 \ll d_2$ by $d_2 = \infty$ needs to be justified more carefully. Second, due to technical reasons, there is an open gap $\frac{2}{n+2} < \frac{p-1}{r} < \frac{2}{n}$ in the condition imposed on the quantity $\frac{p-1}{r}$ that is not covered by **Theorems A** and **B**. This indicates that the results have not completely revealed the dynamics of the shadow system (1.5). Third, in **Theorem B**, it seems that the conditions imposed on the parameters τ, q and r might not be necessary. In fact, the authors require the conditions $p = r$, $\tau = s + 1 - q$ in order to reduce the shadow system (1.5) into a single equation with nonlocal terms. This simplification saves us from further investigation of the interaction between u and ξ in (1.5). The above encourages us to understand the dynamics of the shadow system as much as possible. The current work essentially closes the gap in the previous description in terms of finite time blow-up phenomena. The results are nearly optimal.

By setting $v = e^t u$ and $\hat{\xi} = e^{\frac{t}{\tau}}\xi$, system (1.5) becomes

$$\begin{cases} v_t = d_1 \Delta v + e^{-(p-1)t+\frac{q}{\tau}t}\frac{v^p}{\hat{\xi}^q} & \text{in } \Omega\times(0,T),\\ \tau\hat{\xi}_t = e^{-rt+\frac{s+1}{\tau}t}\frac{1}{|\Omega|}\int_\Omega \frac{v^r}{\hat{\xi}^s}dx & \text{in } (0,T),\\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega\times(0,T),\\ v(x,0) = v_0(x) \geq 0,\ \hat{\xi}(0) = \hat{\xi}_0 > 0 & \text{in } \Omega. \end{cases}$$

This is equivalent to the following form by setting $u = v$ and $\zeta = \hat{\xi}^{s+1}$:

$$\begin{cases} u_t = \Delta u + g_1(t)\zeta^{-q'}u^p & \text{in } \Omega\times(0,T),\\ \zeta_t = g_2(t)\frac{1}{|\Omega|}\int_\Omega u^r dx & \text{in } (0,T),\\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega\times(0,T),\\ u(x,0) = u_0(x) \geq 0,\ \zeta(0) = \zeta_0 > 0 & \text{in } \Omega. \end{cases} \tag{1.6}$$

where

$$g_1(t) = e^{-(p-1)t+\frac{q}{\tau}t},\quad g_2(t) = \frac{s+1}{\tau}e^{-rt+\frac{s+1}{\tau}t},\quad q' = \frac{q}{s+1}, \tag{1.7}$$

and for simplicity we have used the original symbol u for v and taken $d_1 = 1$. We remark that our results ultimately concern blow up time which is very small. Hence without loss of generality, *we will only consider the range* $[0,1]$ *for* t. Then there exist positive constants m_i, M_i, $i = 1,2$ such that

$$m_i \leq g_i(t) \leq M_i,\quad i = 1,2. \tag{1.8}$$

Note also that $\lim_{t\to 0} g_1(t) = 1$ and $\lim_{t\to 0} g_2(t) = \frac{s+1}{\tau}$.

Our main results are stated as follows. Again, we always assume $0 < \frac{p-1}{r} < \frac{q}{s+1} = q'$ and consider only radially symmetric solutions on the domain $\Omega = B_1(0)$.

Theorem 1.1. *Suppose that Ω is the unit ball $B_1(0)$. If $p < r$ and $\frac{p-1}{r} > \frac{2}{n}$, then (1.5), or equivalently* (1.6) *has a finite time blow-up solution for suitable initial data.*

Theorem 1.2. *Suppose that Ω is the unit ball $B_1(0)$. If $p \geq r$ and $\frac{p-1}{r} > \frac{2}{n+2}$, then (1.5), or equivalently* (1.6) *has a finite time blow-up solution for suitable initial data.*

Theorem 1.3. *Suppose that Ω is the unit ball $B_1(0)$. If $\frac{p-1}{r} > \frac{2}{n+2}$ and either $1 < p < \frac{n+2}{n-2}$ or $n \leq 2$, then (1.5), or equivalently* (1.6) *has a finite time blow-up solution for suitable initial data.*

Note that together with **Theorems A** and **B**, in terms of blow-up results for the shadow system, essentially the whole range of $\frac{p-1}{r}$ is covered.

From now on, we will focus on problem (1.6). Some remarks about the theorems are in place.

(1) The technique used to prove **Theorems 1.1** and **1.2** is different from that for **Theorem 1.3**. The former makes use of integral estimates and is quite elementary. The disadvantage is that the results depend very much on the exponents and spatial dimensions. However, it works even for *super-critical* p: $p \geq p^* = \dfrac{n+2}{n-2}$. For the proof of **Theorem 1.3**, we make use of some well-known machinery for the study of blow-up phenomena for semilinear heat equations, in particular the works [6, 7, 8]. The strategy is more transparent but it works only for *sub-critical* p: $p < p^*$.

(2) The main results: **Theorems 1.1**, **1.2** and **1.3** are listed based on the different techniques employed in the proofs. To see the range covered more clearly, we rewrite the main results into the following two parts:
- Assume that $\frac{p-1}{r} > \frac{2}{n}$, then finite time blow-up occurs;
- Assume that $\frac{2}{n+2} < \frac{p-1}{r} \leq \frac{2}{n}$, then finite time blow-up occurs provided that one of the following conditions is satisfied: (i) $p \geq r$; (ii) either $n \leq 2$ or $1 < p < \frac{n+2}{n-2}$, $n \geq 3$.

Therefore, one easily sees that there are still some ranges for the exponents p and r not covered by the current work, precisely,

$$\frac{2}{n+2} < \frac{p-1}{r} \leq \frac{2}{n} \quad \text{with} \quad \frac{n+2}{n-2} \leq p < r \quad \text{and} \quad n \geq 3.$$

The above essentially concerns the supercritical case in higher dimensions. We will return to it in a future paper.

Throughout the proofs, a key quantity to control is ζ. From the form of system (1.6), in order to have blow-up, it is useful to have $\zeta(t)$ *bounded from above*. If this is the case, the equation for u can be written as:

$$u_t = \Delta u + K(t) u^p \tag{1.9}$$

where the coefficient $K(t)$ is *bounded from below*. Then we can borrow some ideas and results for blow-up problems in the *single equation* case. Since

$$\zeta(t) = \zeta(0) + \frac{1}{|B_1(0)|} \int_0^t g_2(\tau) \|u\|_{L^r}^r (\tau)\, d\tau,$$

and $m_2 \leq g_2(t) \leq M_2$, we are lead to investigate the growth of the L^r-norm of u near the blow-up time. **Theorem 1.1** corresponds to the case that $\|u\|_{L^r}$ is bounded while **Theorem 1.2** corresponds to the case that $\|u\|_{L^r}$ goes to infinity but $\displaystyle\int_0^t \|u\|_{L^r}^r (\tau)\, d\tau$ remains bounded. **Theorem 1.3** essentially combines the two theorems into one (in the sub-critical case).

The proofs of our theorems make crucial use of the following two bounds:

Radial-bound:

$$|u(x,t)| \le C z^{-\frac{2}{k-1}} \quad \text{for any } 1<k<p; \tag{1.10}$$

L^∞-bound:

$$\|u\|_{L^\infty} \le C(T-t)^{-\frac{1}{p-1}}. \tag{1.11}$$

In the above, we denote $z = |x|$. Precise statements will be given in **Lemmas 2.3**, **3.1** and **Theorem 3.1**. Both of the above estimates are consistent with the classical estimates for semilinear heat equations [7, 15]. In fact, under fairly general assumption, the asymptotic blow-up profile is given by [15]:

$$\lim_{t\longrightarrow T} u(x,t) \sim C\left[\frac{|\log|x||}{|x|^2}\right]^{\frac{1}{p-1}} \qquad \text{for } |x| \ll 1.$$

Here we heuristically explain the restrictions on the exponents. Given the above bounds, we estimate $\int_0^t \|u(\tau)\|_{L^r}^r\, d\tau$. Let $z_*(t) = (T-t)^{\frac{(k-1)}{2(p-1)}}$, the point where the bounds (1.10) and (1.11) coincide. Then

$$\begin{aligned}
\|u(t)\|_{L^r}^r &\le \frac{1}{(T-t)^{\frac{r}{p-1}}}[z_*(t)]^n + \int_{z_*(t)}^1 \left[z^{-\frac{2}{k-1}}\right]^r z^{n-1}\,dz \\
&\le \frac{1}{(T-t)^{\frac{r}{p-1}}}\left[(T-t)^{\frac{(k-1)}{2(p-1)}}\right]^n + \int_{(T-t)^{\frac{k-1}{2(p-1)}}}^1 \left[z^{-\frac{2}{k-1}}\right]^r z^{n-1}\,dz \\
&\le \frac{1}{(T-t)^{\frac{2r-n(k-1)}{2(p-1)}}}.
\end{aligned}$$

Hence, if it is such that $\int_0^T \|u(\tau)\|_{L^r}^r\, d\tau < \infty$, we would need:

$$\frac{2r-n(k-1)}{2(p-1)} < 1, \quad \text{i.e.} \quad 2r < n(k-1)+2(p-1). \tag{1.12}$$

Now if

$$2r < (n+2)(p-1), \quad \text{i.e.} \quad \frac{p-1}{r} > \frac{2}{n+2},$$

then we can always find a k with $1<k<p$ such that (1.12) is satisfied. Based on this observation, we expect that near the blow-up time T, the behavior of u resembles that of the *single equation, constant coefficient* case:

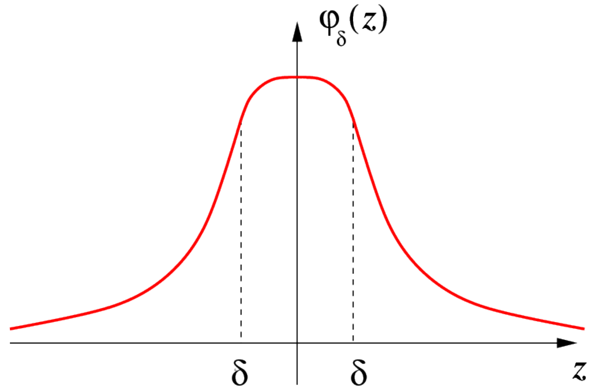
$$u_t \sim \Delta u + K(T)u^p.$$

The rigorous treatment will be provided in our proofs.

To prepare for the proofs, we introduce the initial data for u, $u_0(x) = \sigma\varphi_\delta(z)$:

$$\varphi_\delta(z) = \begin{cases} (1+\frac{\alpha}{2})\delta^{-\alpha} - \frac{\alpha}{2}\delta^{-(\alpha+2)}z^2, & 0 \le z < \delta, \\ z^{-\alpha}, & \delta \le z \le 1, \end{cases} \tag{1.13}$$

where $\alpha = \frac{2}{p-1}$ and δ and σ are positive numbers. Note that φ_δ is C^1 (see figure).



The purpose of σ is to adjust the blow-up time of the solution. The initial value ζ_0 for ζ will be chosen to be small.

We note the following elementary statements about φ_δ:

(1) For $0 < \beta < \frac{n}{\alpha}$,

$$\frac{1}{|B_1(0)|}\int_{B_1(0)} \varphi_\delta^\beta dx = \frac{O(\delta^n)}{\delta^{\alpha\beta}} + n\int_\delta^1 \frac{z^{n-1}}{z^{\alpha\beta}}\,dz = \frac{n}{n-\alpha\beta} + O(\delta^{n-\alpha\beta}). \tag{1.14}$$

(2) For $\delta \le z \le 1$,

$$(\varphi_\delta)_{zz} + \frac{n-1}{z}(\varphi_\delta)_z = \frac{-\alpha(-\alpha-1)}{z^{\alpha+2}} + \frac{(n-1)(-\alpha)}{z^{\alpha+2}} \ge -\frac{\alpha n}{z^{\alpha+2}},$$

while for $0 \le z \le \delta$,

$$(\varphi_\delta)_{zz} + \frac{n-1}{z}(\varphi_\delta)_z = -\frac{\alpha}{2\delta^{\alpha+2}}(2) + \frac{n-1}{z}\left(\frac{-\alpha}{2\delta^{\alpha+2}}2z\right) = -\frac{\alpha n}{\delta^{\alpha+2}}.$$

Hence if K_0 is large and σ is some positive constant, *bounded away from zero but independent of* δ, then it holds that

$$(\sigma\varphi_\delta)_{zz} + \frac{n-1}{z}(\sigma\varphi_\delta)_z + K_0(\sigma\varphi_\delta)^p \ge (-\sigma\alpha n + K_0\sigma^p)\min\left\{\frac{1}{\delta^{\alpha p}}, \frac{1}{z^{\alpha p}}\right\} \ge 0 \tag{1.15}$$

for $0 < z < 1$. (Note that $\alpha p = \alpha + 2$.) From the above, for $u_0 = \sigma\varphi_\delta$, we have

$$\Delta u_0 + K_0 u_0^p \ge 0. \tag{1.16}$$

Now we proceed to prove our theorems.

2. Proofs of Theorems 1.1 and 1.2 by Integral Estimates and Maximum principle

We first state two useful lemmas. The first gives some preliminary estimates of u. Its proof is the same as [13, Lemma 3.2] but is provided here for reader's convenience.

Lemma 2.1. *Let u be the solution of*

$$\begin{cases} u_t = \Delta u + K(t)u^p & \text{in } B_1(0)\times(0,T), \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_1(0)\times(0,T), \\ u(x,0)=\sigma\varphi_\delta(z) & \text{in } B_1(0), \end{cases} \tag{2.1}$$

where $K(t)$ is continuous and positive and $\sigma>0$. Then the following properties hold:

(i) $u \geq \sigma$, *for all* $0<z<1$ *and* $0<t<T$.
(ii) $u_z \leq 0$, *for all* $0<z<1$ *and* $0<t<T$.
(iii) $u^\beta(z,t) \leq \frac{1}{z^n}\frac{1}{|B_1(0)|}\int_{B_1(0)} u^\beta dx$, *for all* $0<z<1$ *and* $0<t<T$, *where* $\beta>0$.
(iv) $u_z(\frac{3}{4},t) \leq -C_0$, *for* $0<t<\min\{1,T\}$.

Proof. Statement (i) follows easily by comparing u with the solution v of the following problem

$$\begin{cases} v_t = \Delta v & \text{in } B_1(0)\times(0,T), \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial B_1(0)\times(0,T), \\ v(x,0)=\sigma & \text{in } B_1(0). \end{cases}$$

For (ii), we consider $\psi = z^{n-1}u_z$. Using the equation for u, we have

$$u_t = \frac{\psi_z}{z^{n-1}} + K(t)u^p$$

from which it is then easy to verify that ψ satisfies the following statements:

$$\begin{array}{rcll} \psi_t &=& \psi_{zz} - \frac{n-1}{z}\psi_z + pK(t)u^{p-1}\psi, & \text{in } B_1(0)\times(0,T); \\ \psi(0,t) &=& 0, & \text{on } \partial B_1(0)\times(0,T); \\ \psi(z,0) &<& 0, & \text{in } B_1(0). \end{array}$$

Then maximum principle gives (ii).

Next we compute:

$$u^\beta(z,t)z^n = u^\beta(z,t)\int_0^z nr^{n-1}\,dr \leq \int_0^z u^\beta(r,t)nr^{n-1}\,dr = \frac{1}{|B_1(0)|}\int_{B_1(0)} u^\beta(x)\,dx$$

giving (iii).

Statement (iv) follows from strong maximum principle for ψ. □

The next lemma gives some useful estimates for the blow-up time of (1.9). The proof resembles [13, p. 1774-1775].

