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Asymptotic analysis and homogenization of invariant measures

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In this paper, we study limiting behavior of the invariant measures for reaction–diffusion equations in the whole space $\mathbb{R}^d$ with regular and singular perturbations. In the regular case, we show the convergence of the unique stationary solution of $du_\varepsilon = (\Delta u_\varepsilon - \varepsilon u)dt + \sigma(x, u_\varepsilon)dW$ to a stationary solution of the limiting equation $du = \Delta u dt + \sigma(x, u)dW$. We also consider the asymptotic behavior of the stationary solution under the perturbations of spectrum. Finally, for the singular perturbation of homogenization type, we show the weak convergence of invariant measure to its homogenized limit.

Keywords: Vanishing dissipation limit; stationary solution; long time behavior, homogenization.

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1. Introduction and Main Results

The theory of reaction–diffusion equations combined the theories of heat conduction and mass diffusion, on the one hand, and equations of chemical and biological kinetics, on the other. A comprehensive study of the deterministic reaction–diffusion equation was performed by Volpert [23]. In this book, one can find numerous examples of application of reaction–diffusion models.

An important class of reaction–diffusion equations describes the dynamics in random media, which leads to stochastic PDEs

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = Du(t,x) + f(x,u(t,x)) + \sigma(x,u(t,x))\dot{W}(t,x), \quad t>0,\ x\in G\subset\mathbb{R}^d; \\ u(0,x) = u_0(x). \end{cases} \tag{1.1}$$

Here, D is an elliptic operator, f and σ are measurable functions, the Gaussian noise $\dot{W}(t,x)$ is white in time and colored in space (see (1.2) for the precise definition of $W(t,x)$), and $G \subseteq \mathbb{R}^d$. A comprehensive analysis of stochastic equations of this type has been performed by Da Prato and Zabczyk (see [10, 11] and references therein). The long time behavior of solutions of (1.1) is a question of a special interest. In particular, long time behavior of solutions of (1.1) is closely related to the existence of invariant measures. The works [10, 11] provide conditions for the existence and uniqueness of invariant measures for (1.1) based on the results of Krylov and Bogoliubov [15] on the tightness of a family of measures. The conditions for the existence of invariant measures were expressed in terms of the coefficients in [16, 13, 6–9] in particular, in bounded domains. An alternative approach to establishing the existence of invariant measures, which is based on the coupling method, was suggested by Bogachev and Rockner [5] and Mueller [20]. This method can be applied even for space-white noise but only in the case when the space dimension is one.

The solutions of (1.1) in unbounded domains are studied in weighted Sobolev spaces. Let ρ be a non-negative continuous $L^1(\mathbb{R}^d)\bigcap L^\infty(\mathbb{R}^d)$ function. Following [21], we call ρ to be an *admissible* weight if for every $T>0$ there exists $C(T)>0$ such that

$$G(t,\cdot)*\rho \le C(T)\rho, \quad \forall\, t\in[0,T], \text{ where } G(t,x) = \frac{1}{(4\pi t)^{d/2}}e^{-\frac{|x|^2}{4t}}.$$

Typically, we set $\rho(x) = \exp(-\gamma|x|)$ for $\gamma>0$, or $\rho(x) = (1+|x|^n)^{-1}$ with $n>d$, which are admissible weights. For such weights, we define

$$L^2_\rho(\mathbb{R}^d) := \left\{ w:\mathbb{R}^d\to\mathbb{R}, \int_{\mathbb{R}^d} |w(x)|^2\rho(x)dx < \infty \right\}.$$

Throughout the paper, we will use the notation

$$\|w\|_\rho^2 := \int_{\mathbb{R}^d} |w(x)|^2\rho(x)dx$$

to denote the weighted L^2 norm in $\mathbb{R}^d$. The $L^2(\mathbb{R}^d)$ norm without weights will be denoted by $\|\cdot\|$. The noise W is given by

$$W(t,x) := \sum_{k=1}^{\infty} \sqrt{a_k}e_k(x)\beta_k(t), \tag{1.2}$$

where $\beta_k(t)$ are independent Wiener processes, $\{e_k(x), k \geq 1\}$ is an orthonormal basis in $L^2(\mathbb{R}^d)$ such that

$$\sup_{k\geq 1} \|e_k\|_{L^\infty(\mathbb{R}^d)} \leq 1$$

and $a_k \geq 0$ satisfy

$$\sum_{k=1}^{\infty} a_k := a < \infty. \tag{1.3}$$

Clearly, in a bounded domain G an elliptic operator with homogeneous Dirichlet boundary conditions has discrete negative spectrum, separated from the origin. Thus

$$\|S(t)u\|_{L^2(G)} \leq Ce^{-\lambda_1 t}\|u\|_{L^2(G)}, \quad u \in L^2(G), \tag{1.4}$$

where $\lambda_1 > 0$ is the first eigenvalue of $-D$ in G (subject to Dirichlet boundary condition $u = 0$ on ∂G), and $S(t)$ is the corresponding semigroup generated by D. Using (1.4) one can easily establish the existence and uniqueness of invariant measures [18] even in certain unbounded domains.

In unbounded domains, however, the spectrum is typically more complicated, thus the validity of the estimate of type (1.4) is a highly nontrivial question. A typical approach to establish the existence of invariant measures in unbounded domains is to require some form of dissipation from the nonlinearity f. An example of such dissipation condition can be found in [2]:

$$uf(u) \leq -ku^2 + c \tag{1.5}$$

for some $k > 0$ and $c \in \mathbb{R}$. The work [18] established the existence of invariant measure for (1.1) with $D = \Delta$ in $\mathbb{R}^d$ under different conditions on f. In particular, the invariant measure for (1.1) exists if f satisfies the global bound:

$$|f(x,u)| \leq \varphi(x) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d), \quad \forall\, u \in \mathbb{R}.$$

In [19], using a version of Ito's formula, the dissipation condition (1.5) is replaced with a weaker one:

$$uf(x,u) \leq \eta(x) \quad \text{for } |u| > M \ \text{ and } \ x \in \mathbb{R}^d$$

for some $M > 0$ and a non-negative $\eta(x) \in L^1(\mathbb{R}^d), d \geq 3$. The question of the existence of invariant measures in unbounded domains was also addressed in [11, 12, 21].