Lemma 2.2. *Consider (1.9) with initial data $u_0(x) = \sigma\varphi_\delta(x)$. Suppose there are positive constants K_1, K_2 such that $K(t)$ satisfies $K_1 \le K(t) \le K_2$ for all $t \ge 0$. Assume $\Delta u_0(x) + \frac{K_1}{2}u_0^p(x) \ge 0$. Then the solution u will blow up at some finite time T which satisfies:*

$$\frac{C_*\delta^2}{\sigma^{p-1}K_2} \le T \le \frac{2C_*\delta^2}{\sigma^{p-1}K_1} \tag{2.2}$$

where $C_* = \left[(p-1)\left(1+\frac{\alpha}{2}\right)^{p-1}\right]^{-1}$.

Recall that by (1.16), the assumption on u_0 can be satisfied if K_1 is large enough and σ is some positive constant.

Proof. First, as u attains its maximum at $z = 0$, we have

$$u_t(0,t) \;=\; \Delta u(0,t) + K(t)u^p(0,t) \le K_2 u^p(0,t).$$

Integrating the above gives:

$$u(0,t) \le \left[\frac{1}{u^{p-1}(0,0)} - (p-1)K_2 t\right]^{-\frac{1}{p-1}} = \left[\frac{\delta^2}{\sigma^{p-1}\left(1+\frac{\alpha}{2}\right)^{p-1}} - (p-1)K_2 t\right]^{-\frac{1}{p-1}}$$

leading to the lower bound for T.

The proof of the upper bound is more involved. By the assumption on $K(\cdot)$, we have that for $t \ge 0$,

$$u_t = \Delta u + K(t)u^p \ge \Delta u + K_1 u^p.$$

Then the comparison principle gives

$$u(x,t) \ge u_*(x,t), \qquad \text{on } B_1(0) \times (0, T_1) \tag{2.3}$$

where $u_*(x,t)$ satisfies:

$$\begin{cases} u_{*t} = \Delta u_* + K_1 u_*^p & \text{in } B_1(0) \times (0,T_1), \\ \frac{\partial u_*}{\partial \nu} = 0 & \text{on } \partial B_1(0) \times (0,T_1), \\ u_*(x,0) = u_0(x) & \text{in } B_1(0), \end{cases}$$

where $T_1 \ge T$. Now set $\psi = u_{*t} - K_3 u_*^p$ (K_3 is to be determined). Differentiating in time the equation for u_* gives,

$$u_{*tt} = \Delta u_{*t} + K_1 p u_*^{p-1} u_{*t}.$$

Hence

$$\begin{aligned}\psi_t &= (u_*)_{tt} - K_3 p u_*^{p-1}(u_*)_t = \Delta u_{*t} + K_1 p u_*^{p-1} u_{*t} - K_3 p u_*^{p-1} u_{*t} \\ &= \Delta\psi + K_3\Delta(u_*^p) + K_1 p u_*^{p-1} u_{*t} - K_3 p u_*^{p-1} u_{*t} \\ &= \Delta\psi + K_3 p(p-1) u_*^{p-1} |\nabla u_*|^2 + K_3 p u_*^{p-1}\Delta u_* + K_1 p u_*^{p-1} u_{*t} - K_3 p u_*^{p-1} u_{*t} \\ &\geq \Delta\psi + K_3 p u_*^{p-1}(u_{*t} - K_1 u_*^p) + (K_1 - K_3) p u_*^{p-1} u_{*t} \\ &= \Delta\psi + p u_*^{p-1}\left[K_3 u_{*t} - K_3 K_1 u_*^p + (K_1 - K_3) u_{*t}\right] \\ &= \Delta\psi + p u_*^{p-1}\left[K_1 u_{*t} - K_3 K_1 u_*^p\right] \\ &= \Delta\psi + K_1 p u_*^{p-1}\left[u_{*t} - K_3 u_*^p\right] = \Delta\psi + K_1 p u_*^{p-1}\psi.\end{aligned}$$

Now consider

$$\psi(x,0) = u_{*t}(x,0) - K_3 u_*^p(x,0) = \Delta u_0(x) + (K_1 - K_3) u_0^p(x).$$

So upon choosing $K_3 = \frac{K_1}{2}$ and making use of the assumption, we have $\psi(x,0) \geq 0$. In addition, $\frac{\partial\psi}{\partial\nu} = 0$ on $\partial B_1(0) \times (0, T_1)$. Hence the maximum principle implies that,

$$\psi = u_{*t} - K_3 u_*^p > 0 \quad \text{in } B_1(0) \times (0, T_1).$$

Straightforward integrating (similar to the lower bound calculation) then yields

$$u_*(0,t) \geq \left[\frac{1}{u_0^{p-1}(0)} - (p-1)K_3 t\right]^{-\frac{1}{p-1}} = \left[\frac{\delta^2}{\sigma^{p-1}\left(1+\frac{\alpha}{2}\right)^{p-1}} - (p-1)K_3 t\right]^{-\frac{1}{p-1}}$$

leading to the stated upper bound for T. □

To prove **Theorems 1.1** and **1.2**, we apply the previous result by setting

$$K(t) = g_1(t)\zeta^{-q'}(t) = e^{-(p-1)t + \frac{q}{\tau}t}\zeta^{-q'}(t).$$

The following claim clearly leads to a blow-up solution of (1.6):

(2.4) *there exists a* t_1 *independent of* δ *such that* $\zeta_0 \leq \zeta(t) \leq 2\zeta_0$ *for* $0 < t < t_1$.

For the above, we are working with the understanding that $\zeta_0,\ \delta \ll 1$, and $\sigma = 1$. These will be assumed for the rest of this section.

The next lemma is one of the key ingredients in establishing (2.4). Its proof has some similarity to [2, Lemma 2.2].

Lemma 2.3. *Let* $u(x,t)$ *be the solution of (2.1). Define for* $0 < \beta \leq 1$,

$$h(t) = \frac{1}{|B_1(0)|}\int_{B_1(0)} u^\beta dx.$$

If for $0 < \delta < \delta_0$ *and* $0 < t < t(\delta)$, *there exists a* K_0 *independent of* δ *such that* $K(t) \geq K_0$ *for* $0 < t < t(\delta)$, *then for any* $1 < k < p$ *and* $\ell \geq \frac{k}{\beta}$, *there exists an* $\epsilon > 0$, *sufficiently small but independent of* δ, *such that the following estimate holds:*

$$u(z,t) \le \left[\frac{2h^\ell(t)}{\epsilon(k-1)}\right]^{\frac{1}{k-1}} z^{-\frac{2}{k-1}} \qquad \text{for } 0<z<\frac{3}{4} \text{ and } 0<t<t(\delta). \tag{2.5}$$

Proof. Define the operator

$$\mathcal{L}[\eta] = \eta_t - \eta_{zz} + \frac{n-1}{z}\eta_z - pK(t)u^{p-1}\eta,$$

and set

$$\eta = z^{n-1}u_z(z,t) + \epsilon z^n \frac{u^k(z,t)}{h^\ell(t)} = \psi(z,t) + \epsilon z^n \frac{u^k(z,t)}{h^\ell(t)}.$$

The constants ℓ, k, ϵ will be chosen later. Recall that $\mathcal{L}[\psi]=0$. Then direct computation gives that $\mathcal{L}[\eta]$ equals the following,

$$\begin{aligned}
&\mathcal{L}\left[\epsilon z^n \frac{u^k(z,t)}{h^\ell(t)}\right]\\
=\;& -2\epsilon k z^{n-1}\frac{u^{k-1}}{h^\ell}u_z - \epsilon(p-k)K(t)z^n\frac{u^{p-1+k}}{h^\ell} - \epsilon k(k-1)z^n\frac{u^{k-2}}{h^\ell}u_z^2\\
&+\epsilon\ell z^n\frac{u^k}{h^{\ell+1}}\frac{\beta(\beta-1)}{|B_1(0)|}\int_{B_1(0)}u^{\beta-2}|\nabla u|^2dx - \epsilon\ell\beta K(t)z^n\frac{u^k}{h^{\ell+1}}\frac{1}{|B_1(0)|}\int_{B_1(0)}u^{\beta+p-1}dx\\
\le\;& -2\epsilon k z^{n-1}\frac{u^{k-1}}{h^\ell}u_z - \epsilon(p-k)K(t)z^n\frac{u^{p-1+k}}{h^\ell} \qquad (\text{note: } 0<\beta\le 1)\\
=\;& -2\epsilon k\frac{u^{k-1}}{h^\ell}\eta + \frac{\epsilon z^n u^k}{h^{2\ell}}\left[2\epsilon k u^{k-1} - (p-k)K(t)h^\ell u^{p-1}\right].
\end{aligned}$$

By the facts that $u \ge 1$ (**Lemma 2.1**(i)), $h(t)\ge 1$, $1<k<p$, and $K(t)\ge K_0$ for $0<t<t(\delta)$, there exists an $\epsilon>0$, *independent of* $0<\delta\le\delta_0$, such that

$$\mathcal{L}[\eta] \le -2\epsilon k\frac{u^{k-1}}{h^\ell}\eta \quad \text{for } 0<z<\frac{3}{4} \text{ and } 0<t<t(\delta).$$

Next, note that $\eta(0,t)=0$ for $0<t<t(\delta)$. By **Lemma 2.1** and the imposed condition $\ell \ge \frac{k}{\beta}$, for ϵ small enough, we have that

$$\begin{aligned}
\eta\left(\frac{3}{4},t\right) &= \left(\frac{3}{4}\right)^{n-1}u_z\left(\frac{3}{4},t\right) + \epsilon\left(\frac{3}{4}\right)^n\frac{u^k\left(\frac{3}{4},t\right)}{h^\ell(t)}\\
&\le -C_0\left(\frac{3}{4}\right)^{n-1} + \epsilon\left(\frac{3}{4}\right)^{n-n\frac{k}{\beta}} < 0,
\end{aligned}$$

Now at $t=0$, for $0\le z<\delta$,

$$\eta(z,0) \le \left[-\alpha\delta^{-(\alpha+2)} + \epsilon\left(1+\frac{\alpha}{2}\right)^k\frac{1}{h^\ell(0)}\delta^{-\alpha k}\right]z^n,$$

while for $\delta \le z \le 1$,

$$\eta(z,0) = \left[-\alpha z^{-(\alpha+2)} + \epsilon \frac{1}{h^\ell(0)} z^{-\alpha k}\right] z^n.$$

As $\alpha + 2 = \alpha p > \alpha k$, then $\eta(z,0) < 0$ for sufficiently small ϵ.

Consequently, it follows that

$$\begin{cases} \mathcal{L}[\eta] \le -2\epsilon k \frac{u^{k-1}}{h^\ell}\eta & \text{in } 0 < z < \frac{3}{4}, 0 < t < t(\delta), \\ \eta \le 0 & \text{at } z = 0, \frac{3}{4} \text{ and } 0 < t < t(\delta), \\ \eta \le 0 & \text{in } 0 < z < \frac{3}{4} \text{ and } t = 0. \end{cases}$$

Thus for $0 < z < \frac{3}{4}$ and $0 < t < t(\delta)$, the maximum principle yields that

$$\eta = z^{n-1} u_z(z,t) + \epsilon z^n \frac{u^k(z,t)}{h^\ell(t)} \le 0.$$

Integrating in z in the above inequality gives

$$u \le \left[\frac{2h^\ell}{\epsilon(k-1)}\right]^{\frac{1}{k-1}} z^{-\frac{2}{k-1}} \quad \text{in } 0 < z < \frac{3}{4}, \;\; 0 < t < t(\delta).$$

The proof is thus complete. □

Now we are ready to prove **Theorems 1.1** and **1.2**.

2.1. **Proof of (2.4) for Theorem 1.1:** $p < r$, $\frac{p-1}{r} > \frac{2}{n}$**.** We repeat the remark that this range corresponds to the case that $\zeta_t = g_2(t)\frac{1}{|B_1(0)|}\|u(t)\|_{L^r}^r$ *is bounded.*

Since $\zeta_t = g_2(t)\dfrac{1}{|B_1(0)|}\displaystyle\int_{B_1(0)} u^r dx > 0$, we have that

$$\zeta(t) \ge \zeta_0. \tag{2.6}$$

For each $0 < \delta \le \delta_0$, let $(0, t_1(\delta))$ be the maximal time interval for which $\zeta(t) < 2\zeta_0$ holds. Define

$$\bar{u}(t) = \frac{1}{|B_1(0)|}\int_{B_1(0)} u dx.$$

Now choose $K(t) = g_1(t)\zeta^{-q'}$ and $\ell = k \in (1,p)$ in **Lemma 2.3**. Then we have $K(t) = g_1(t)\zeta^{-q'} \ge m_1(2\zeta_0)^{-q'}$ for $0 < t < t_1(\delta)$. Hence

$$u(z,t) \le \left[\frac{2\bar{u}^k(t)}{\epsilon(k-1)}\right]^{\frac{1}{k-1}} z^{-\frac{2}{k-1}}, \tag{2.7}$$

for $0 < z < \frac{3}{4}$ and $0 < t < t_1(\delta)$, with some small ϵ but independent of δ.

For $\bar{u}(t)$, by **Lemma 2.1** and (2.6), it is obvious that

$$\begin{aligned}\bar{u}_t(t) &= g_1(t)\zeta^{-q'}\frac{1}{|B_1(0)|}\int_{B_1(0)} u^p dx \\ &\leq M_1\zeta_0^{-q'}\frac{1}{|B_1(0)|}\int_{B_1(0)} u^r dx = M_1\zeta_0^{-q'}\frac{\zeta_t}{g_2(t)} \leq \frac{M_1}{m_2}\zeta_0^{-q'}\zeta_t.\end{aligned}$$

(In the above, we have used the fact $p < r$.) By (1.14), this immediately yields that for $0 < t < t_1(\delta)$,

$$\bar{u}(t) \leq \bar{u}_0 + \frac{M_1}{m_2}\zeta_0^{-q'}(\zeta(t) - \zeta_0) \leq \frac{2n}{n-\alpha} + \frac{M_1}{m_2}\zeta_0^{-q'+1}, \tag{2.8}$$

where $n - \alpha > 0$ since $p < r$ and $\frac{p-1}{r} > \frac{2}{n}$.

Going back to equation (1.6) which is satisfied by ζ, by **Lemma 2.1** and (2.7) again, we have

$$\begin{aligned}\zeta_t &= g_2(t)\frac{1}{|B_1(0)|}\int_{B_1(0)} u^r dx \\ &= g_2(t)\left\{\frac{1}{|B_1(0)|}\int_{B_{\frac{1}{2}}(0)} u^r dx + \frac{1}{|B_1(0)|}\int_{B_1(0)\setminus B_{\frac{1}{2}}(0)} u^r dx\right\} \\ &\leq M_2\left\{n\left[\frac{2}{\epsilon(k-1)}\right]^{\frac{r}{k-1}}\left(\frac{1}{2}\right)^{n-\frac{2r}{k-1}}\frac{1}{n-\frac{2r}{k-1}}\bar{u}^{\frac{kr}{k-1}} + 2^{nr}\bar{u}^r\right\},\end{aligned}$$

for $0 < t < t_1(\delta)$. In the above, $\frac{p-1}{r} > \frac{2}{n}$ is used to guarantee the existence of $k \in (1, p)$ such that $n - \frac{2r}{k-1} > 0$. Therefore, by (2.8)

$$\zeta_t \leq C_1(\epsilon, k)$$

independent of $0 < \delta \leq \delta_0$ and we easily derive that

$$\zeta_0 = \zeta(t_1(\delta)) - \zeta_0 \leq C_1(\epsilon, k)t_1(\delta),$$

that is

$$t_1(\delta) \geq \frac{\zeta_0}{C_1(\epsilon, k)}.$$

Finally, by setting

$$t_1 = \frac{\zeta_0}{C_1(\epsilon, k)},$$

we have for $0 < t < t_1$ that $\zeta(t) \leq 2\zeta_0$, concluding the proof of (2.4).

2.2. **Proof of (2.4) for Theorem 1.2:** $p \geq r$, $\frac{p-1}{r} > \frac{2}{n+2}$**.** Even though the statement is the same as that in the previous section, the key difference in terms of behavior is that $\zeta_t = g_2(t)\frac{1}{|B_1(0)|}\|u(t)\|_{L^r}^r$ *can become unbounded but the time integral* $\int_0^t \|u(s)\|_{L^r}^r\, ds$ *remains bounded.* Hence in this case we will control directly $\zeta(t)$ instead of $\zeta_t(t)$.

We have again that $\zeta(t) \geq \zeta_0$ as $\zeta(t)$ is increasing. Recall the definition:

$$h(t) = \frac{1}{|B_1(0)|}\int_{B_1(0)} u^\beta dx.$$

Here we require that $0 < \beta \leq 1$ and $\beta \in [r-p+1, \frac{n}{\alpha})$, where $\alpha = \frac{2}{p-1}$. Such β exists as $p \geq r$ and $\frac{p-1}{r} > \frac{2}{n+2}$. It is routine to check that

$$\begin{aligned} h_t(t) &= \frac{\beta}{|B_1(0)|}\int_{B_1(0)} u^{\beta-1}\left(\Delta u + g_1(t)\zeta^{-q'}u^p\right)dx \\ &= -\frac{\beta(\beta-1)}{|B_1(0)|}\int_{B_1(0)} u^{\beta-2}|\nabla u|^2 dx + g_1(t)\frac{\beta\zeta^{-q'}}{|B_1(0)|}\int_{B_1(0)} u^{\beta+p-1}dx \\ &\geq m_1\frac{\beta\zeta^{-q'}}{|B_1(0)|}\int_{B_1(0)} u^r dx, \end{aligned} \tag{2.9}$$

because of $u \geq 1$ (**Lemma 2.1**(i)) and the choice of β: $0 < \beta \leq 1$, $\beta + p - 1 \geq r$. Note that by (1.14), we get

$$h(t) \geq h(0) = \frac{1}{|B_1(0)|}\int_{B_1(0)} u_0^\beta dx = \frac{n}{n-\alpha\beta} + O(\delta^{n-\alpha\beta}), \tag{2.10}$$

where $n - \alpha\beta > 0$ is guaranteed by the choice of β.

For each $0 < \delta \leq \delta_0$, let $(0, t_2(\delta))$ be the maximal time interval for which

$$h(t) < h(0) + \frac{m_1}{M_2}\beta(2\zeta_0)^{-q'}\zeta_0, \tag{2.11}$$

and $(0, t_1(\delta))$ be the maximal time interval for which $\zeta(t) < 2\zeta_0$. We claim that $t_2(\delta) \leq t_1(\delta)$. Clearly, both $h(t)$ and $\zeta(t)$ are strictly increasing. For $0 < t < t_1(\delta)$, it follows immediately from (2.9) and the equation satisfied by ζ that

$$h_t(t) > \frac{m_1}{M_2}\beta(2\zeta_0)^{-q'}\zeta_t.$$

Thus at $t = t_1(\delta)$,

$$h(t_1(\delta)) > h(0) + \frac{m_1}{M_2}\beta(2\zeta_0)^{-q'}\zeta_0.$$

Therefore, $t_2(\delta) \leq t_1(\delta)$, i.e., for $0 < t < t_2(\delta)$

$$h(t) \leq h(0) + \frac{m_1}{M_2}\beta(2\zeta_0)^{-q'}\zeta_0 \text{ and } \zeta(t) \leq 2\zeta_0. \tag{2.12}$$

Now choose $K(t) = g_1(t)\zeta^{-q'}$ and $\ell = \frac{k}{\beta}$ in **Lemma 2.3**. The above then gives

$$u(z,t) \le \left[\frac{2h^\ell(t)}{\epsilon(k-1)}\right]^{\frac{1}{k-1}} z^{-\frac{2}{k-1}} \tag{2.13}$$

for $0 < z < \frac{3}{4}$ and $0 < t < t_2(\delta)$, where ϵ is independent of δ.

For convenience, define

$$h_{\text{in}}(t) = \frac{1}{|B_1(0)|}\int_{B_R(0)} u^\beta dx \quad \text{and} \quad h_{\text{out}}(t) = \frac{1}{|B_1(0)|}\int_{B_1(0)\setminus B_R(0)} u^\beta dx,$$

where $R > 0$ will be chosen later.

For the inside part, using (2.12) and (2.13), we get

$$\begin{aligned} h_{\text{in}}(t) &= \frac{1}{|B_1(0)|}\int_{B_R(0)} u^\beta(x,t)dx \\ &\le \left[\frac{2h^\ell}{\epsilon(k-1)}\right]^{\frac{\beta}{k-1}} \frac{1}{|B_1(0)|}\int_{B_R(0)} z^{-\frac{2\beta}{k-1}}dx \le C_2(\epsilon,k)R^{n-\frac{2\beta}{k-1}} \end{aligned} \tag{2.14}$$

for $0 < t < t_2(\delta)$. Since $\beta \in [r-p+1, \frac{n}{\alpha})$, it is possible to choose $1 < k < p$ such that $n - \frac{2\beta}{k-1} > 0$.

For the outside part, first calculate

$$\begin{aligned} \frac{d}{dt}h_{\text{out}}(t) &= \frac{d}{dt}\frac{1}{|B_1(0)|}\int_{B_1(0)\setminus B_R(0)} u^\beta(x,t)dx \\ &= \frac{\beta}{|B_1(0)|}\int_{B_1(0)\setminus B_R(0)} u^{\beta-1}\left(\Delta u + g_1(t)\zeta^{-q'}u^p\right)dx \\ &= -n\beta R^{n-1}u^{\beta-1}(R,t)u_z(R,t) - \frac{\beta(\beta-1)}{|B_1(0)|}\int_{B_1(0)\setminus B_R(0)} u^{\beta-2}|\nabla u|^2 dx \\ &\quad + g_1(t)\beta\zeta^{-q'}\frac{1}{|B_1(0)|}\int_{B_1(0)\setminus B_R(0)} u^{\beta+p-1}dx. \end{aligned}$$

Consider the equation satisfied by u in (1.6):

$$u_t = \Delta u + g_1(t)\zeta^{-q'}u^p$$

in $\frac{R}{2} \le z \le 1$, $0 < t < t_2(\delta)$. Differentiating with respect to z, by **Lemma 2.1** and (2.12), we see that u_z solves a parabolic equation with bounded coefficients in $\frac{R}{2} \le z \le 1$, $0 < t < t_2(\delta)$. Observe that at $t = 0$,

$$|u_z(z,0)| = |\varphi_\delta'(z)| = \alpha z^{-\alpha-1} \le 2^{\alpha+1}\alpha R^{-\alpha-1}$$

for $\frac{R}{2} \le z \le 1$. Therefore, by parabolic regularity, we have

$$|u_z(z,t)| \le C_3(R) \ \text{ in } \ R \le z \le 1, 0 < t < t_2(\delta)$$

which, combined with **Lemma 2.1** and (2.12), implies that

$$\left|\frac{d}{dt}h_{out}(t)\right| \leq C_4(R) \text{ in } 0 < t < t_2(\delta). \tag{2.15}$$

Due to the estimates (2.14), (2.15) and the definition of $t_2(\delta)$ in (2.11), it is easy to see that for $0 < \delta \leq \delta_0$

$$\begin{aligned}\frac{m_1}{M_2}\beta(2\zeta_0)^{-q'}\zeta_0 &= h(t_2(\delta)) - h(0) \\ &= h_{\text{in}}(t_2(\delta)) - h_{\text{in}}(0) + h_{\text{out}}(t_2(\delta)) - h_{\text{out}}(0) \\ &\leq C_2(\epsilon, k)R^{n-\frac{2\beta}{k-1}} + C_4(R)t_2(\delta).\end{aligned}$$

Choose $R > 0$ small enough such that

$$C_2(\epsilon, k)R^{n-\frac{2\beta}{k-1}} = \frac{1}{2}\frac{m_1}{M_2}\beta(2\zeta_0)^{-q'}\zeta_0,$$

which immediately tells us that

$$t_2(\delta) \geq \frac{1}{2C_4(R)}\frac{m_1}{M_2}\beta(2\zeta_0)^{-q'}\zeta_0.$$

Finally, setting

$$t_2 = \frac{1}{2C_4(R)}\frac{m_1}{M_2}\beta(2\zeta_0)^{-q'}\zeta_0,$$

by (2.12), we have for $0 < t < t_2$, $\zeta(t) \leq 2\zeta_0$. This completes the proof of (2.4).

3. Proof of Theorem 1.3 by Schauder fixed point theorem

We first recall that in this Theorem, p and r satisfy: $\frac{p-1}{r} > \frac{2}{n+2}$. To start the proof, we re-write (1.6) in the following form:

$$\begin{cases} u_t = \Delta u + \dfrac{g_1(t)}{\left[\zeta_0 + \frac{1}{|B_1(0)|}\int_0^t g_2(\tau)\|u(\cdot,\tau)\|_{L^r}^r\, d\tau\right]^{q'}}u^p & \text{in } B_1(0)\times(0,T), \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_1(0)\times(0,T), \\ u(x,0) = \sigma\varphi_\delta(z) & \text{in } B_1(0). \end{cases} \tag{3.1}$$

We will employ the Schauder Fixed Point Theorem to prove the existence of a blow-up solution. The idea is to analyze a single semilinear heat equation with time dependent coefficient:

$$u_t = \Delta u + K(t)u^p. \tag{3.2}$$

By **Lemma 2.2**, if $K(\cdot)$ is bounded from below, u will blow up at some finite time T. Now define:

$$\widetilde{K}(t) := g_1(t)\left[\zeta_0 + \frac{1}{|B_1(0)|}\int_0^t g_2(\tau)\left\|u(\cdot,\tau)\right\|_{L^r}^r\, d\tau\right]^{-q'} \quad \text{for } 0 \leq t \leq T. \tag{3.3}$$

Our strategy is to show that the map:

$$\mathcal{F} : K(\cdot) \longrightarrow \widetilde{K}(\cdot) \tag{3.4}$$

has a fixed point, leading to a blow-up solution for (3.1) or equivalently (1.6).

To implement the above, we need the continuity and compactness property of the map $\mathcal{F}$. Our method to prove them relies crucially on the two estimates (1.10) and (1.11). The former gives a global bound about the blow-up profile while the latter gives an estimate near the blow-up point. These are established using the techniques from [2] and [7]. However, due to the combination of the Neumann boundary condition and the time dependent coefficient, we need to make some non-trivial adjustment, in particular for the integral estimates in [7]. They are first proved for a truncated version of (3.2).

We now give the precise conditions on the coefficient $K(\cdot)$ which appears in (3.2).

Definition 1. *Let M and γ be two positive numbers and $a(\cdot)$ be a positive function such that $\lim_{\delta\to 0} a(\delta) = 0$. We define $\mathcal{H}_\delta(\zeta_0, T, M, a, \gamma)$ to be the collection of functions $K : [0, T] \longrightarrow \mathbb{R}_+$ satisfying the following conditions:*

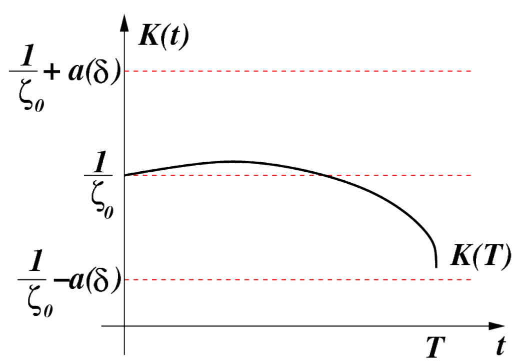
(1) $K(0) = \dfrac{1}{\zeta_0^{q'}}$;

(2) $\dfrac{1}{\zeta_0^{q'}} - a(\delta) \le K(\cdot) \le \dfrac{1}{\zeta_0^{q'}} + a(\delta)$ *for* $t \in [0, T]$;

(3) for all $0 \le t_1 \le t_2 \le T$*, it holds that*

$$|K(t_1) - K(t_2)| \le M(t_2 - t_1)^\gamma. \tag{3.5}$$

Later on, we will choose $M \gg 1$ and $0 < \gamma \ll 1$. For simplicity, we sometimes use $\mathcal{H}_\delta$ to denote $\mathcal{H}_\delta(\zeta_0, T, M, a, \gamma)$. It is clear that $\mathcal{H}_\delta$ is a bounded convex set. A caricature of a typical element of this set is depicted in the following figure:



Our approach is to treat (3.2) as a perturbation of an equation with constant coefficient:

$$\begin{aligned} u_t &= \Delta u + K(t)u^p = \Delta u + K(T)u^p + (K(t) - K(T))u^p \\ &\approx \Delta u + K(T)u^p \quad \text{for} \quad 0 < T - t \ll 1, \end{aligned}$$

where T is the blow-up time of u. In order to derive the necessary estimates of u, we introduce a modification of (3.2). Let D be some positive constant to be specified later. Define $G_D : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ to be a smooth and increasing function satisfying

$$G_D(\theta) = \begin{cases} \theta & \text{for } 0 \le \theta \le D, \\ D+1 & \text{for } \theta \ge D+2, \end{cases} \quad \text{and} \quad G(\theta) \le \theta \text{ for all } \theta. \tag{3.6}$$

Instead of (3.2), we first consider the following problem

$$\begin{cases} u_t = \Delta u + K(T)u^p + (K(t) - K(T))\frac{G_D^p((T-t)^{\frac{1}{p-1}}u)}{(T-t)^{\frac{p}{p-1}}} & \text{in } B_1(0) \times (0,T), \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_1(0) \times (0,T), \\ u(x,0) = \sigma \varphi_\delta(z) & \text{in } B_1(0). \end{cases} \tag{3.7}$$

The steps of the proof, including the determination of T and σ, are outlined in the following:

(i) Find u_1 which solves the following equation in $B_1(0)$:

$$u_t = \Delta u + \zeta_0^{-q'} u^p, \quad u(x,0) = \varphi_\delta(x), \quad \frac{\partial u}{\partial \nu}\Big|_{\partial B_1(0)} = 0. \tag{3.8}$$

Note that here the σ in the initial data u_0 is taken to be 1. By **Lemma 2.2**, for ζ_0 small enough, u_1 blows up at some finite time T. *This T is fixed for the rest of this paper.*

(ii) Take $K \in \mathcal{H}_\delta$, solve (3.7). Using the fact that $\frac{G(x)}{x} \le 1$, we have

$$\begin{aligned} &K(T)u^p + (K(t) - K(T))\frac{G_D^p((T-t)^{\frac{1}{p-1}}u)}{(T-t)^{\frac{p}{p-1}}} \\ &\quad \ge K(T)u^p - |K(t) - K(T)|\, u^p \ge (K(T) - 2a(\delta))u^p \ge (K(0) - 3a(\delta))u^p, \end{aligned}$$

and

$$\begin{aligned} &K(T)u^p + (K(t) - K(T))\frac{G_D^p((T-t)^{\frac{1}{p-1}}u)}{(T-t)^{\frac{p}{p-1}}} \\ &\quad \le K(T)u^p + |K(t) - K(T)|u^p = (K(T) + 2a(\delta))u^p \le (K(0) + 3a(\delta))u^p \end{aligned}$$

Applying **Lemma 2.2** to (3.8) with $\sigma = 1$ and $K(t) \equiv \zeta_0^{-q'}$, we have that the blow-up time $T_1 = T$ for u_1 satisfies

$$\frac{C_*\delta^2}{\zeta_0^{-q'}} \ (:= a) \le T_1 \le \frac{2C_*\delta^2}{\zeta_0^{-q'}} \ (:= b)$$

while applying to (3.7), its blow-up time T_2 satisfies ($K(0) = \zeta_0^{-q'}$):

$$\frac{C_*\delta^2}{\sigma^{p-1}(\zeta_0^{-q'} + 3a(\delta))} \ (:= c) \le T_2 \le \frac{2C_*\delta^2}{\sigma^{p-1}(\zeta_0^{-q'} - 3a(\delta))} \ (:= d).$$

By equating $c = b$ and $d = a$, we expect to find a σ in the following range:

$$\left[\left(\frac{\zeta_0^{-q'}}{2(\zeta_0^{-q'} + 3a(\delta))}\right)^{\frac{1}{p-1}}, \quad \left(\frac{2\zeta_0^{-q'}}{(\zeta_0^{-q'} - 3a(\delta))}\right)^{\frac{1}{p-1}}\right] \tag{3.9}$$

such that (3.7) also blows up at $T = T_1$. This will be shown rigorously by the next two lemmas.