The uniqueness of invariant measure is a question of a separate interest [11]. In general, to show uniqueness, one needs the recurrence property for the solutions, as well as the strong Feller property. However, if the semigroup satisfies an estimate of type (1.4), the uniqueness of the invariant measure follows easily, e.g. using the method developed in [18]. More precisely, we have the following theorem for the Eq. (1.1) with solution u taking values in a Hilbert space H.

Theorem 1.1. ([18]) *Let $S(t)$ be the semigroup for the operator D such that*

$$\|S(t)u\|_H \le Ce^{-\lambda t}\|u\|_H, \quad \forall\, u \in H, \;\; \lambda > 0. \tag{1.6}$$

Then, if $f(x,u)$ and $\sigma(x,u)$ have sufficiently small Lipschitz constants in u, there exists a unique stationary solution $u^* \in H$ for (1.1). *This stationary solution defines the unique invariant measure for* (1.1) *in H. Moreover, this solution can be constructed as a limit of an iteration scheme.*

Remark 1.1. The semigroup, generated by $\Delta u - \varepsilon u$ in $H = L^2(\mathbb{R}^d)$ satisfies (1.6). The other examples of semigroups satisfying (1.6) are provided in [18].

However, there is a large class of practically important operators, e.g. the Laplace operator in $\mathbb{R}^d$, that do not satisfy (1.6) yet for which (1.1) has a stationary solution. In contrast with the previous theorem, the existence results for stationary solutions are rather abstract and not constructive ([2]). Moreover, we cannot say anything about the uniqueness of the stationary solution in this case. Nevertheless, in certain cases we can approximate the original operator D with the family of operators D_ε, which satisfy (1.6) with $\lambda = \lambda(\varepsilon) \to 0, \varepsilon \to 0$. This way we can efficiently approximate one of the stationary solutions of Eq. (1.1). We have the following results:

Theorem 1.2. (Vanishing dissipation limit) *Consider*

$$u_t^\varepsilon = \Delta u^\varepsilon - \varepsilon u^\varepsilon + \sigma(x, u^\varepsilon)\dot{W}(t) \tag{1.7}$$

and the limiting equation

$$u_t^* = \Delta u^* + \sigma(x, u^*)\dot{W}(t). \tag{1.8}$$

Assume $\sigma(x,0) \in L^2(\mathbb{R}^d)$ with $d \ge 3$, and σ is Lipschitz in u with a sufficiently small Lipschitz constant L satisfying

$$2a\left(1 + \frac{1}{(8\pi)^{d/2}\left(\frac{d}{2} - 1\right)}\right)L^2 < 1. \tag{1.9}$$

Let u_ε^* be the unique stationary solution of (1.7) (*whose existence follows from Theorem* 1.1). *Then there exists u_0^*, which is a stationary solution of* (1.8), *such that*

$$E\|u_\varepsilon^*(t) - u_0^*(t)\|^2_{L^2(\mathbb{R}^d)} \to 0, \quad \varepsilon \to 0, \;\; t \ge 0.$$

Note that while the limiting equation may have more than one stationary solution, the previous theorem enables us to approximate one of them as a vanishing dissipation limit. It will be interesting to investigate in the future more precisely which limit is to be selected. A closely related result on the convergence of invariant measures to the invariant measure of the limiting equation was obtained in [17] in the finite-dimensional case.

In the next theorem, we study the behavior of the invariant measures under the regular perturbations of differential operators in $\mathbb{R}^d$ with discrete spectrum. We will make use of the results from the spectral theory.

Theorem 1.3. ([22, Chap. 19]) *Consider the spectral problem*

$$\Delta u - q(x)u + \lambda u = 0, \quad u \in L^2(\mathbb{R}^d),\ \ \lambda \in \mathbb{R} \tag{1.10}$$

and the corresponding perturbed problem

$$\Delta u_\varepsilon - q(x)u_\varepsilon - \varepsilon q_1(x)u_\varepsilon + \lambda_\varepsilon u_\varepsilon = 0, \quad u_\varepsilon \in L^2(\mathbb{R}^d),\ \ \lambda_\varepsilon \in \mathbb{R}. \tag{1.11}$$

Suppose q and q_1 are such that

(i) *q and q_1 are continuous;*
(ii) *$q(x) \geq 0$, $q(x) \to +\infty$ as $|x| \to \infty$.*
Then (1.10) has discrete spectrum $\{\lambda_n, n \geq 1\}$, with $\lambda_1 < 0$. The corresponding eigenfunctions are denoted with $\{\Psi_n\}$. If, in addition,
(iii) *there exists $M > 0$ such that $q_1(x) \geq -Mq(x)$;*
(iv) *for all $n \geq 1$ we have $q_1(x)\Psi_n(x) \in L^2(\mathbb{R}^d)$, then (1.11) has discrete spectrum $\{\lambda_n^\varepsilon, n \geq 1\}$ as well, with the corresponding eigenfunctions Ψ_n^ε. Moreover, for every $n \geq 1$ we have*

$$\lambda_n^\varepsilon = \lambda_n + O(\varepsilon), \quad \varepsilon \to 0$$

and

$$\int_{\mathbb{R}^d} (\Psi_n^\varepsilon(x) - \Psi_n(x))^2 dx = O(\varepsilon^2), \quad \varepsilon \to 0.$$

We establish the next theorem.