As $a(\delta) \ll 1$ for $\delta \ll 1$, we can simply consider the range (3.9) for σ to be

$$\left[3^{-\frac{1}{p-1}}, \;\; 3^{\frac{1}{p-1}}\right]. \tag{3.10}$$

Hence we only need to consider bounded range of σ which is also bounded away from zero.

(iii) **Lemma 3.2** shows that for $D \gg 1$, $\delta \ll 1$, and $\sigma \in [3^{-\frac{1}{p-1}}, 3^{\frac{1}{p-1}}]$, any solution of (3.7) that exists up to time T in fact satisfies $(T-t)^{\frac{1}{p-1}} u \le D$. Hence the truncation function G_D is *not active* so that the solution actually solves (3.2).

(iv) **Lemma 3.3** gives a unique value of σ_2 such that the solution u_2 of (3.7) (or actually (3.2)) with initial data $\sigma_2 \varphi_\delta$ *blows up at* T.

With the above, we re-define the map $\mathcal{F}$ as:

$$\mathcal{F} : K \longrightarrow \widetilde{K}(t) := g_1(t)\left[\zeta_0 + \frac{1}{|B_1(0)|}\int_0^t g_2(\tau)\left\|u_2(\cdot,\tau)\right\|_{L^r}^r \, d\tau\right]^{-q'} \tag{3.11}$$

for $t \in [0, T]$. In order to apply the Schauder Fixed Point Theorem on $\mathcal{F}$, we need:

(A): $\mathcal{F} : \mathcal{H}_\delta \longrightarrow \mathcal{H}_\delta$;

(B): $\mathcal{F}$ is a *continuous* and *compact* map.

The rest of the paper is devoted to proving the above two properties.

3.1. Crucial Estimates for (3.2).

From now on, for simplicity we will choose δ small enough such that $a(\delta)(D+1) \le \frac{1}{10}$ (where $a(\cdot)$ is the function that appears in the definition of $\mathcal{H}_\delta$). The initial data ζ_0 for ζ is also chosen to be small (in order for **Lemma 2.2** to be applicable). By step (ii) in page 18, σ is taken from $[3^{-\frac{1}{p-1}}, 3^{\frac{1}{p-1}}]$. The key result of this section is the following statement concerning solutions of (3.2).

Theorem 3.1. *Given any δ small enough and any $K \in \mathcal{H}_\delta$, there exists a solution u of (3.2) that blows up at T. Furthermore, this solution satisfies the following estimates:*
(i) for any $1 < k < p$, there exists $C_{M,k}$ such that

$$u(x,t) \le C_{M,k} z^{-\frac{2}{k-1}}; \tag{3.12}$$

(ii) there exists a C_M such that

$$\left\|u(\cdot,t)\right\|_{L^\infty} \le \frac{C_M}{(T-t)^{\frac{1}{p-1}}}. \tag{3.13}$$

The above theorem follows from the next three lemmas. As we mentioned earlier, we first consider the truncated version (3.7) and then choose D large enough.

Lemma 3.1. *For any $1 < k < p$, there exist δ_0 and $C_{M,k} > 0$ such that for $0 < \delta < \delta_0$ and for all $K \in \mathcal{H}_\delta$, if the solution u of (3.7) exists up to time T^-, then it satisfies:*

$$u(x,t) \le C_{M,k} z^{-\frac{2}{k-1}}. \tag{3.14}$$

Here the constants δ_0 is independent of K and $C_{M,k}$ is independent of δ and K.

Lemma 3.2. *There exists $C_M > 0$ such that for δ small enough and for all $K \in \mathcal{H}_\delta$, if the solution u of (3.7) exists up to time T^-, then it satisfies:*

$$\|u(\cdot,t)\|_{L^\infty} \le \frac{C_M}{(T-t)^{\frac{1}{p-1}}}. \tag{3.15}$$

Here the constant C_M is independent of D, δ and K. In particular, for any $D > C_M$ and all small enough δ, the solution of (3.7) in fact solves (3.2).

Lemma 3.3. *There exists a* unique $\sigma_* \in [3^{-\frac{1}{p-1}}, 3^{\frac{1}{p-1}}]$ *such that the solution of (3.7) with initial data $u(x,0) = \sigma_*\varphi_\delta(x)$ exists for $0 < t < T$ and* blows up at T.

The proof of **Lemma 3.1** is basically the same as **Lemma 2.3** and will be given in this section. The proofs of **Lemmas 3.2** and **3.3** are quite lengthy. Their proofs rely on the techniques from [7] and will be given in **Section 4**. Properties **(A)** and **(B)** will be proved in **Sections 3.2** and **3.3**.

Proof of **Lemma 3.1**. By the condition of the function $a(\delta)$, we can choose δ small enough such that for all $K \in \mathcal{H}_\delta$,

$$K(t) \ge \frac{1}{2\zeta_0^{q'}} \quad \text{for } 0 < t < T.$$

Then by **Lemma 2.2**, for ζ_0 small enough and σ some order one positive constant (independent of ζ_0 and δ), we have

$$T \le \frac{4C_*\zeta_0^{q'}\delta^2}{\sigma^{p-1}}. \tag{3.16}$$

Also note that the properties in **Lemma 2.1** still hold for the solution of (3.7) by applying almost the same arguments.

Next, we fix

$$0 < \beta < \min\left\{1,\ \frac{n}{\alpha},\ \frac{n(k-1)}{2}\right\}. \tag{3.17}$$

Similar to **Lemma 2.3**, we define

$$h(t) = \frac{1}{|B_1(0)|}\int_{B_1(0)} u^\beta dx,$$

$$\mathcal{L}[\eta] = \eta_t - \eta_{zz} + \frac{n-1}{z}\eta_z - \hat{p}K(T)u^{p-1}\eta,$$

where $k < \hat{p} < p$ and set

$$\eta = z^{n-1}u_z(z,t) + \epsilon z^n \frac{u^k(z,t)}{h^\ell(t)} = \psi(z,t) + \epsilon z^n \frac{u^k(z,t)}{h^\ell(t)},$$

where $\ell \geq \frac{k}{\beta}$, $1 < k < p$ and ϵ will be chosen later. For convenience, we introduce

$$\mathcal{R} = (K(t) - K(T))\frac{G_D^p((T-t)^{\frac{1}{p-1}}u)}{(T-t)^{\frac{p}{p-1}}}.$$

For what follows, we make frequent use of:

$$\frac{G(x)}{x} \leq 1 \text{ for } x \geq 0, \quad \text{and} \quad |K(t) - K(T)| \leq 2a(\delta).$$

By direct computation, we get

$$\begin{aligned}
&\mathcal{L}[\eta] = \mathcal{L}\left[z^{n-1}u_z(z,t)\right] + \mathcal{L}\left[\epsilon z^n \frac{u^k(z,t)}{h^\ell(t)}\right] \\
= \ & (p-\hat{p})K(T)z^{n-1}u^{p-1}u_z + p(K(t)-K(T))\frac{G_D^{p-1}((T-t)^{\frac{1}{p-1}}u)}{T-t}G_D' z^{n-1}u_z \\
& -2\epsilon k z^{n-1}\frac{u^{k-1}}{h^\ell}u_z - \epsilon(\hat{p}-k)K(T)z^n\frac{u^{p-1+k}}{h^\ell} - \epsilon k(k-1)z^n\frac{u^{k-2}}{h^\ell}u_z^2 \\
& +\epsilon \ell z^n \frac{u^k}{h^{\ell+1}}\frac{\beta(\beta-1)}{|B_1(0)|}\int_{B_1(0)} u^{\beta-2}|\nabla u|^2 dx \\
& -\epsilon\ell\beta K(T)z^n\frac{u^k}{h^{\ell+1}}\frac{1}{|B_1(0)|}\int_{B_1(0)} u^{\beta+p-1}dx \\
& +\epsilon k z^n \frac{u^{k-1}}{h^\ell}\mathcal{R} - \epsilon\ell\beta z^n\frac{u^k}{h^{\ell+1}}\frac{1}{|B_1(0)|}\int_{B_1(0)} u^{\beta-1}\mathcal{R}dx \\
\leq \ & (p-\hat{p})K(T)z^{n-1}u^{p-1}u_z - 2pa(\delta)z^{n-1}u^{p-1}u_z \\
& -2\epsilon k z^{n-1}\frac{u^{k-1}}{h^\ell}u_z - \epsilon(\hat{p}-k)K(T)z^n\frac{u^{p-1+k}}{h^\ell} \\
& -\epsilon\ell\beta K(T)z^n\frac{u^k}{h^{\ell+1}}\frac{1}{|B_1(0)|}\int_{B_1(0)} u^{\beta+p-1}dx \\
& +2a(\delta)\epsilon k z^n\frac{u^{p+k-1}}{h^\ell} + 2a(\delta)\epsilon\ell\beta z^n\frac{u^k}{h^{\ell+1}}\frac{1}{|B_1(0)|}\int_{B_1(0)} u^{\beta+p-1}dx \\
& (\text{note that } 0 \leq G_D' \leq 1,\ u_z \leq 0,\ 0 < \beta \leq 1 \text{ and } k > 1 \text{ are used}) \\
= \ & [(p-\hat{p})K(T) - 2pa(\delta)]z^{n-1}u^{p-1}u_z \\
& -2\epsilon k z^{n-1}\frac{u^{k-1}}{h^\ell}u_z - \epsilon(\hat{p}-k)K(T)z^n\frac{u^{p-1+k}}{h^\ell} + 2a(\delta)\epsilon k z^n\frac{u^{p+k-1}}{h^\ell}
\end{aligned}$$

$$
\begin{aligned}
&+[2a(\delta)-K(T)]\epsilon\ell\beta z^n\frac{u^k}{h^{\ell+1}}\frac{1}{|B_1(0)|}\int_{B_1(0)}u^{\beta+p-1}dx\\
\leq\;& -2\epsilon kz^{n-1}\frac{u^{k-1}}{h^\ell}u_z-\epsilon(\hat p-k)K(T)z^n\frac{u^{p-1+k}}{h^\ell}+2a(\delta)\epsilon kz^n\frac{u^{p+k-1}}{h^\ell}\\
=\;& -2\epsilon k\frac{u^{k-1}}{h^\ell}\eta+\epsilon z^n\frac{u^k}{h^{2\ell}}\left[2\epsilon ku^{k-1}-(\hat p-k)K(T)h^\ell u^{p-1}+2ka(\delta)h^\ell u^{p-1}\right]\\
\leq\;& -2\epsilon k\frac{u^{k-1}}{h^\ell}\eta+\epsilon z^n\frac{u^k}{h^{2\ell}}\left[2\epsilon ku^{k-1}-\frac{1}{2}(\hat p-k)K(T)h^\ell u^{p-1}\right].
\end{aligned}
$$

Here for the last two inequalities, we need choose δ small enough such that

$$
2a(\delta)\leq\min\left\{\frac{p-\hat p}{2p\zeta_0^{q'}},\frac{1}{2\zeta_0^{q'}}\right\}\leq\min\left\{\frac{p-\hat p}{p}K(T),K(T)\right\},
$$

$$
2ka(\delta)\leq(\hat p-k)\frac{1}{4\zeta_0^{q'}}\leq\frac{1}{2}(\hat p-k)K(T).
$$

Then repeat the arguments in the proof of **Lemma 2.3**, we obtain that for $1<k<p$ and $\ell\geq\frac{k}{\beta}$, there exists $\epsilon>0$, independent of δ and $K\in\mathcal{H}_\delta$, such that

$$
u(z,t)\leq\left[\frac{2h^\ell(t)}{\epsilon(k-1)}\right]^{\frac{1}{k-1}}z^{-\frac{2}{k-1}}\quad\text{for } 0<z<\frac{3}{4},\text{ and } 0<t<T. \tag{3.18}
$$

Note that by (1.14), we get

$$
h(0)=\frac{1}{|B_1(0)|}\int_{B_1(0)}u_0^\beta dx=\frac{\sigma^\beta n}{n-\alpha\beta}+O(\delta^{n-\alpha\beta}),
$$

where $n-\alpha\beta>0$ is guaranteed by the choice of β and $\sigma\in[3^{-\frac{1}{p-1}},3^{\frac{1}{p-1}}]$.

We claim that there exists a $\delta_0>0$ such that for $0<\delta<\delta_0$, and $K\in\mathcal{H}_\delta$,

$$
h(t)\leq\frac{4^{\frac{\beta}{p-1}}n}{n-\alpha\beta}+\frac{2n}{n-\alpha\beta}\qquad\text{for } 0<t<T. \tag{3.19}
$$

Suppose that there exist $\delta^*>0$ and $K^*\in\mathcal{H}_{\delta^*}$ such that (3.19) does not hold. Let $(0,t_3(\delta^*,K^*))$ denote the maximal interval for which

$$
h(t)\leq\frac{4^{\frac{\beta}{p-1}}n}{n-\alpha\beta}+\frac{2n}{n-\alpha\beta}
$$

is satisfied.

Similar to the proof in **Section 2.2**, we decompose $h(t)$ into

$$
h_{\text{in}}(t)=\frac{1}{|B_1(0)|}\int_{B_R(0)}u^\beta dx\quad\text{and}\quad h_{\text{out}}(t)=\frac{1}{|B_1(0)|}\int_{B_1(0)\setminus B_R(0)}u^\beta dx,
$$

where $R > 0$ will be chosen later. For the inside part, by (3.18), direct computation yields that for $0 < t < t_3(\delta^*, K^*)$

$$h_{\text{in}}(t) \le \left[\frac{2h^\ell}{\epsilon(k-1)}\right]^{\frac{\beta}{k-1}} \frac{n}{n-\frac{2\beta}{k-1}} R^{n-\frac{2\beta}{k-1}} = C_1 R^{n-\frac{2\beta}{k-1}}. \tag{3.20}$$

Here $n - \frac{2\beta}{k-1} > 0$ is due to the choice of β in (3.17). For the outside part, similarly we have,

$$\left|\frac{d}{dt} h_{\text{out}}(t)\right| \le C_2(R) \ \text{ in } \ 0 < t < t_3(\delta^*, K^*). \tag{3.21}$$

Putting together the above estimates (3.20) and (3.21), we get

$$\begin{aligned} h(t_3(\delta^*, K^*)) - h(0) &= h_{\text{in}}(t_3(\delta^*, K^*)) - h_{\text{in}}(0) + h_{\text{out}}(t_3(\delta^*, K^*)) - h_{\text{out}}(0) \\ &\le C_1 R^{n-\frac{2\beta}{k-1}} + C_2(R) t_3(\delta^*, K^*). \end{aligned} \tag{3.22}$$

On the other hand side, due to the choice of $t_3(\delta^*, K^*)$, it is easy to see that for δ small,

$$h(t_3(\delta^*, K^*)) - h(0) \ge \frac{n}{n-\alpha\beta}. \tag{3.23}$$

Therefore, we can first choose R small enough such that

$$C_1 R^{n-\frac{2\beta}{k-1}} = \frac{1}{2}\frac{n}{n-\alpha\beta},$$

then (3.22) and (3.23) imply that

$$t_3(\delta^*, K^*) \ge \frac{1}{2C_2(R)} \frac{n}{n-\alpha\beta}.$$

Combined with (3.16), we have

$$\frac{4C_* \zeta_0^{q'} (\delta^*)^2}{\sigma^{p-1}} \ge T_{\delta^*}(K^*) \ge t_3(\delta^*, K^*) \ge \frac{1}{2C_2(R)} \frac{n}{n-\alpha\beta},$$

which immediately gives

$$\delta^* \ge C_3(R, p, \beta).$$

Consequently, the claim is proved by setting $\delta_0 = \frac{1}{2} C_3(R, p, \beta)$.

Clearly (3.18) together with this claim completes the proof of the **Lemma**. □

With the above and granted **Theorem 3.1**, we are ready to prove statements **(A)** and **(B)**.

3.2. **Proof of (A):** $\mathcal{F}:\mathcal{H}_\delta\longrightarrow\mathcal{H}_\delta$**.** The precise statement we will show is:

for some $M\gg 1$, $\gamma\ll 1$, there exists a δ_0 such that for all $\delta\le\delta_0$, $\mathcal{F}$ maps $\mathcal{H}_\delta$ into $\mathcal{H}_\delta$.

We recall the map $\mathcal{F}$ defined in (3.11) and that T is the blow-up time of u_2 (obtained from step (iv) in page 18). In the following, we simply denote u_2 by u. Sometimes during the derivation, we will use $\lesssim$ to indicate that the inequality is true up to a multiplicative constant C_M which is independent of δ and $K\in\mathcal{H}_\delta$.

Step I. Combining (3.12) and (3.13), we have that

$$u(r,t)\le C_M\min\left\{r^{-\frac{2}{k-1}},\ \ (T-t)^{-\frac{1}{p-1}}\right\}. \tag{3.24}$$

From the above, we claim that: *there exists a $0<\kappa=\kappa(n,p,r)<1$ and $C_M>0$ such that*

$$\|u(\cdot,t)\|_{L^r}^r\le C_M(T-t)^{-\kappa}. \tag{3.25}$$

Proof. Let $z_*(t)$ be $z_*(t)=(T-t)^{\frac{k-1}{2(p-1)}}$. Then

$$\begin{aligned}\|u(\cdot,t)\|_{L^r}^r &\lesssim \int_{|r|\le z_*(t)}\|u(\cdot,\tau)\|_{L^r}^r\,dx+\int_{|r|\ge z_*(t)}\|u(\cdot,\tau)\|_{L^r}^r\,dx\\ &\lesssim (T-t)^{-\frac{r}{p-1}}(T-t)^{n\left(\frac{k-1}{2(p-1)}\right)}+\int_{z_*(t)}^1 r^{-\frac{2r}{k-1}}r^{n-1}\,dr\\ &\lesssim (T-t)^{\frac{n(k-1)-2r}{2(p-1)}}+(T-t)^{\left(\frac{k-1}{p-1}\right)\left(\frac{n(k-1)-2r}{2(k-1)}\right)}\\ &\le C_M(T-t)^{-\kappa}\end{aligned}$$

where κ is chosen such that

$$\kappa=\frac{2r-n(k-1)}{2(p-1)}. \tag{3.26}$$

In order for κ to be less than one, we therefore need

$$\frac{2r-n(k-1)}{2(p-1)}<1,\quad\text{i.e.}\quad 2r<n(k-1)+2(p-1). \tag{3.27}$$

By the standing condition imposed on p and r in this section: $\frac{p-1}{r}>\frac{2}{n+2}$ i.e. $2r<(n+2)(p-1)$, we can always choose a $k\in(1,p)$ so that (3.27) is satisfied. □

Step II. Claim: *the value of $\displaystyle\int_0^T g_2(\tau)\,\|u(\cdot,\tau)\|_{L^r}^r\,d\tau$ can be made as small as possible by choosing δ small enough.*

Proof. We compute:

$$\begin{aligned}\int_0^T g_2(\tau)\,\|u(\cdot,\tau)\|_{L^r}^r\,d\tau &\lesssim \int_0^T (T-\tau)^{-\kappa}\,d\tau \lesssim \int_0^{\delta^2}(\delta^2-\tau)^{-\kappa}\,d\tau\\ &\lesssim \frac{1}{\delta^{2\kappa}}\int_0^1 \frac{\delta^2}{(1-\tau)^\kappa}\,d\tau \le C_M\delta^{2-2\kappa}.\end{aligned}$$

Hence the claim follows. □

Recall the definition (3.3) of $\widetilde{K}(t)$:

$$\widetilde{K}(t) = g_1(t)\left[\zeta_0 + \frac{1}{|B_1(0)|}\int_0^t g_2(\tau)\,\|u(\cdot,\tau)\|_{L^r}^r\,d\tau\right]^{-q'} \quad \text{for } t<T.$$

Note that $|g_1(t)-1| \lesssim \delta^2$ for $t \lesssim \delta^2$. Hence upon choosing $a(\delta)=C_M\delta^\eta$ for $\eta < 2-2\kappa$, we have

$$\widetilde{K}(0)-a(\delta) \le \widetilde{K}(t) \le \widetilde{K}(0)+a(\delta) \quad \text{for } 0<t<T.$$

Step III. By the form of $\widetilde{K}$, for $t_1<t_2<T$, we have:

$$\begin{aligned}
&\left|\widetilde{K}(t_1)-\widetilde{K}(t_2)\right|\\
=\;& \Bigg|g_1(t_1)\left[\zeta_0+\frac{1}{|B_1(0)|}\int_0^{t_1} g_2(\tau)\,\|u(\cdot,\tau)\|_{L^r}^r\,d\tau\right]^{-q'}\\
&-g_1(t_2)\left[\zeta_0+\frac{1}{|B_1(0)|}\int_0^{t_2} g_2(\tau)\,\|u(\cdot,\tau)\|_{L^r}^r\,d\tau\right]^{-q'}\Bigg|\\
=\;& |g_1(t_1)-g_1(t_2)|\left[\zeta_0+\frac{1}{|B_1(0)|}\int_0^{t_1} g_2(\tau)\,\|u(\cdot,\tau)\|_{L^r}^r\,d\tau\right]^{-q'}\\
&+g_1(t_2)\left|\frac{\left[\zeta_0+\frac{1}{|B_1(0)|}\int_0^{t_2} g_2(\tau)\,\|u(\cdot,\tau)\|_{L^r}^r\,d\tau\right]^{q'}-\left[\zeta_0+\frac{1}{|B_1(0)|}\int_0^{t_1} g_2(\tau)\,\|u(\cdot,\tau)\|_{L^r}^r\,d\tau\right]^{q'}}{\left[\zeta_0+\frac{1}{|B_1(0)|}\int_0^{t_1} g_2(\tau)\,\|u(\cdot,\tau)\|_{L^r}^r\,d\tau\right]^{q'}\left[\zeta_0+\frac{1}{|B_1(0)|}\int_0^{t_2} g_2(\tau)\,\|u(\cdot,\tau)\|_{L^r}^r\,d\tau\right]^{q'}}\right|\\
\le\;& \zeta_0^{-q'}\,|g_1(t_1)-g_1(t_2)|+\zeta_0^{-2q'}C_M\int_{t_1}^{t_2}\|u(\cdot,\tau)\|_{L^r}^r\,d\tau\\
\le\;& \zeta_0^{-q'}\,|g_1(t_1)-g_1(t_2)|+\zeta_0^{-2q'}C_M\int_{t_1}^{t_2}(T-t)^{-\kappa}\,dt\\
\le\;& \zeta_0^{-q'}\,|g_1(t_1)-g_1(t_2)|+\zeta_0^{-2q'}C_M\Big[(T-t_1)^{1-\kappa}-(T-t_2)^{1-\kappa}\Big]\\
\le\;& C_M(\zeta_0)(t_2-t_1)^{1-\kappa} \le C_M(\zeta_0)(t_2-t_1)^{\gamma},
\end{aligned}$$

where a newly re-defined constant $C_M(\zeta_0)$ is used and γ is chosen such that:

$$0<\gamma\leq 1-\kappa=1-\frac{2r-n(k-1)}{2(p-1)}=\frac{n(k-1)+2(p-1)-2r}{2(p-1)}. \tag{3.28}$$

Observe that,

$$\left|\widetilde{K}(t_1)-\widetilde{K}(t_2)\right|\leq C_M(\zeta_0)(t_2-t_1)^{\gamma}=C_M(\zeta_0)(t_2-t_1)^{\epsilon}(t_2-t_1)^{\gamma-\epsilon}$$

for sufficiently small $\epsilon>0$. As $t_2-t_1\leq T\leq C\delta^2\ll 1$. So $C_M(\zeta_0)(t_2-t_1)^{\epsilon}\leq C_M(\zeta_0)\delta^{2\epsilon}$. Hence, we can always choose δ small that $C_M(\zeta_0)(t_2-t_1)^{\epsilon}\leq M$, i.e. $\widetilde{K}$ satisfies (3.5).

Thus all the conditions in **Definition 1** are satisfied for $\widetilde{K}$. So **(A)** is proved.

3.3. **Proof of (B): $\mathcal{F}$ is a continuous and compact map.** We will show that:

given $\{K_n\}\subseteq\mathcal{H}_\delta$, *such that* $K_n\longrightarrow_{C^\gamma}K_*$, *then* $\widetilde{K_n}\longrightarrow_{C^\gamma}\widetilde{K_*}$.

For this, let u_n and u_* be the solutions of (3.2) (or (3.7)) with K equal to K_n and K_*. Recall that these solutions are obtained through the steps (ii), (iii) and (iv) in page 18 (after the blow-up time T is determined from step (i)). The desired convergence follows from the compactness of the function space (using the L^∞-bound (3.13) from **Theorem 3.1**) and the uniqueness of σ in **Lemma 3.3**.

Let w_n, σ_n be the rescaled solution w and the σ (in the initial condition) for u_n. Let also w_* and σ_* the corresponding variables for u_*. Using (3.13), all the w_n's are uniformly bounded. In addition, all the choices of σ_n lie in the compact set $[3^{-\frac{1}{p-1}},3^{\frac{1}{p-1}}]$ ((ii) in page 18). Hence by parabolic regularity, the w_n's will converge (up to a subsequence) to some function w'_*. Similarly, the σ_n's will converge to some σ'_*. Then it is easy to see that w'_* solves (4.2) with $K(\cdot)=K_*(\cdot)$ and $w'_*(\cdot,s_0)=\sigma'_*\Phi_\delta(\cdot)$.

Next, we will show that $\lim_{s\to\infty}w'_*(\cdot,s)=L_{K_*}$. First, by the lower-semicontinuity of the intergral functional in (4.23), **Proposition 4.2** and hence **Proposition 4.3** also hold for w'_*. Suppose that $\lim_{s\to\infty}w'_*(\cdot,s)=0$. Then for some s_*, $w'_*(\cdot,s_*)\ll 1$. By the convergence of w_n's to w'_*, this leads to that for $n\gg 1$, $w_n(\cdot,s_*)\ll 1$. Then we have that $\lim_{s\to\infty}w_n(\cdot,s)=0$. (This is similar to the argument in **Step I (Existence)** in page 34.) This then contradicts the choice of the σ_n's for w_n in (4.26).

Now, we have a solution w'_* of (4.2) such that its long time limit equals L_{K_*}. By the uniqueness of σ, we must have $\sigma'_*=\sigma_*$. Hence we have that $w_n\longrightarrow w_*$.

With the above, the required convergence follows from uniform estimates and compactness. For any $t<T-\epsilon$, we have for all n,

$$\|u_n(\cdot,t)\|_\infty\leq\frac{C_M}{(T-t)^{\frac{1}{p-1}}}\leq\frac{C_M}{\epsilon^{\frac{1}{p-1}}}.$$

This leads to $u_n\longrightarrow u_*$ and hence $\widetilde{K_n}\longrightarrow\widetilde{K_*}$ uniformly for $t\in[0,T-\epsilon]$. By the uniform Hölder continuity in t for all of the $\widetilde{K_n}$'s and $\widetilde{K_*}$, for $t\in[T-\epsilon,T]$, the following quantities

$$\left|\widetilde{K_n}(t) - \widetilde{K_n}(T-\epsilon)\right| \quad \text{and} \quad \left|\widetilde{K_*}(t) - \widetilde{K_*}(T-\epsilon)\right|$$

can also be made as small as possible. Hence $\widetilde{K_n} \longrightarrow \widetilde{K_*}$ uniformly on the whole interval $[0,T]$.

To improve the convergence of $\widetilde{K}_n$ to $\widetilde{K}_*$ in C^γ-norm, we note that the Hölder-exponent γ of the $\widetilde{K}_n$'s is determined purely by the growth rate of $\|u\|_{L^r}^r$. Hence we can choose $0 < \gamma_1 < \gamma_2$ such that both of them satisfy (3.28). Then all the $\widetilde{K}_n$'s and $\widetilde{K}_*$ belong to C^{γ_1} and C^{γ_2}. By the facts that the $\widetilde{K}_n$'s converge uniformly to $\widetilde{K}_*$, and C^{γ_2} is compactly embedded in C^{γ_1}, it follows that the convergence can be taken in C^{γ_1}. Hence it follows that $\mathcal{F}$ is a continuous map from C^{γ_1} into itself.

The compactness of the map also follows, as $\mathcal{F}$ essentially maps C^{γ_1} into C^{γ_2}.

4. Integral Estimates for the Proof of Lemmas 3.2 and 3.3

Some remarks are in place. **Lemma 3.2** is the well-known estimate for type-I blow-up for semilinear heat equation. The phenomena is quite robust. See [6, 7, 8, 11, 14, 15]. However, care needs to be observed here due to the time dependent coefficient. The imposed condition on K to be Hölder continuous in time is sufficient for our purpose. It gives that in some rescaled time variable, K converges to its limiting value (at the blow-up time) exponentially fast. (Currently we do not know how to prove the estimate for general continuous K). In addition, the results in the literature are usually established for the Dirichlet boundary condition. Hence it requires some non-trivial steps to extend the usual approach to our case. But overall, we follow the technique developed in [7]. The key idea is the use of rescaled variable and the following two results from [7] for the solutions of the rescaled equation:

(1) Proposition 2.2 (uniform space time integral estimates) leading to the L^∞ bound in Theorem 3.7;
(2) Proposition 4.3 (integral estimate for derivative) leading to the dichotomy of the large time limit in Theorem 5.1.

For completeness, we will derive all the integral estimates and also prove the corresponding L^∞-bound. Theorem 5.1 follows word by word from [7].