Theorem 1.4. (Perturbation analysis) *Consider*

$$du = [\Delta u - q(x)u - \varepsilon q_1(x)u + f(x,u)]dt + \sigma(x,u)dW(t) \tag{1.12}$$

and

$$du = [\Delta u - q(x)u + f(x,u)]dt + \sigma(x,u)dW(t). \tag{1.13}$$

Assume that q and q_1 satisfy the conditions of Theorem 1.3, and $f(x,u)$ and $\sigma(x,u)$ are continuous in x and Lipschitz in u, with $f(x,0) \in L^2(\mathbb{R}^d)$, $\sigma(x,0) \in L^2(\mathbb{R}^d)$, and the Lipschitz constant L satisfying

$$L^2\left(\frac{16}{\lambda_1^2} - \frac{4a}{\lambda_1}\right) < 1 \tag{1.14}$$

and

$$\frac{\lambda_1}{2} - 24(1+a)\frac{L^2}{\lambda_1} < 0. \tag{1.15}$$

Here, $\lambda_1 < 0$ is the first eigenvalue of (1.10), and a is given by (1.3). Let u_ε and u^* be the unique stationary solutions of (1.12) and (1.13), respectively. Then, for every $t \geq 0$ we have

$$\mathbb{E}\|u_\varepsilon(t) - u^*(t)\|^2_{L^2(\mathbb{R}^d)} \to 0, \quad \varepsilon \to 0.$$

Example 1.1. Suitable q and q_1 in $\mathbb{R}^2$ are $q(x,y) = x^2 + y^2$ and $q_1(x,y) = xy$.

A separate class of problems involves the study of porous media, subject to the influence of random noise. This leads us to the homogenization problems for stochastic differential equations. A comprehensive study of linear homogenization problems in both elliptic and parabolic settings, as well as certain homogenization results for elliptic operators in random media, may be found in [25]. In particular, the work [25] addresses the homogenization problem in presence of regular random perturbations. In this case, it is possible to treat the corresponding stochastic equations for almost every realization of the random field, and obtain the results in spirit of Central Limit Theorem. This approach however cannot be extended to the singularly perturbed stochastic equation with white noise.

One of the earlier works, which studies the nonlinear homogenization problem for parabolic stochastic equations in bounded domains is [3]. The consequent work [14] considers the same problem with an oscillating coefficient in the noise term. The long time behavior of stochastic homogenization problems was considered in [24] in the linear case in a bounded domain. The work [4] studies the homogenization problem for Brinkman flows. In this case, the oscillations are present in the reaction terms, and the authors establish the convergence in probability.

In this paper, we characterize the convergence of invariant measures in the homogenization limit. More precisely, we establish the mean square convergence of the unique stationary solution of the nonlinear stochastic equation with rapidly oscillating coefficients to the corresponding unique stationary solution of the homogenized problem. This, in turn, implies the weak convergence of invariant measures. The two-scale convergence [1] and Nash–Aronson estimates play an important role in our analysis.

Consider the following homogenization problem:

$$du_\varepsilon = \left[\operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) - u_\varepsilon + f\left(\frac{x}{\varepsilon}, u_\varepsilon\right)\right]dt + \sigma\left(\frac{x}{\varepsilon}, u_\varepsilon\right)dW(t). \tag{1.16}$$

Here $x \in \mathbb{R}^d$, $u \in L^2_\rho(\mathbb{R}^d)$ with

$$\rho(x) = e^{-\kappa|x|},$$

$A(x)$ is a measurable periodic and uniformly elliptic matrix, $f(x,u)$ and $\sigma(x,u)$ are uniformly bounded, continuous, periodic functions in x and Lipschits in u. Assume that the Lipschitz constant L and the parameter κ are small enough (more precisely, they satisfy the smallness conditions (4.4), (4.6)–(4.8) below). Similar conditions may be formulated for

$$\rho(x) = \frac{1}{1+\kappa|x|^r}, \quad \kappa > 0, \;\; r > d.$$

Along with (1.16), we consider the homogenized problem

$$du_0 = [\operatorname{div}(A_0\nabla u_0) - u_0 + f_0(u_0)]dt + \sigma_0(u_0)dW(t), \tag{1.17}$$

where A_0 is the homogenized matrix described in [25, Sec. 1.2].

$$f_0(u) := \int_\Pi f(x,u)dx, \quad \sigma_0(u) := \int_\Pi \sigma(x,u)dx,$$

with $\Pi = [0,1]^d$.

Theorem 1.5. *Under the above assumptions on* A, ρ, f *and* σ, *for every* $t \geq 0$ *we have*

$$\mathbb{E}\|u_\varepsilon(t) - u_0(t)\|_\rho^2 \to 0, \quad \varepsilon \to 0,$$

where u_ε *and* u_0 *are the corresponding unique stationary solutions of* (1.16) *and* (1.17), *whose existence follows from Theorem* 1.1.

Finally, we note that a similar convergence result holds in bounded domains with Dirichlet boundary conditions. In particular, for bounded $G \subset \mathbb{R}^d$, let u_ε be the unique stationary solution of

$$\begin{cases} du_\varepsilon = \left[\mathrm{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) + f\left(\frac{x}{\varepsilon}, u_\varepsilon\right)\right]dt + \sigma\left(\frac{x}{\varepsilon}, u_\varepsilon\right)dW(t), \quad x \in G, \\ u = 0 \quad \text{on } \partial G. \end{cases} \tag{1.18}$$

where $A(x)$, $f(x,u)$ and $\sigma(x,u)$ are the same as in Theorem 1.5. Assume the Lipschitz constant L for f and σ satisfy the smallness assumptions (4.6)–(4.8) with possibly different C_ν and γ. Finally, let u_0 be the corresponding stationary solution of the limiting equation

$$\begin{cases} du_0 = [\mathrm{div}(A_0\nabla u_0) + f_0(u_0)]dt + \sigma_0(u_0)dW(t), \quad x \in G, \\ u = 0 \quad \text{on } \partial G. \end{cases} \tag{1.19}$$

Theorem 1.6. *For every* $t \geq 0$, *we have*

$$\mathbb{E}\|u_\varepsilon(t) - u_0(t)\|_{L^2(G)}^2 \to 0, \quad \varepsilon \to 0.$$

2. Vanishing Dissipation Limit

In this section, we prove Theorem 1.2. The proof follows from the two lemmas below.

Lemma 2.1. *Under the conditions of Theorem* 1.2, *Eq.* (1.7) *has a unique stationary solution* u_ε^*, *defined for* $t \in \mathbb{R}$, *such that*

$$\sup_{t\in R} \mathbb{E}\|u_\varepsilon^*(t)\|_{L^2(\mathbb{R}^d)} \leq C < \infty.$$

with C *independent of* ε.