Consider the following change of variables for the solution u of (3.7):

$$s = -\log(T-t), \quad y = \frac{x}{\sqrt{T-t}}, \quad w(y,s) = (T-t)^{\frac{1}{p-1}}u(x,t) \tag{4.1}$$

where the solution u exists up to time T^-. Then the rescaled function w satisfies:

$$w_s = \Delta w - \frac{1}{2}y\cdot\nabla w - \frac{1}{p-1}w + K(T)w^p + b(s)G_D^p(w), \quad w(y,s_0) = \sigma\Phi_\delta(y) \tag{4.2}$$

where $\Phi_\delta(y) = T^{\frac{1}{p-1}}\varphi_\delta(x)$, $s_0 = -\log T$, and $b(s) = K(t) - K(T)$. Without loss of generality, we set $K(T) = 1$. The domain of definition for w is the following

$$\Omega(s) = \Big\{(y, s) : s \geq s_0,\ \ ye^{-\frac{s}{2}} \in \Omega\Big\} \tag{4.3}$$

which in the present paper is explicitly given by:

$$\Omega(s) = B_{\eta(s)}(0),\ \text{ with }\ \eta(s) = e^{\frac{s}{2}}. \tag{4.4}$$

Note that by **Definition 1**, for any $K \in \mathcal{H}_\delta$, the function $b(\cdot)$ satisfies:

$$|b(s)| = |K(t) - K(T)| \leq C(T-t)^\gamma = Ce^{-\gamma s}; \tag{4.5}$$

$$\lim_{s\to\infty} b(s) = 0 \quad \text{and} \quad \textstyle\int_0^\infty |b(s)|^c\, ds \leq C < \infty \ \text{ for any } c > 0; \tag{4.6}$$

$$\lim_{\delta\to 0} \|b(\cdot)\|_\infty = 0. \tag{4.7}$$

Next we will derive a series of integral estimates for w. We will mainly follow [7, Prop. 2.1, 2.2, 6.1]. We first define the following rescaled energy functional:

$$E[w] = \int_{\Omega(s)} \Big[\frac{|\nabla w|^2 + \frac{1}{p-1}w^2}{2} - \frac{w^{p+1}}{p+1}\Big]\rho\, dy, \quad \text{where } \ \rho(y) = e^{-\frac{|y|^2}{4}}. \tag{4.8}$$

The following are some simple observations useful for the later computations:

(1) Both $\|w(s_0)\|_\infty$, and $E[w](s_0)$ are order one constants, independent of δ. The first follows from:

$$w(0, s_0) = \sqrt{T}^{\frac{1}{p-1}}\varphi_\delta(0) \sim \delta^{\frac{2}{p-1}}\delta^{-\frac{2}{p-1}}$$

while the second follows from:

$$\int_{\Omega(s_0)} |\nabla w(y, s_0)|^2\, \rho\, dy = T^{\frac{2}{p-1}+1-\frac{n}{2}} \int_{B_1(0)} |\nabla_x u|^2\, e^{-\frac{|x|^2}{T}}\, dx$$

which can be easily seen not to depend on T or δ provided that either $1 < p < \frac{n+2}{n-2}$ or $n \leq 2$. Note that in both cases, we have used the fact $T = O(\delta^2)$ (**Lemma 2.2**).

(2) Let V_n be the velocity of the motion of $\partial\Omega(s)$, i.e.

$$V_n = \frac{d}{ds}\eta(s) = \frac{1}{2}e^{\frac{s}{2}}. \tag{4.9}$$

(3) The function w satisfies the Neumann boundary condition on $\partial\Omega(s)$:

$$\frac{\partial w}{\partial n}\Big|_{\partial\Omega(s)} = (T-t)^{\frac{1}{p-1}-\frac{1}{2}}\frac{\partial u}{\partial n}\Big|_{\partial B_1(0)} = 0. \tag{4.10}$$

(4) In order to handle some boundary integral terms, we will estimate w on the boundary $\partial\Omega(s)$. Note that by the radial bound (3.14), we have that $u(\cdot, t)$ is

uniformly bounded for $\left\{|x| > \frac{1}{2},\ \ t < T\right\}$. By parabolic regularity, $\nabla u(\cdot, t)$ is also uniformly bounded on $\left\{|x| > \frac{1}{2},\ \ t < T\right\}$. Hence, we have that

$$|w|\Big|_{\partial\Omega(s)} = (T-t)^{\frac{1}{p-1}}|u| \leq e^{-\frac{s}{p-1}}|u| \leq Ce^{-\frac{s}{p-1}}; \tag{4.11}$$

$$\begin{aligned} |\nabla_y w|\Big|_{\partial\Omega(s)} &= (T-t)^{\frac{1}{p-1}}\left|\nabla_x u(x,t)\right|\frac{\partial x}{\partial y} \\ &= (T-t)^{\frac{1}{p-1}+\frac{1}{2}}\left|\nabla_x u(x,t)\right| = e^{-(\frac{1}{p-1}+\frac{1}{2})s}\left|\nabla_x u(x,t)\right| \\ &\leq Ce^{-(\frac{1}{p-1}+\frac{1}{2})s}. \end{aligned} \tag{4.12}$$

Note also that

$$\rho\Big|_{\partial\Omega(s)} = e^{-\frac{|y|^2}{4}}\Big|_{\partial\Omega(s)} = e^{-\frac{e^s}{4}}, \qquad \text{and} \qquad |\partial\Omega(s)| = Ce^{\frac{n-1}{2}s}.$$

Then it follows that,

$$\left|\int_{\partial\Omega(s)} w^2\rho V_n da\right|, \quad \left|\int_{\partial\Omega(s)} |\nabla w|^2\rho V_n da\right| \leq Ce^{As-Be^s} \tag{4.13}$$

for some positive constants A, B and C.

Now we proceed to present the necessary integral estimates for w.

Estimate for $\int_{\Omega(s)} w^2\rho\, dy$.

$$\frac{d}{ds}\frac{1}{2}\int_{\Omega(s)} w^2\rho\, dy = \int_{\Omega(s)} ww_s\rho\, dy + \int_{\partial\Omega(s)}\frac{1}{2}w^2\rho V_n\, da$$

$$= \int_{\Omega(s)} w\left[\Delta w - \frac{1}{2}y\cdot\nabla w - \frac{1}{p-1}w + w^p + b(s)G_D^p(w)\right]\rho\, dy + \int_{\partial\Omega(s)}\frac{1}{2}w^2\rho V_n\, da$$

$$= \int_{\Omega(s)}\left[-|\nabla w|^2 - \frac{1}{p-1}w^2 + w^{p+1} + b(s)G_D^p(w)w\right]\rho\, dy + \int_{\partial\Omega(s)}\frac{1}{2}w^2\rho V_n\, da$$

so that

$$\begin{aligned} &\frac{d}{ds}\frac{1}{2}\int_{\Omega(s)} w^2\rho\, dy \\ &= -2E[w] + \int_{\Omega(s)}\frac{p-1}{p+1}w^{p+1}\rho\, dy + b(s)\int_{\Omega(s)} G_D^p(w)w\rho\, dy + \int_{\partial\Omega(s)}\frac{1}{2}w^2\rho V_n\, da. \end{aligned} \tag{4.14}$$

As V_n is positive quantity, the above also leads to

$$\frac{d}{ds}\frac{1}{2}\int_{\Omega(s)} w^2\rho\, dy \geq -2E[w] + \frac{1}{2}\frac{p-1}{p+1}\int_{\Omega(s)} w^{p+1}\rho\, dy, \tag{4.15}$$

where we have used that

$$|b(s)|\int_{\Omega(s)} G_D^p(w)w\rho\, dy \leq Ce^{-\gamma s}\int_{\Omega(s)} w^{p+1}\rho\, dy \leq \frac{1}{2}\frac{p-1}{p+1}\int_{\Omega(s)} w^{p+1}\rho\, dy$$

provided that $s \geq s_0$ and s_0 is large enough.

Estimate for $E[w]$.

$$
\begin{aligned}
&\frac{d}{ds}E[w](s)\\
= &\frac{d}{ds}\int_{\Omega(s)}\left[\frac{|\nabla w|^2+\frac{1}{p-1}w^2}{2}-\frac{w^{p+1}}{p+1}\right]\rho\,dy\\
= &\int_{\Omega(s)}\Big[(\nabla w\cdot\nabla w_s)+\frac{1}{p-1}ww_s-w^pw_s\Big]\rho\,dy+\int_{\partial\Omega(s)}\left[\frac{w^2}{2(p-1)}-\frac{w^{p+1}}{p+1}\right]\rho V_n\,da\\
= &\int_{\Omega(s)}w_s\Big[-\operatorname{div}(\rho\nabla w)+\frac{1}{p-1}w\rho-w^p\rho-b(s)G_D^p(w)\rho+b(s)G_D^p(w)\rho\Big]\,dy\\
&+\int_{\partial\Omega(s)}\left[\frac{w^2}{2(p-1)}-\frac{w^{p+1}}{p+1}\right]\rho V_n\,da\\
= &-\int_{\Omega(s)}w_s^2\rho\,dy+b(s)\int_{\Omega(s)}w_sG_D^p(w)\rho\,dy+\int_{\partial\Omega(s)}\left[\frac{w^2}{2(p-1)}-\frac{w^{p+1}}{p+1}\right]\rho V_n\,da
\end{aligned}
$$

so that

$$
\begin{aligned}
\frac{d}{ds}E[w] \;&\leq\; -\int_{\Omega(s)}w_s^2\rho\,dy+|b(s)|\int_{\Omega(s)}\left[\frac{1}{2}w_s^2+\frac{1}{2}G_D^{2p}(w)\right]\rho\,dy\\
&\quad+\int_{\partial\Omega(s)}\left[\frac{w^2}{2(p-1)}-\frac{w^{p+1}}{p+1}\right]\rho V_n\,da\\
&\leq\; -\frac{1}{2}\int_{\Omega(s)}w_s^2\rho\,dy+C\sqrt{|b(s)|}+Ce^{As-Be^s}. \qquad (4.16)
\end{aligned}
$$

Note that so far, all the constants do not depend on δ and the solution. Equipped with the above result, we proceed to prove uniform lower and upper bounds for the rescaled energy $E[w]$.

Lemma 4.1. *For all $s \geq s_0$, $E[w](s)$ is bounded from above and below, i.e. there are $C_1, C_2 > 0$ such that*

$$-C_2 < E[w](s) < C_1. \qquad (4.17)$$

Proof. Due to (4.16), it is clear that for $s > s_1 \geq s_0$, we have

$$E[w](s) - E[w](s_1) \leq C_0.$$

On one side, as explained at the beginning of this subsection, $E[w](s_0)$ is some order one constant, independent of δ. Thus

$$E[w](s) \leq E[w](s_0) + C_0 < C_1.$$

On the other side, if there exists $s_1^* > s_0$ such that $E[w](s_1^*) < -C_0 - 1$, then

$$E[w](s) < -1 \quad \text{for any } \; s > s_1^*.$$

This, combined with (4.15), yields that

$$\frac{d}{ds}\frac{1}{2}\int_{\Omega(s)} w^2\rho\,dy \geq 2 + \frac{1}{2}\frac{p-1}{p+1}\int_{\Omega(s)} w^{p+1}\rho\,dy \geq 2 + c\left(\int_{\Omega(s)} w^2\rho\,dy\right)^{\frac{p+1}{2}},$$

which leads to the finite time blow-up for the quantity $\int_{\Omega(s)} w^2\rho\,dy$ and hence contradicting the definition of T. The proof is complete. □

Next we show how the estimates (4.15) and (4.17) lead to the following:

Proposition 4.1. *There exists an $N_1 < \infty$, independent of δ and $K \in \mathcal{H}_\delta$, such that*

$$\int_{s_0}^{\infty}\int_{\Omega(s)} |w_s|^2\rho\,dy\,ds \leq N_1; \tag{4.18}$$

$$\int_{\Omega(s)} |w|^2\,\rho\,dy \leq N_1, \quad \text{for } s \geq s_0; \tag{4.19}$$

$$\int_{s}^{s+1}\left(\int_{\Omega(\tau)} |w|^{p+1}\,\rho\,dy\right)^2 d\tau \leq N_1, \quad \text{for } s \geq s_0. \tag{4.20}$$

Proof. ([7, Prop. 2.2, 6.1]) Estimate (4.18) follows from integrating in (4.16) in s.

For (4.19), we use (4.15) in the following way:

$$\begin{aligned}\frac{d}{ds}\frac{1}{2}\int_{\Omega(s)} w^2\rho\,dy &\geq -2E[w] + \frac{1}{2}\frac{p-1}{p+1}\int_{\Omega(s)} w^{p+1}\rho\,dy\\ &\geq -2C_1 + c\left(\int_{\Omega(s)} w^2\rho\,dy\right)^{\frac{p+1}{2}}\\ &= \left(\int_{\Omega(s)} w^2\rho\,dy\right)^{\frac{p+1}{2}}\left(c - 2C_1\left(\int_{\Omega(s)} w^2\rho\,dy\right)^{-\frac{p+1}{2}}\right).\end{aligned}$$

Now if $\int_{\Omega(s)} w^2\rho\,dy$ is bigger than $\left(\frac{4C_1}{c}\right)^{\frac{2}{p+1}}$ for some s, then $\frac{d}{ds}\int_{\Omega(s)} w^2\rho\,dy$ will be positive and hence $\int_{\Omega(s)} w^2\rho\,dy$ will keep increasing leading to

$$\frac{d}{ds}\frac{1}{2}\int_{\Omega(s)} w^2\rho\,dy \geq \frac{c}{2}\left(\int_{\Omega(s)} w^2\rho\,dy\right)^{\frac{p+1}{2}}.$$

The above will again lead to finite time blow-up for the quantity $\int_{\Omega(s)} w^2\rho\,dy$. Thus the estimate (4.19) follows.

Statement (4.20) can be derived from (4.15) and **Lemma 4.1** as follows:

$$\begin{aligned}\frac{1}{2}\frac{p-1}{p+1}\int_{\Omega(s)} w^{p+1}\rho\,dy &\leq 2E[w]+\frac{d}{ds}\int_{\Omega(s)}\frac{1}{2}w^2\rho\,dy\\ &\leq 2C_1+\int_{\Omega(s)} ww_s\rho\,dy+\int_{\partial\Omega(s)}\frac{1}{2}w^2\rho V_n\,da\end{aligned}$$

so that

$$\left(\int_{\Omega(s)} w^{p+1}\rho\,dy\right)^2 \leq C+C\left(\int_{\Omega(s)} w^2\rho\,dy\right)\left(\int_{\Omega(s)} w_s^2\rho\,dy\right)+Ce^{2As-2Be^s}.$$

By (4.18) and (4.19) just proved, integrating in s gives the result. □

With the above estimates, we now proceed to the

Proof of **Lemma 3.2**. The strategy follows mostly from the proof of [7, Theorem 3.7]. Suppose there exist $D_i \longrightarrow \infty$, $\delta_i \longrightarrow 0$, $K_i \in \mathcal{H}_{\delta_i}$ such that the solution u_i of (3.7) (which exists up to T^-) satisfies $(T-t_i)^{\frac{1}{p-1}}u_i(0,t_i) = D_i$ at some first time $t_i < T$. (Note that each of the u_i's is radially symmetric and decreasing in $|x|$.) We will derive a contradiction in the following manner.

Consider the rescaled function w_i which solves (4.2). Then $S_i = -\log(T-t_i)$ is the first time such that $w_i(0,S_i) = D_i$. Note that for all i, $S_i - s_0 \geq C$ for some positive constant C. This is because w_i solves (4.2) and

$$w_i(0,s_0) = \sigma\varphi_{\delta_i}(0)T^{\frac{1}{p-1}} = \frac{\sigma}{\delta_i^{\frac{2}{p-1}}}(\delta_i^2)^{\frac{1}{p-1}} \leq O(1)$$

so that it takes at least $O(1)$ time for $w_i(0,s)$ to reach the value $D_i \gg 1$. Hence the domain of definition of w_i contains the cylinder

$$Q_i = \left\{(y,s) : |y| \leq \eta, -\eta^2 < s - S_i \leq 0\right\}$$

for some fixed $\eta \ll 1$.

Now consider the following rescaled functions;

$$v_i(z,\tau) = \lambda_i^{\frac{2}{p-1}} w_i(\lambda_i z, \lambda_i^2\tau + S_i) \tag{4.21}$$

where $\lambda_i^{\frac{2}{p-1}} D_i = 1$ and hence $\lambda_i \longrightarrow 0$. Note the following statements:

(1) Each of v_i is defined on the rescaled cylinder $\widetilde{Q}(\frac{\eta}{\lambda_i})$ where

$$\widetilde{Q}(r) = \left\{(z,\tau) : |z| \leq r, -r^2 < \tau \leq 0\right\}.$$

(2) We have that $0 \leq v_i \leq 1$ on $\widetilde{Q}(\frac{\eta}{\lambda_i})$ and $v_i(0,0) = 1$.

(3) For each i, the function v_i satisfies:

$$(v_i)_\tau - \Delta v_i - v_i^p - b_i(s)\lambda_i^{\frac{2p}{p-1}} G_D^p\left(\frac{v_i}{\lambda_i^{\frac{2}{p-1}}}\right) = -\frac{1}{2}\lambda_i^2\left(z\cdot\nabla v_i + \frac{2}{p-1}v_i\right). \tag{4.22}$$

The last term of the left hand side can be handled as follows:

$$|b_i(s)|\lambda_i^{\frac{2p}{p-1}} G_D^p\left(\frac{v_i}{\lambda_i^{\frac{2}{p-1}}}\right) = |b_i(s)| v_i^p\left[\frac{\lambda_i^{\frac{2}{p-1}}}{v_i} G_D\left(\frac{v_i}{\lambda_i^{\frac{2}{p-1}}}\right)\right]^p \leq 2a(\delta_i) \longrightarrow 0.$$

(Note that $\frac{G_D(x)}{x} \leq 1$ and recall the choice of δ_i.)

(4) The estimates (4.18), (4.19) and (4.20) remains true for $0 \leq s \leq S_i$.

Arguing as in [7, p. 18-19], we have that as $i \longrightarrow \infty$, v_i converges to a function v defined on $\mathbb{R}^n \times (-\infty, 0)$ satisfying

$$v_\tau = \Delta v + v^p, \quad 0 \leq v \leq 1, \quad v(0,0) = 1.$$

Furthermore,

$$\iint_{\widetilde{Q}(\frac{\eta}{\lambda_i})} |(v_i)_\tau|^2 \, dz\, d\tau \leq \lambda_i^c \int_{s_0}^{S_i}\int_{Q_i} |(w_i)_s|^2 \, dy\, ds$$

with $c = -n + 2 + \frac{4}{p-1} > 0$ since either $1 < p < \frac{n+2}{n-2}$ or $n \leq 2$. On the other hand, for $\eta \ll 1$, we have

$$\int_{s_0}^{S_i}\int_{Q_i} |(w_i)_s|^2 \, dy\, ds \leq C(\eta)\int_{s_0}^{S_i}\int_{Q_i} |(w_i)_s|^2 \rho\, dy\, ds \leq N \ \text{ (independent of } i).$$

Hence it holds that as $i \longrightarrow \infty$,

$$\iint_{\widetilde{Q}(\frac{\eta}{\lambda_i})} |(v_i)_\tau|^2 \, dz\, d\tau \longrightarrow 0.$$

This leads to a nonnegative solution v of

$$\Delta v + v^p = 0, \quad \text{with } v(0) = 1,$$

contradicting the non-existence result of [3, 4]. □

Next we state the following estimate and its consequence which are crucial to the proof of **Lemma 3.3**.

Proposition 4.2. *There exists an $N_2 < \infty$, independent of δ and $K \in \mathcal{H}_\delta$, such that*

$$\int_{s_0}^{\infty}\int_{\Omega(s)} (|w_s|^2 + |\nabla w|^2)(1 + |y|^2)\rho\, dy\, ds \leq N_2. \tag{4.23}$$

Proposition 4.3. *Let $w \in L^\infty(y \in \mathbb{R}^n, s \geq s_0)$. Let w be the solution of (4.2) which exists (i.e. is defined) for all $s \geq s_0$. Then for $\delta \ll 1$, we have*

$$\lim_{s\to\infty} w(y,s) = 0, \quad \text{or } L_K := [K(T)(p-1)]^{-\frac{1}{p-1}}, \tag{4.24}$$

where the limit holds in compact sets of $y \in \mathbb{R}^n$.

The proofs of the above are basically the same as [7, Prop. 4.3] and [7, Theorem 5.1]. For completeness, we prove (4.23) in the appendix. Once we have this, as in (4.2), $b(s)G_D^p(w) \leq w^p$ and $\lim_{s\to\infty} b(s) = 0$, the proof of [7, Theorem 5.1] is applicable basically word by word. Hence we omit the detailed proof of **Proposition 4.3**.

Now we are ready to prove the final **Lemma 3.3**. To emphasize the parameter σ, we denote the solution of (4.2) as w_σ. The desired result is clearly equivalent to the following statement:

There exists a solution of (4.2) with initial data $\sigma\Phi_\delta$ such that

$$\lim_{s\to\infty} w_\sigma(y,s) = L_K. \tag{4.25}$$

The above limit holds in compact sets of y. The choice of σ is unique.

Proof. The key is to choose an appropriate value of σ. For simplicity again, we set $K(T) = 1$.

Step I (Existence). First, for σ small enough, the solution w_σ will tend to zero as $s \longrightarrow \infty$. This is because the $-\frac{1}{p-1}w$ term will dominate w^p and $b(s)G_D^p(w)$: w^p, $b(s)G_D^p(w) \ll w$ at $s = s_0$. Hence by using a spatially constant function as super-solution, it follows that w will converge to zero (exponentially fast).

Second, define:

$$\sigma_* = \sup\left\{\sigma \in [3^{-\frac{1}{p-1}}, 3^{\frac{1}{p-1}}] : \text{ the solution } w_\sigma \text{ of (4.2) satisfies } \lim_{s\to\infty} w(y,s) = 0.\right\} \tag{4.26}$$

We remark that the choice of the range of σ is from step (ii) in page 18. Furthermore, for any $\sigma < \sigma_*$, the solution w_σ exists for all time. By the L^∞-bound (3.13) for w_σ, the following limit exists for all s in the respective domain:

$$w_{\sigma_*}(y,s) = \lim_{\sigma\to\sigma_*} w_\sigma(y,s). \tag{4.27}$$

With the above, we claim that $w_{\sigma_*}(y,s) \longrightarrow L_K$ as $s \longrightarrow \infty$. By **Proposition 4.3**, $\lim_{s\to\infty} w_{\sigma_*}(y,s)$ equals 0 or L_K. If it is zero, we can increase the value of σ_* and still have a solution w that tends to zero, contradicting the definition of σ_*. (Note that as our function is monotonically decreasing in $|y|$, we only need to consider the convergence at $y = 0$.)

Step II (Uniqueness). Suppose there are two values σ_1 and σ_2 satisfying the conclusion of the Lemma. Then by comparison principle, *all* w_σ with $\sigma \in [\sigma_1, \sigma_2]$

has the property also. Differentiating (4.2) with respect to σ and letting $\eta(y,s) = \frac{\partial}{\partial\sigma} w_\sigma(y,s)$ lead to:

$$\eta_s = \Delta\eta - \frac{1}{2} y \cdot \nabla\eta - \frac{1}{p-1}\eta + p w_\sigma^{p-1}\eta + b(s) p G_D^{p-1}(w_\sigma) G_D'(w_\sigma)\eta, \tag{4.28}$$

$$\eta(y,s_0) = \Phi_\delta(y) \ \ (>0). \tag{4.29}$$

Note that for $\sigma \in [\sigma_1,\sigma_2]$, w_σ is a monotonic increasing function of σ. Then we must have $\eta(y,s) > 0$ for all $y \in \Omega(s)$ and $s \geq s_0$.

Now we rewrite (4.28) as:

$$\eta_s = \frac{1}{\rho}\nabla\cdot(\rho\nabla\eta) + V(y,s)\eta \tag{4.30}$$

where $V(y,s) = p w_\sigma^{p-1}(y,s) + b(s) p G_D^{p-1}(w_\sigma) G_D'(w_\sigma) - \frac{1}{p-1}$. Using the convergence property of w: $w \longrightarrow L_K$ as $s \longrightarrow \infty$, we have that

$$\lim_{s\to\infty} V(y,s) = p L_K^{p-1} - \frac{1}{p-1} = 1, \quad \text{on compact } y\text{'s}.$$

(Note that $L_K = (p-1)^{-\frac{1}{p-1}}$.)

(The heuristic argument for what follows is that equation (4.30) for η as $s \longrightarrow \infty$ roughly becomes $\eta_s \approx \frac{1}{\rho}\nabla\cdot(\rho\nabla\eta) + \eta$ which has $\lambda = 1$ and $\eta \equiv 1$ as a positive eigenvalue and eigenfunction for the operator on the right hand side. Then we can use $\underline{\eta}(y,s) = \epsilon e^s$ (with $\epsilon \ll 1$) as a subsolution so that $\eta(y,s)$ will be *uniformly positive* for all s and y in any compact sets. This then contradicts the fact that $\lim_{s\to\infty} w_{\sigma_1}(y,s) - \lim_{s\to\infty} w_{\sigma_2}(y,s) = L_K - L_K = 0$. The rigorous proof follows.)

Let $B_R = \{y : |y| \leq R\}$ be the ball of radius R centered at the origin. Consider the following eigenvalue problem:

$$\frac{1}{\rho}\nabla\cdot(\rho\nabla\varphi) + \frac{1}{2}\varphi \;=\; \lambda\varphi, \quad y \in B_R(y), \tag{4.31}$$

$$\varphi \;=\; 0, \quad y \in \partial B_R(y). \tag{4.32}$$

The *principal (maximum)* eigenvalue λ_p of the above is given by the following Rayleigh quotient formulation:

$$\lambda_p = -\inf_{\varphi} \frac{\int_{B_R}\left(|\nabla\varphi|^2 - \frac{1}{2}\varphi^2\right)\rho\, dy}{\int_{B_R}\varphi^2\rho\, dy}.$$

For $R \gg 1$, by using a smooth test function ψ of the form $\psi(y) \equiv 1$ for $|y| \leq R-1$ and $\psi(y) = 0$ on $|y| = R$, we have that $\lambda_p > 0$. Let $\varphi_p(y)$ be the corresponding principal eigenfunction which is *automatically positive.*

Now for s large enough, on B_R, we have:

$$\eta_s = \frac{1}{\rho}\nabla\cdot(\rho\nabla\eta) + V(y,s)\eta \geq \frac{1}{\rho}\nabla\cdot(\rho\nabla\eta) + \frac{1}{2}\eta.$$

Then we can use:

$$\underline{\eta}(y,s) = \begin{cases} \epsilon e^{\lambda_p s}\varphi_p(y), & |y| \le R \\ 0, & |y| \ge R \end{cases}$$

with $\epsilon \ll 1$ as a *subsolution* for η. This leads to that $\eta(y,s)$ is *uniformly positive* for all $s > 0$ in compact y's. Hence

$$0 = L_K - L_K = \lim_{s\to\infty} \big(w_{\sigma_2}(y,s) - w_{\sigma_1}(y,s)\big) = \lim_{s\to\infty} \int_{\sigma_1}^{\sigma_2} \eta(y,s)\,d\sigma > 0$$

which is clearly a contradiction. Hence it is not possible that $\sigma_1 < \sigma_2$. □

Appendix A. Proof of Proposition 4.2

As a preparation, we would need the following weighted energy

$$E_2[w] = \int_{\Omega(s)} \Big[\frac{|\nabla w|^2 + \frac{1}{p-1}w^2}{2} - \frac{w^{p+1}}{p+1}\Big]\rho|y|^2\,dy.$$

This is the analogue of $E[w]$ with ρ replaced by $\rho|y|^2$.

First, we prepare a series of necessary integral estimates for w. For the following, we freely make use of the fact that the boundary integrals are exponentially small of the form $O(e^{As-Be^s})$.

Estimate for $\int_{\Omega(s)} w^2\rho|y|^2\,dy$**.**

$$\begin{aligned}
&\frac{d}{ds}\int_{\Omega(s)} \frac{1}{2}w^2\rho|y|^2\,dy \\
= &\int_{\Omega(s)} w\left[\Delta w - \frac{1}{2}y\cdot\nabla w - \frac{1}{p-1}w + w^p + b(s)G_D^p(w)\right]\rho|y|^2\,dy \\
&+\int_{\partial\Omega(s)} \frac{1}{2}w^2\rho|y|^2 V_n\,da \\
= &\int_{\Omega(s)} \left[-|\nabla w|^2 - \frac{1}{p-1}w^2 + w^{p+1}\right]\rho|y|^2\,dy - \int_{\Omega(s)} w(\nabla(|y|^2)\cdot\nabla w)\rho\,dy \\
&+b(s)\int_{\Omega(s)} G_D^p(w)w\rho|y|^2\,dy + \int_{\partial\Omega(s)} \frac{1}{2}w^2\rho|y|^2 V_n\,da \\
= &-2E_2[w] + \int_{\Omega(s)} \frac{p-1}{p+1}w^{p+1}\rho|y|^2\,dy + \int_{\Omega(s)}\left(n - \frac{1}{2}|y|^2\right)w^2\rho\,dy \\
&+b(s)\int_{\Omega(s)} G_D^p(w)w\rho|y|^2\,dy + \int_{\partial\Omega(s)}\left(\frac{1}{2}|y|^2V_n - |y|\right)w^2\rho\,da
\end{aligned}$$

leading to

$$\frac{d}{ds}\int_{\Omega(s)} \frac{1}{2}w^2\rho|y|^2\,dy$$

$$
\begin{aligned}
&= -2E_2[w] + \frac{p-1}{p+1}\int_{\Omega(s)} w^{p+1}\rho|y|^2\,dy - \int_{\Omega(s)}\left(\frac{|y|^2}{2}-n\right)w^2\rho\,dy \\
&\quad +b(s)\int_{\Omega(s)} G_D^p(w)w\rho|y|^2\,dy + O(e^{As-Be^s}).
\end{aligned} \tag{A.1}
$$

Estimate for $\int_{\Omega(s)} w_s^2\rho|y|^2\,dy$.

$$
\begin{aligned}
&\frac{d}{ds}E_2[w](s) \\
=\ & \frac{d}{ds}\int_{\Omega(s)}\left[\frac{|\nabla w|^2+\frac{1}{p-1}w^2}{2}-\frac{w^{p+1}}{p+1}\right]\rho|y|^2\,dy \\
=\ & \int_{\Omega(s)}\left(\nabla w\cdot\nabla w_s+\frac{1}{p-1}ww_s-w^pw_s\right)\rho|y|^2\,dy \\
& +\int_{\partial\Omega(s)}\left[\frac{w^2}{2(p-1)}-\frac{w^{p+1}}{p+1}\right]\rho|y|^2V_n\,da \\
=\ & \int_{\Omega(s)} w_s\Big[-\operatorname{div}(\rho\nabla w)+\frac{1}{p-1}w\rho-w^p\rho-b(s)G_D^p(w)\rho+b(s)G_D^p(w)\rho\Big]|y|^2\,dy \\
& -2\int_{\Omega(s)}(y\cdot\nabla w)w_s\rho\,dy+\int_{\partial\Omega(s)}\left[\frac{w^2}{2(p-1)}-\frac{w^{p+1}}{p+1}\right]\rho|y|^2V_n\,da \\
=\ & -\int_{\Omega(s)} w_s^2\rho|y|^2\,dy-2\int_{\Omega(s)}(y\cdot\nabla w)w_s\rho\,dy \\
& +b(s)\int_{\Omega(s)} w_sG_D^p(w)\rho|y|^2\,dy+\int_{\partial\Omega(s)}\left[\frac{w^2}{2(p-1)}-\frac{w^{p+1}}{p+1}\right]\rho|y|^2V_n\,da
\end{aligned}
$$

leading to

$$
\begin{aligned}
&\frac{d}{ds}E_2[w](s) \\
&= -\int_{\Omega(s)} w_s^2\rho|y|^2\,dy-2\int_{\Omega(s)}(y\cdot\nabla w)w_s\rho\,dy \\
&\quad +b(s)\int_{\Omega(s)} w_sG_D^p(w)\rho|y|^2\,dy+O(e^{As-Be^s}).
\end{aligned} \tag{A.2}
$$