Remark 2.1. As we mentioned earlier, the semigroup S_ε for the operator $D_\varepsilon u := \Delta u - \varepsilon u$ satisfies the estimate (1.4). Thus, for fixed $\varepsilon > 0$, the existence and uniqueness of a bounded stationary solution u_ε^*, defined for $t \in \mathbb{R}$, follows from Theorem 1.1. However, the novelty of the present result is that this stationary solution is bounded uniformly in ε.

Proof. The desired solution will be constructed as a limit of an iteration scheme. Set $u_0 \equiv 0$ and let u_{n+1} be the unique stationary solution of

$$du_{n+1} = (\Delta u_{n+1} - \varepsilon u_{n+1})dt + \sigma(x, u_n(t))dW(t).$$

Arguing as in [18, Theorem 5], such u_{n+1} is defined correctly, since at every step we have $\sup_{t\in\mathbb{R}} \mathbb{E}\|\sigma(\cdot, u_n)\|^2_{L^2(\mathbb{R}^d)} < \infty$. Indeed, $\mathbb{E}\|\sigma(\cdot, 0)\|^2 < C$, hence there exists $u_1(t, x)$ satisfying

$$\sup_{t\in\mathbb{R}} \mathbb{E}\|u_1(t)\|_{L^2(\mathbb{R}^d)} < \infty.$$

Then, using the Lipschitz property of σ

$$\mathbb{E}\|\sigma(\cdot, u_1)\|^2_{L^2(\mathbb{R}^d)} \leq 2\int_{\mathbb{R}^d} |\sigma(x, 0)|^2 dx + 2L^2 \int_{\mathbb{R}^d} \mathbb{E}|u_1(t, x)|^2 dx < \infty$$

hence u_2 is defined correctly, and so on. Write

$$\begin{aligned} u_{n+1}(t) &= \int_{-\infty}^{t} S_\varepsilon(t-s)\sigma(\cdot, u_n)dW(s) \\ &= \sum_k \sqrt{a_k}\int_{-\infty}^{t} e^{-\varepsilon(t-s)}\int_{\mathbb{R}^d} G(t-s, x-y)\sigma(y, u_n(s, y))e_k dy d\beta_k(s). \end{aligned}$$

Then

$$\begin{aligned} &\mathbb{E}\|u_{n+1}(t)\|^2_{L^2(\mathbb{R}^d)} \\ &\quad = \mathbb{E}\int_{\mathbb{R}^d}\left|\sum_k \sqrt{a_k}\int_{-\infty}^{t} e^{-\varepsilon(t-s)}\int_{\mathbb{R}^d} G(t-s, x-y)\sigma(y, u_n)e_k(y)dy d\beta_k(s)\right|^2 dx \\ &\quad \leq \sum_k a_k \int_{\mathbb{R}^d}\int_{-\infty}^{t} e^{-2\varepsilon(t-s)}\mathbb{E}\left|\int_{\mathbb{R}^d} G(t-s, x-y)\sigma(y, u_n)e_k(y)dy\right|^2 ds dx. \end{aligned}$$

We proceed with the change variables in the kth term in the above summation:

$$\begin{aligned} &\int_{\mathbb{R}^d}\int_{-\infty}^{t} e^{-2\varepsilon(t-s)}\mathbb{E}\left|\int_{\mathbb{R}^d} G(t-s, x-y)\sigma(y, u_n)e_k(y)dy\right|^2 ds dx \\ &\quad \leq \int_0^\infty ds\int_{\mathbb{R}^d} dx\mathbb{E}\left(\int_{\mathbb{R}^d} G(s, x-y)\sigma(y, u_n(t-s, y))e_k(y)dy\right)^2 \\ &\quad \leq \int_0^\infty ds\int_{\mathbb{R}^d} dx\mathbb{E}\left(\int_{\mathbb{R}^d} G(s, x-y)(\sigma(y, 0) + L|u_n(t-s, y)|)|e_k(y)|dy\right)^2 \\ &\quad \leq 2\int_0^\infty ds\int_{\mathbb{R}^d} dx\left(\int_{\mathbb{R}^d} G(s, x-y)|\sigma(y, 0)||e_k(y)|dy\right)^2 \\ &\qquad + 2\int_0^\infty ds\int_{\mathbb{R}^d} dx\mathbb{E}\left(\int_{\mathbb{R}^d} G(s, x-y)L|u_n(t-s, y)||e_k(y)|dy\right)^2 \\ &\quad := 2J_1 + 2J_2. \end{aligned}$$

To estimate the first integral, we write

$$\begin{aligned}J_1 &:= \int_0^\infty ds \int_{\mathbb{R}^d} dx \left(\int_{\mathbb{R}^d} G(s, x-y)|\sigma(y,0)||e_k(y)|dy \right)^2 \\ &= \int_0^1 [\cdots]ds + \int_1^\infty [\cdots]ds \\ &:= J_{11} + J_{12}.\end{aligned}$$

Since $|e_k(y)| \le 1$, we immediately get

$$J_{11} \le \int_0^1 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(s, x-y)dx|\sigma(y,0)|^2 dy \le \int_{\mathbb{R}^d} |\sigma(y,0)^2|dy := C_\sigma.$$

As for the estimate for J_{12}, we have

$$\begin{aligned}J_{12} &\le \int_1^\infty ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} G^2(s, x-y)e_k^2(y)|\sigma(y,0)|^2 dy \\ &\le \int_{\mathbb{R}^d} |\sigma(0,y)|^2 dy \cdot \int_1^\infty \int_{\mathbb{R}^d} G^2(s,x)dxds \\ &:= C_\sigma C_d,\end{aligned}$$

where

$$C_d := \int_1^\infty \int_{\mathbb{R}^d} G^2(s,x)dxds = \frac{1}{(8\pi)^{d/2}\left(\frac{d}{2}-1\right)}, \quad d \ge 3.$$

Similarly, we write

$$J_2 = \int_0^\infty [\cdots]ds = \int_0^1 [\cdots]ds + \int_1^\infty [\cdots]ds := J_{21} + J_{22},$$

with

$$\begin{aligned}J_{21} &\le \int_0^1 ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} G(s, x-y)dy \int_{\mathbb{R}^d} G(s, x-y)L^2\mathbb{E}|u_n(t-s,y)|^2|e_k(y)|^2 dy \\ &\le L^2 \sup_{t\in\mathbb{R}} \mathbb{E}\|u_n(t)\|^2_{L^2(\mathbb{R}^d)}\end{aligned}$$

and

$$\begin{aligned}J_{22} &\le L^2 \int_1^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G^2(s, x-y)|e_k(y)|^2 dxdy\mathbb{E} \int_{\mathbb{R}^d} |u_n(t-s,y)|^2 dy \\ &\le L^2 C_d \sup_{t\in\mathbb{R}} \mathbb{E}\|u_n(t)\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Combining the estimates above, we have

$$\begin{aligned}&\int_{-\infty}^t \int_{\mathbb{R}^d} e^{-2\varepsilon(t-s)}\mathbb{E}\left| \int_{\mathbb{R}^d} G(t-s, x-y)\sigma(y, u_n)e_k(y) \right|^2 ds \\ &\qquad \le 2(C_\sigma + C_\sigma C_d) + 2L^2(1+C_d) \sup_{t\in\mathbb{R}} \mathbb{E}\|u_n(t)\|^2_{L^2(\mathbb{R}^d)}\end{aligned}$$

and hence, summing up in k, we have

$$\sup_{t\in\mathbb{R}} \mathbb{E}\|u_{n+1}(t)\|^2_{L^2(\mathbb{R}^d)} \leq 2a(C_\sigma + C_\sigma C_d) + 2aL^2(1+C_d)\sup_t \mathbb{E}\|u_n(t)\|^2_{L^2(\mathbb{R}^d)}.$$

Consequently,

$$\sup_{t\in\mathbb{R}} \mathbb{E}\|u_{n+1}(t)\|^2_{L^2(\mathbb{R}^d)} \leq \frac{2aC_\sigma(1+C_d)}{1-2a(1+C_d)L^2} := C, \tag{2.1}$$

provided L satisfies (1.9).

We next proceed with establishing the convergence of u_n.

$$\begin{aligned}
&\mathbb{E}\|u_{n+1}(t) - u_n(t)\|^2_{L^2(\mathbb{R}^d)} \\
&\quad\leq \sum_k a_k \int_{\mathbb{R}^d}\int_{-\infty}^t e^{-2\varepsilon(t-s)}\mathbb{E}\left(\int_{\mathbb{R}^d} G(t-s,x-y)\right. \\
&\quad\quad \left.\times[\sigma(y,u_n(y)) - \sigma(y,u_{n-1}(y))]e_k(y)dy\right)^2 dsdx \\
&\quad\leq L^2\sum_k a_k \int_{\mathbb{R}^d}\int_0^\infty \mathbb{E}\left(\int_{\mathbb{R}^d} G(s,x-y)|u_n(t-s,y) - u_{n-1}(t-s,y)|\right. \\
&\quad\quad \left.\times|e_k(y)|dy\right)^2 dsdx.
\end{aligned}$$

Once again, fix $k \geq 1$. We have

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}^d}\mathbb{E}\left(\int_{\mathbb{R}^d} G(s,x-y)|u_n(t-s,y) - u_{n-1}(t-s,y)||e_k(y)|dy\right)^2 dxds \\
&\quad= \int_0^1 [\cdots]ds + \int_1^\infty [\cdots]ds := I_1 + I_2.
\end{aligned}$$

Analogous estimates yield

$$I_1 \leq \sup_{t\in\mathbb{R}} \mathbb{E}\|u_n(t) - u_{n-1}(t)\|^2_{L^2(\mathbb{R}^d)}$$

and

$$I_2 \leq C_d \sup_{t\in\mathbb{R}} \mathbb{E}\|u_n(t) - u_{n-1}(t)\|^2_{L^2(\mathbb{R}^d)}.$$

Thus

$$\mathbb{E}\|u_{n+1}(t) - u_n(t)\|^2_{L^2(\mathbb{R}^d)} \leq L^2 a(1+C_d)\sup_t \mathbb{E}\|u_n(t) - u_{n-1}(t)\|^2_{L^2(\mathbb{R}^d)}. \tag{2.2}$$

If L satisfies (1.9), iterating (2.2) we deduce that u_n is a Cauchy sequence in $L^2(\mathbb{R}^d)$. Thus, there exists a limiting function $u^*_\varepsilon(t,\cdot) \in L^2(\mathbb{R}^d)$. It is straightforward to verify (see, e.g. [18, Theorem 3]) that $u^*_\varepsilon(t,\cdot)$ is a mild solution to (1.7) and satisfies

$$\sup_{t\in\mathbb{R}} \mathbb{E}\|u^*_\varepsilon(t)\|^2_{L^2(\mathbb{R}^d)} \leq C < \infty.$$

where C is given by (2.1) and is independent of ε. ◻

Lemma 2.2. *There exists* $u_0(t)$, *which solves* (1.8), *such that for all* $t \geq 0$

$$\mathbb{E}\|u_\varepsilon^*(t) - u_0(t)\|_{L^2(\mathbb{R}^d)} \to 0, \quad \varepsilon \to 0,$$

where u_ε^* *is the solution constructed in Lemma* 2.1.

Proof. We first show that for fixed $t \geq 0$, $u_\varepsilon^*(t)$ is a Cauchy sequence in ε. For $0 < \varepsilon < \varepsilon_1$, we have

$$\begin{aligned}&\mathbb{E}\|u_\varepsilon^*(t) - u_{\varepsilon_1}^*(t)\|_{L^2(\mathbb{R}^d)}^2 \\ &\quad = \sum_k a_k \int_{\mathbb{R}^d} \mathbb{E}\int_{-\infty}^t \left[\int_{\mathbb{R}^d} [e^{-\varepsilon(t-s)} G\sigma(u_\varepsilon^*)e_k - e^{-\varepsilon_1(t-s)} G\sigma(u_{\varepsilon_1}^*)e_k]dy\right]^2 dsdx\end{aligned}$$