Estimate for $\int_{\Omega(s)}(y\cdot\nabla w)w_s\rho\,dy$**.**

$$
\begin{aligned}
&\int_{\Omega(s)}(y\cdot\nabla w)w_s\rho\,dy\\
=\;&\int_{\Omega(s)}(y\cdot\nabla w)\left(\Delta w-\frac{1}{2}y\cdot\nabla w-\frac{1}{p-1}w+w^p+b(s)G_D^p(w)\right)\rho\,dy\\
=\;&\int_{\Omega(s)}(y\cdot\nabla w)\left[\operatorname{div}(\rho\nabla w)-\frac{1}{p-1}w\rho+w^p\rho+b(s)G_D^p(w)\rho\right]\,dy\\
=\;&-\int_{\Omega(s)}\nabla(y\cdot\nabla w)\cdot(\nabla w)\rho\,dy-\frac{1}{p-1}\int_{\Omega(s)}(y\cdot\nabla w)w\rho\,dy+\int_{\Omega(s)}(y\cdot\nabla w)w^p\rho\,dy\\
&+b(s)\int_{\Omega(s)}(y\cdot\nabla w)G_D^p(w)\rho\,dy\\
=\;&-\int_{\Omega(s)}\left(1-\frac{n}{2}+\frac{|y|^2}{4}\right)|\nabla w|^2\rho\,dy\\
&-\frac{1}{p-1}\int_{\Omega(s)}\left(-\frac{n}{2}+\frac{|y|^2}{4}\right)w^2\rho\,dy-\frac{1}{p-1}\int_{\partial\Omega(s)}\frac{1}{2}w^2|y|\rho\,da\\
&+\int_{\Omega(s)}\frac{1}{p+1}\left(-n+\frac{|y|^2}{2}\right)w^{p+1}\rho\,dy+\int_{\partial\Omega(s)}\frac{1}{p+1}w^{p+1}|y|\rho\,da\\
&+b(s)\int_{\Omega(s)}(y\cdot\nabla w)G_D^p(w)\rho\,dy\\
=\;&-\int_{\Omega(s)}|\nabla w|^2\rho\,dy-\frac{1}{2}\int_{\Omega(s)}\left(\frac{|y|^2}{2}-n\right)|\nabla w|^2\rho\,dy-\frac{1}{2(p-1)}\int_{\Omega(s)}\left(\frac{|y|^2}{2}-n\right)w^2\rho\,dy\\
&+\frac{1}{p+1}\int_{\Omega(s)}\left(\frac{|y|^2}{2}-n\right)w^{p+1}\rho\,dy+b(s)\int_{\Omega(s)}(y\cdot\nabla w)G_D^p(w)\rho\,dy\\
&+\int_{\partial\Omega(s)}\left(-\frac{1}{2(p-1)}w^2|y|+\frac{1}{p+1}w^{p+1}|y|\right)\rho\,da
\end{aligned}
$$

leading to

$$
\begin{aligned}
\text{(A.3)}\qquad &\int_{\Omega(s)}(y\cdot\nabla w)w_s\rho\,dy\\
=\;&-\int_{\Omega(s)}|\nabla w|^2\rho\,dy-\frac{1}{2}\int_{\Omega(s)}\left(\frac{|y|^2}{2}-n\right)|\nabla w|^2\rho\,dy\\
&-\frac{1}{2(p-1)}\int_{\Omega(s)}\left(\frac{|y|^2}{2}-n\right)w^2\rho\,dy+\frac{1}{p+1}\int_{\Omega(s)}\left(\frac{|y|^2}{2}-n\right)w^{p+1}\rho\,dy
\end{aligned}
$$

$$+b(s)\int_{\Omega(s)}(y\cdot\nabla w)G_D^p(w)\rho\,dy+O(e^{As-Be^s}).$$

Next, we will show how the above estimates lead to the following identity, which corresponds to the result [7, Prop. 4.2]:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\int_{\Omega(s)}\left(\frac{|y|^2}{2}-n\right)w^2\rho-(p+1)\int_{\Omega(s)}(y\cdot\nabla w)w_s\rho\,dy\\
\text{(A.4)}\quad =\;&\int_{\Omega(s)}\left(\frac{p-1}{4}|y|^2+c_1\right)|\nabla w|^2\rho\,dy\\
&+b(s)\int_{\Omega(s)}\left[\frac{1}{2}w|y|^2-nw-(p+1)(y\cdot\nabla w)\right]G_D^p(w)\rho\,dy+O(e^{As-Be^s}),
\end{aligned}$$

where $c_1=\frac{1}{2}[n+2-(n-2)p]>0$ when either $1<p<\frac{n+2}{n-2}$ or $n\le 2$.

To prove the above, by (4.14), (A.1) and (A.3), we have:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\int_{\Omega(s)}\left(\frac{|y|^2}{2}-n\right)w^2\rho\,dy-(p+1)\int_{\Omega(s)}(y\cdot\nabla w)w_s\rho\,dy\\
=\;&-E_2[w]+\frac{1}{2}\int_{\Omega(s)}\frac{p-1}{p+1}w^{p+1}\rho|y|^2\,dy-\frac{1}{2}\int_{\Omega(s)}\left(\frac{|y|^2}{2}-n\right)w^2\rho\,dy\\
&+\frac{1}{2}b(s)\int_{\Omega(s)}G_D^p(w)w\rho|y|^2\,dy\\
&+2nE[w]-n\int_{\Omega(s)}\frac{p-1}{p+1}w^{p+1}\rho\,dy-nb(s)\int_{\Omega(s)}G_D^p(w)w\rho\,dy\\
&+(p+1)\int_{\Omega(s)}|\nabla w|^2\rho\,dy+\frac{p+1}{2}\int_{\Omega(s)}\left(\frac{|y|^2}{2}-n\right)|\nabla w|^2\rho\,dy\\
&+\frac{1}{2}\frac{p+1}{p-1}\int_{\Omega(s)}\left(\frac{|y|^2}{2}-n\right)w^2\rho\,dy-\int_{\Omega(s)}\left(\frac{|y|^2}{2}-n\right)w^{p+1}\rho\,dy\\
&-(p+1)b(s)\int_{\Omega(s)}(y\cdot\nabla w)G_D^p(w)\rho\,dy+O(e^{As-Be^s})\\
=\;&-E_2[w]+2nE[w]\\
&-\frac{2}{p+1}\int_{\Omega(s)}\left(\frac{|y|^2}{2}-n\right)w^{p+1}\rho\,dy+\frac{1}{p-1}\int_{\Omega(s)}\left(\frac{|y|^2}{2}-n\right)w^2\rho\,dy\\
&+(p+1)\int_{\Omega(s)}|\nabla w|^2\rho\,dy+\frac{p+1}{2}\int_{\Omega(s)}\left(\frac{|y|^2}{2}-n\right)|\nabla w|^2\rho\,dy\\
&+b(s)\int_{\Omega(s)}\left[\frac{1}{2}w|y|^2-nw-(p+1)(y\cdot\nabla w)\right]G_D^p(w)\rho\,dy+O(e^{As-Be^s}).
\end{aligned}$$

Moreover, according to the definitions of $E[w]$ and $E_2[w]$, we get

$$-E_2[w]+2nE[w]=-2\int_{\Omega(s)}\left(\frac{|y|^2}{2}-n\right)\Big[\frac{|\nabla w|^2+\frac{1}{p-1}w^2}{2}-\frac{w^{p+1}}{p+1}\Big]\rho\,dy.$$

Hence, it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\int_{\Omega(s)}\left(\frac{|y|^2}{2}-n\right)w^2\rho dy-(p+1)\int_{\Omega(s)}(y\cdot\nabla w)w_s\rho\,dy\\
=\;&(p+1)\int_{\Omega(s)}|\nabla w|^2\rho\,dy+\frac{p-1}{2}\int_{\Omega(s)}\left(\frac{|y|^2}{2}-n\right)|\nabla w|^2\rho\,dy\\
&+b(s)\int_{\Omega(s)}\left[\frac{1}{2}w|y|^2-nw-(p+1)(y\cdot\nabla w)\right]G_D^p(w)\rho\,dy+O(e^{As-Be^s})
\end{aligned}$$

giving (A.4) immediately.

Now we are ready for the

Proof of **Proposition 4.2**. Define

$$\tilde{E}_2[w]=E_2[w]-\frac{1}{2}\int_{\Omega(s)}\left(\frac{|y|^2}{2}-n\right)w^2\rho\,dy.$$

Using (A.2) and (A.4), it can be estimated as follows

$$\begin{aligned}
&\frac{d}{ds}\tilde{E}_2[w]\\
=\;&-\int_{\Omega(s)}w_s^2\rho|y|^2\,dy-2\int_{\Omega(s)}(y\cdot\nabla w)w_s\rho\,dy\\
&-(p+1)\int_{\Omega(s)}(y\cdot\nabla w)w_s\rho\,dy-\int_{\Omega(s)}\left(\frac{p-1}{4}|y|^2+c_1\right)|\nabla w|^2\rho\,dy\\
&-b(s)\int_{\Omega(s)}\left[\frac{1}{2}w|y|^2-w_s|y|^2-nw-(p+1)(y\cdot\nabla w)\right]G_D^p(w)\rho\,dy+O(e^{As-Be^s})\\
=\;&-\int_{\Omega(s)}w_s^2\rho|y|^2\,dy-(p+3)\int_{\Omega(s)}(y\cdot\nabla w)w_s\rho\,dy-\int_{\Omega(s)}\left(\frac{p-1}{4}|y|^2+c_1\right)|\nabla w|^2\rho\,dy\\
&-b(s)\int_{\Omega(s)}\left[\frac{1}{2}w|y|^2-w_s|y|^2-nw-(p+1)(y\cdot\nabla w)\right]G_D^p(w)\rho\,dy+O(e^{As-Be^s}).\\
\le\;&-\frac{1}{2}\int_{\Omega(s)}w_s^2\rho|y|^2\,dy+(p+3)\int_{\Omega(s)}\left(\epsilon|y|^2|\nabla w|^2+\frac{1}{4\epsilon}w_s^2\right)\rho\,dy\\
&-\frac{p-1}{4}\int_{\Omega(s)}|y|^2|\nabla w|^2\rho\,dy-\frac{c_1}{2}\int_{\Omega(s)}|\nabla w|^2\rho\,dy+C\sqrt{|b(s)|}+O(e^{As-Be^s}).
\end{aligned}$$

Next choose ϵ such that

$$\frac{p-1}{4}-(p+3)\epsilon=\frac{p-1}{8}.$$

Then we have

$$\begin{aligned}&\frac{d}{ds}\tilde{E}_2[w]\\ \le\ & -\frac{1}{2}\int_{\Omega(s)} w_s^2\rho|y|^2\,dy-\frac{p-1}{8}\int_{\Omega(s)}|y|^2|\nabla w|^2\rho\,dy-\frac{c_1}{2}\int_{\Omega(s)}|\nabla w|^2\rho\,dy\\ & +c_2\int_{\Omega(s)} w_s^2\rho\,dy+C\sqrt{|b(s)|}+O(e^{As-Be^s}).\end{aligned}$$

This, together with (4.16), yields that

$$\begin{aligned}&\frac{d}{ds}\left(\tilde{E}_2[w]+c_3E[w]\right)\\ \le\ & -\frac{1}{2}\int_{\Omega(s)} w_s^2\rho|y|^2\,dy-\frac{p-1}{8}\int_{\Omega(s)}|y|^2|\nabla w|^2\rho\,dy-\frac{c_1}{2}\int_{\Omega(s)}|\nabla w|^2\rho\,dy\\ & -\int_{\Omega(s)} w_s^2\rho\,dy+C\sqrt{|b(s)|}+O(e^{As-Be^s})\\ \le\ & -c_4\int_{\Omega(s)}\left(w_s^2+|\nabla w|^2\right)\left(1+|y|^2\right)\rho\,dy+C\sqrt{|b(s)|}+O(e^{As-Be^s}),\end{aligned}$$

where $c_3=2c_2+2$. Hence

$$\begin{aligned}\text{(A.5)}\quad &\int_{s_1}^{s}\int_{\Omega(s)}(|w_s|^2+|\nabla w|^2)(1+|y|^2)\rho\,dy\,ds\\ &\qquad\le\left(\tilde{E}_2[w]+c_3E[w]\right)(s_1)-\left(\tilde{E}_2[w]+c_3E[w]\right)(s)+C_4,\end{aligned}$$

where $s_1>s_0$ and the constant C_4 is independent of δ and $K\in\mathcal{H}_\delta$.

Now, it remains to bound $\tilde{E}_2[w]+c_3E[w]$ from below. Suppose this is not true. By (4.14), (A.1) and Hölder's inequality, we have

$$\begin{aligned}&\frac{d}{ds}\int_{\Omega(s)}\frac{1}{2}(|y|^2+c_3)w^2\rho\,dy\\ =\ & -2E_2[w]+\int_{\Omega(s)}\frac{p-1}{p+1}w^{p+1}\rho|y|^2\,dy-\int_{\Omega(s)}\left(\frac{|y|^2}{2}-n\right)w^2\rho\,dy\\ & +b(s)\int_{\Omega(s)}G_D^p(w)w\rho|y|^2\,dy-2c_3E[w]+c_3\int_{\Omega(s)}\frac{p-1}{p+1}w^{p+1}\rho\,dy\\ & +c_3b(s)\int_{\Omega(s)}G_D^p(w)w\rho\,dy+O(e^{As-Be^s})\\ \ge\ & -2\left(\tilde{E}_2[w]+c_3E[w]\right)+\int_{\Omega(s)}\frac{p-1}{p+1}(|y|^2+c_3)w^{p+1}\rho\,dy\end{aligned}$$

$$
\begin{aligned}
&\quad -|b(s)|\int_{\Omega(s)}(|y|^2+c_3)\frac{1}{2}\left[G_D^{2p}(w)+w^2\right]\rho\,dy\\
&\quad -2\int_{\Omega(s)}\left(\frac{|y|^2}{2}-n\right)w^2\rho\,dy+O(e^{As-Be^s})\\
\geq&\quad -2\left(\tilde{E}_2[w]+c_3E[w]\right)+c_5\left(\int_{\Omega(s)}(|y|^2+c_3)w^2\rho\,dy\right)^{\frac{p+1}{2}}\\
&\quad -2\int_{\Omega(s)}(|y|^2+c_3)w^2\rho\,dy+O(e^{As-Be^s})+O(e^{-\gamma s}).
\end{aligned}\tag{A.6}
$$

Furthermore, recall that $s_0=-\log T$ and $T=\delta^2$, thus when δ is small enough, $|O(e^{As-Be^s})|<1$ and $|O(e^{-\gamma s})|<1$ in (A.6).

Now for simplicity, set

$$\ell(s)=\int_{\Omega(s)}(|y|^2+c_3)w^2\rho\,dy$$

and rewrite (A.6) as follows

$$\frac{1}{2}\frac{d}{ds}\ell(s)\geq -2\left(\tilde{E}_2[w]+c_3E[w]\right)+c_5[\ell(s)]^{\frac{p+1}{2}}-2\ell(s)+O(e^{As-Be^s})+O(e^{-\gamma s}).\tag{A.7}$$

Let

$$c_6^*=\sup_{\ell\geq 0}\left(2\ell-\frac{1}{2}c_5\ell^{\frac{p+1}{2}}\right).$$

Due to (A.5), if there exists $s_1>s_0$ such that

$$\left(\tilde{E}_2[w]+c_3E[w]\right)(s_1)<-C_4-\frac{1}{2}(c_6^*+2),$$

then $\left(\tilde{E}_2[w]+c_3E[w]\right)(s)<-\frac{1}{2}(c_6^*+2)$ for any $s>s_1$.

Therefore, it is clear that (A.7) implies that for $s>s_1$

$$\frac{1}{2}\frac{d}{ds}\ell(s)\geq\frac{1}{2}c_5[\ell(s)]^{\frac{p+1}{2}},$$

whose solution $\ell(s)$ blows up in finite time. Hence

$$\left(\tilde{E}_2[w]+c_3E[w]\right)(s)\geq -C_4-\frac{1}{2}(c_6^*+2)$$

for any $s>s_0$. Applying the above to (A.5) completes the proof. □

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