Adding and subtracting $e^{-\varepsilon_1(t-s)}G\sigma(u_\varepsilon^*)e_k$ to each term in the summation above, we get

$$\begin{aligned}&\mathbb{E}\int_{\mathbb{R}^d}\int_{-\infty}^t \left[\int_{\mathbb{R}^d} [e^{-\varepsilon(t-s)} G\sigma(u_\varepsilon^*)e_k - e^{-\varepsilon_1(t-s)} G\sigma(u_{\varepsilon_1}^*)e_k]dy\right]^2 dsdx \\ &\quad \leq 2\mathbb{E}\int_{\mathbb{R}^d}\int_{-\infty}^t \left[(e^{-\varepsilon(t-s)} - e^{-\varepsilon_1(t-s)})\int_{\mathbb{R}^d} G\sigma(u_\varepsilon^*)e_k dy\right]^2 dsdx \\ &\qquad + 2\mathbb{E}\int_{\mathbb{R}^d}\int_{-\infty}^t \left[e^{-\varepsilon_1(t-s)}\int_{\mathbb{R}^d} G[\sigma(u_\varepsilon^*) - \sigma(u_{\varepsilon_1}^*)]e_k dy\right]^2 dsdx \\ &\quad := 2S_1 + 2S_2. \end{aligned} \tag{2.3}$$

After the change of variables in t, the first term in (2.3) can be further estimated as follows:

$$\begin{aligned} S_1 &\leq \int_{\mathbb{R}^d}\int_0^\infty [e^{-\varepsilon s} - e^{-\varepsilon_1 s}]^2 \mathbb{E}\left[\int_{\mathbb{R}^d} G(s, x-y)|\sigma(y,0) + Lu_*^\varepsilon(t-s,y)||e_k|dy\right]^2 dsdx \\ &\leq 2\int_{\mathbb{R}^d}\int_0^\infty [e^{-\varepsilon s} - e^{-\varepsilon_1 s}]^2 \mathbb{E}\left[\int_{\mathbb{R}^d} G(s,x-y)|\sigma(y,0)||e_k|dy\right]^2 dsdx \\ &\quad + 2L^2\int_{\mathbb{R}^d}\int_0^\infty [e^{-\varepsilon s} - e^{-\varepsilon_1 s}]^2 \mathbb{E}\left[\int_{\mathbb{R}^d} G(s,x-y)|u_*^\varepsilon(t-s,y)||e_k|dy\right]^2 dsdx \\ &:= S_{11} + S_{12}. \end{aligned}$$

Using the bound $\sup_t \mathbb{E}\|u_\varepsilon^*\|^2 \leq C < \infty$, by Dominated Convergence Theorem, $S_{11} \to 0$ and $S_{12} \to 0$ as $\varepsilon \to 0$, $\varepsilon_1 \to 0$. It remains to estimate S_2. A calculation, analogous to (2.2), yields

$$S_2 \leq (1 + C_d)L^2 \sup_{t\in\mathbb{R}} \mathbb{E}\|u_\varepsilon^*(t) - u_{\varepsilon_1}^*(t)\|_{L^2(\mathbb{R}^d)}^2.$$

Altogether,

$$\mathbb{E}\|u_\varepsilon^*(t) - u_{\varepsilon_1}^*(t)\|_{L^2(\mathbb{R}^d)}^2 \leq \varphi(\varepsilon,\varepsilon_1) + 2aL^2(1 + C_d)\sup_{t\in\mathbb{R}}\mathbb{E}\|u_\varepsilon^*(t) - u_{\varepsilon_1}^*(t)\|_{L^2(\mathbb{R}^d)}^2,$$

where $\varphi(\varepsilon, \varepsilon_1) \to 0, \varepsilon, \varepsilon_1 \to 0$. In other words,

$$\sup_{t\in\mathbb{R}} \mathbb{E}\|u_\varepsilon^*(t) - u_{\varepsilon_1}^*(t)\|^2_{L^2(\mathbb{R}^d)} \le \frac{\varphi(\varepsilon, \varepsilon_1)}{1 - 2aL^2(1 + C_d)} \to 0, \quad \varepsilon, \varepsilon_1 \to 0.$$

Thus there is u_0^* such that $u_\varepsilon^* \to u_0^*$ as $\varepsilon \to 0$.

It remains to show that the sequence u_ε^* converges to the solution of the limiting equation

$$\begin{cases} du_0 = \Delta u_0 dt + \sigma(x, u_0) dW(t), \\ u_0(0) = u_0. \end{cases}$$

Denoting

$$S_0(t)u(\cdot) := \int_{\mathbb{R}^d} G(t, x - y)u(y)dy,$$

we have

$$\begin{aligned} &\mathbb{E}\|u_\varepsilon^*(t) - u^0(t)\|^2_{L^2(\mathbb{R}^d)} \\ &\quad \le 3\mathbb{E}\|u_\varepsilon^*(0) - u^0\|^2_{L^2(\mathbb{R}^d)} + \varepsilon^2 \mathbb{E}\left\|\int_0^t S_0(t-s)u_\varepsilon(s)ds\right\|^2_{L^2(\mathbb{R}^d)} \\ &\qquad + \mathbb{E}\left\|\int_0^t S_0(t-s)[\sigma(\cdot, u_\varepsilon^*) - \sigma(\cdot, u^0)]dW(s)\right\|^2_{L^2(\mathbb{R}^d)} \\ &\quad \le 3E\|u_\varepsilon^*(0) - u^0\|^2_{L^2(\mathbb{R}^d)} + B\varepsilon^2 + L^2 a \int_0^t \mathbb{E}\|u_\varepsilon^*(s) - u^0(s)\|^2_{L^2(\mathbb{R}^d)} ds \end{aligned}$$

for some constant $B > 0$. By Gronwall's inequality

$$\mathbb{E}\|u_\varepsilon^*(t) - u^0(t)\|^2_{L^2(\mathbb{R}^d)} \to 0, \quad \varepsilon \to 0.$$

In particular, since the stationary solutions u_ε^* strongly converge to u^0, u^0 is also stationary. This concludes the proof of the lemma, and thus the statement of the Theorem 1.2 follows. □

3. Perturbation Analysis. Proof of Theorem 1.4

Let $\lambda_1^\varepsilon < 0$ and $\lambda_1 < 0$ be the principle eigenvalues of $D_\varepsilon u := \Delta u - qu - \varepsilon q_1 u$ and $Du := \Delta u - qu$, respectively. Choose $\lambda := \frac{\lambda_1}{2} < 0$. Since $\lambda_1^\varepsilon = \lambda_1 + o_1(\varepsilon)$, we have $\lambda_1^\varepsilon < \lambda$ for sufficiently small ε. Denote $S_\varepsilon(t)$ and $S(t)$ to be the corresponding semigroups for D_ε and D. Then

$$\|S_\varepsilon(t)\|_{\mathcal{L}} \le e^{\lambda t}, \quad \|S(t)\|_{\mathcal{L}} \le e^{\lambda t}, \tag{3.1}$$

where $\|D\|_{\mathcal{L}}$ stands for the operator norm of D. By Theorem 1.1, (1.12) and (1.13) have unique stationary solutions, denoted with u_ε and u respectively, defined for

$t \in \mathbb{R}$. Moreover, arguing as in the proof of Lemma 2.1, it is straightforward to verify that

$$\sup_{t\in\mathbb{R}} \mathbb{E}\|u_\varepsilon(t)\|^2_{L^2(\mathbb{R}^d)} \leq \frac{\frac{4}{\lambda^2}\|f(\cdot,0)\|^2_{L^2(\mathbb{R}^d)} - \frac{4a}{\lambda}\|\sigma(\cdot,0)\|^2_{L^2(\mathbb{R}^d)}}{1 - L^2\left(\frac{4}{\lambda^2} - \frac{2a}{\lambda}\right)} := C$$

provided L satisfies the smallness condition (1.14). The following lemma establishes the convergence properties of the semigroup.

Lemma 3.1. *Assume q and q_1 satisfy the assumptions of Theorem* 1.3. *Then*

(i) *For any $u_0 \in L^2(\mathbb{R}^d)$ and $t \geq 0$ we have*

$$\|S_\varepsilon(t)u_0 - S(t)u_0\|_{L^2(\mathbb{R}^d)} \to 0, \quad \varepsilon \to 0.$$

(ii) *If, in addition, $q_1 \in L^\infty(\mathbb{R}^d)$, then*

$$\|S_\varepsilon(t)u_0 - S(t)u_0\|_{L^2(\mathbb{R}^d)} \leq c_\lambda \varepsilon \|q_1\|_{L^\infty(\mathbb{R}^d)} e^{\frac{\lambda}{2}t}\|u_0\|_{L^2(\mathbb{R}^d)}.$$

Proof. We start with the proof of (i). Write

$$u_0 = \sum_{k=1}^\infty c_k \Psi_k = \sum_{k=1}^\infty c_k^\varepsilon \Psi_k^\varepsilon$$

with

$$c_k = \int_{\mathbb{R}^d} u_0(x)\Psi_k(x)dx \quad \text{and} \quad c_k^\varepsilon = \int_{\mathbb{R}^d} u_0(x)\Psi_k^\varepsilon(x)dx.$$

In particular,

$$|c_k - c_k^\varepsilon| \leq \int_{\mathbb{R}^d} |\Psi_k^\varepsilon(x) - \Psi_k(x)||u_0(x)|dx \leq O(\varepsilon)\|u_0\|_{L^2(\mathbb{R}^d)}.$$

Then

$$\begin{aligned}
&\|S_\varepsilon(t)u_0 - S(t)u_0\|^2_{L^2(\mathbb{R}^d)} \\
&\quad = \left\|\sum_{k=1}^\infty e^{\lambda_k^\varepsilon t} c_k^\varepsilon \Psi_k^\varepsilon - e^{\lambda_k t} c_k \Psi_k\right\|^2_{L^2(\mathbb{R}^d)} \\
&\quad \leq 2\left\|\sum_{k=1}^N e^{\lambda_k^\varepsilon t} c_k^\varepsilon \Psi_k^\varepsilon - e^{\lambda_k t} c_k \Psi_k\right\|^2_{L^2(\mathbb{R}^d)} + 2\left\|\sum_{k=N}^\infty e^{\lambda_k^\varepsilon t} c_k^\varepsilon \Psi_k^\varepsilon - e^{\lambda_k t} c_k \Psi_k\right\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned} \tag{3.2}$$

Fix $\mu > 0$. For this μ we can find $N > 0$ such that

$$\left\|\sum_{k=N+1}^\infty e^{\lambda_k t} c_k \Psi_k\right\|^2_{L^2(\mathbb{R}^d)} \leq \sum_{k=N+1}^\infty |c_k|^2 \leq \frac{\mu}{4},$$

as well as

$$e^{2\lambda_N t}\|u_0\|^2_{L^2(\mathbb{R}^d)} \le \frac{\mu}{8}. \tag{3.3}$$

Since

$$\left\|\sum_{k=N+1}^{\infty} e^{\lambda_k^\varepsilon} c_k^\varepsilon \Psi_k^\varepsilon\right\|^2_{L^2(\mathbb{R}^d)} \le e^{2\lambda_N^\varepsilon t}\|u_0\|^2_{L^2(\mathbb{R}^d)},$$

from (3.3) and the convergence $\lambda_N^\varepsilon \to \lambda_N$ we may conclude that there is $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$

$$\left\|\sum_{k=N+1}^{\infty} e^{\lambda_k^\varepsilon} c_k^\varepsilon \Psi_k^\varepsilon\right\|^2_{L^2(\mathbb{R}^d)} \le \frac{\mu}{4},$$

thus the second term in (3.2) is less than $\frac{\mu}{2}$. As for the first term in (3.2), we have

$$\begin{aligned}
&\left\|\sum_{k=1}^{N} e^{\lambda_k^\varepsilon t} c_k^\varepsilon \Psi_k^\varepsilon - e^{\lambda_k t} c_k \Psi_k\right\|^2_{L^2(\mathbb{R}^d)} \\
&\quad \le 2\left\|\sum_{k=1}^{N} e^{\lambda_k t} c_k(\Psi_k^\varepsilon - \Psi_k)\right\|^2_{L^2(\mathbb{R}^d)} + 2\left\|\sum_{k=1}^{N} (e^{\lambda_k^\varepsilon t} c_k^\varepsilon - e^{\lambda_k t} c_k)\Psi_k^\varepsilon\right\|^2_{L^2(\mathbb{R}^d)} \\
&\quad \le 2N\sum_{k=1}^{N} e^{2\lambda_k t}|c_k|^2\|\Psi_k^\varepsilon - \Psi_k\|^2_{L^2(\mathbb{R}^d)} + 2N\sum_{k=1}^{N} |e^{\lambda_k^\varepsilon t} c_k^\varepsilon - e^{\lambda_k t} c_k|^2 \le \frac{\mu}{2}
\end{aligned}$$

for ε small enough. Thus, we have (i). To show (ii), note that

$$\begin{aligned}
&\|S_\varepsilon(t)u_0 - S_0(t)u_0\|_{L^2(\mathbb{R}^d)} \\
&\quad = \left\|\int_0^t S_0(t-s)\varepsilon q_1(\cdot)u_\varepsilon(s)ds\right\|_{L^2(\mathbb{R}^d)} \\
&\quad \le \varepsilon\|q_1\|_{L^\infty(\mathbb{R}^d)}\|u_0\|_{L^2(\mathbb{R}^d)}\int_0^t e^{\lambda(t-s)+\lambda s}ds = \varepsilon\|q_1\|_{L^\infty(\mathbb{R}^d)}\|u_0\|_{L^2(\mathbb{R}^d)} t e^{\lambda t} \\
&\quad \le \varepsilon c_\lambda\|q_1\|_{L^\infty(\mathbb{R}^d)}\|u_0\|_{L^2(\mathbb{R}^d)} e^{\frac{\lambda}{2}t}.
\end{aligned}$$

□

Introduce

$$u_\varepsilon^n := \int_{-n}^t S_\varepsilon(t-s)f(\cdot, u_\varepsilon(s))ds + \int_{-n}^t S_\varepsilon(t-s)\sigma(\cdot, u_\varepsilon(s))dW(s)$$

and

$$u^n := \int_{-n}^t S(t-s)f(\cdot, u(s))ds + \int_{-n}^t S(t-s)\sigma(\cdot, u(s))dW(s)$$

Then, for fixed $t \in \mathbb{R}$

$$\begin{aligned}
&\mathbb{E}\|u_\varepsilon(t) - u_\varepsilon^n(t)\|^2_{L^2(\mathbb{R}^d)} \\
&\quad \le 2\left[E\left\|\int_{-\infty}^{-n} S_\varepsilon(t-s)f(\cdot, u_\varepsilon(s))ds\right\|^2_{L^2(\mathbb{R}^d)}\right. \\
&\qquad \left. + E\left\|\int_{-\infty}^{-n} S_\varepsilon(t-s)\sigma(\cdot, u_\varepsilon(s))dW(s)\right\|^2_{L^2(\mathbb{R}^d)}\right] \\
&\quad \le 4\left(\int_{-\infty}^{-n} e^{\lambda(t-s)}ds\right)\int_{-\infty}^{-n} e^{\lambda(t-s)}(\|f(\cdot,0)\|^2_{L^2(\mathbb{R}^d)} + L^2\mathbb{E}\|u_\varepsilon\|^2_{L^2(\mathbb{R}^d)})ds \\
&\qquad + 4a\int_{-\infty}^{-n} e^{2\lambda(t-s)}(\|\sigma(\cdot,0)\|^2_{L^2(\mathbb{R}^d)} + L^2\mathbb{E}\|u_\varepsilon\|^2_{L^2(\mathbb{R}^d)})ds \\
&\quad \le 4(\|f(\cdot,0)\|^2_{L^2(\mathbb{R}^d)} + L^2C)\left(\int_{-\infty}^{-n} e^{\lambda(t-s)}ds\right)^2 \\
&\qquad + 4a(\|\sigma(\cdot,0)\|^2_{L^2(\mathbb{R}^d)} + L^2C)\int_{-\infty}^{-n} e^{\lambda(t-s)}ds \\
&\quad \le B_1 e^{2\lambda(t+n)} \to 0, \quad n \to \infty, \qquad (3.4)
\end{aligned}$$

with B_1 independent of ε. Similarly,

$$\mathbb{E}\|u(t) - u^n(t)\|^2_{L^2(\mathbb{R}^d)} \le B_1 e^{2\lambda(t+n)} \to 0, \quad n \to \infty.$$

For fixed $t \ge 0$, we have

$$\begin{aligned}
&\mathbb{E}\|u_\varepsilon^n(t) - u^n(t)\|^2_{L^2(\mathbb{R}^d)} \\
&\quad \le 2\mathbb{E}\left\|\int_{-n}^{t}(S_\varepsilon(t-s)f(\cdot, u_\varepsilon(s)) - S(t-s)f(\cdot, u(s)))ds\right\|^2_{L^2(\mathbb{R}^d)} \\
&\qquad + 2\mathbb{E}\left\|\int_{-n}^{t}(S_\varepsilon(t-s)\sigma(\cdot, u_\varepsilon(s)) - S(t-s)\sigma(\cdot, u(s)))dW(s)\right\|^2_{L^2(\mathbb{R}^d)} \\
&\quad := 2(J_1 + J_2). \qquad (3.5)
\end{aligned}$$

Adding and subtracting $S_\varepsilon(t-s)f(\cdot, u_\varepsilon^n(s))$ and $S(t-s)f(\cdot, u^n(s))$, and using the contraction property (3.1) together with the Lipschitz property of f, we have

$$\begin{aligned}
J_1 \le \mathbb{E}\Bigg[\int_{-n}^{t}&(e^{\lambda(t-s)}L\|u_\varepsilon(s) - u_\varepsilon^n(s)\|_{L^2(\mathbb{R}^d)} + e^{\lambda(t-s)}L\|u(s) - u^n(s)\|_{L^2(\mathbb{R}^d)} \\
&+ \|S_\varepsilon(t-s)f(\cdot, u_\varepsilon^n(s)) - S(t-s)f(\cdot, u^n(s))\|_{L^2(\mathbb{R}^d)})ds\Bigg]^2
\end{aligned}$$

$$\leq 3\mathbb{E}\left(\int_{-n}^{t} e^{\lambda(t-s)} L\|u_\varepsilon(s) - u_\varepsilon^n(s)\|_{L^2(\mathbb{R}^d)} ds\right)^2$$

$$+ 3\mathbb{E}\left(\int_{-n}^{t} e^{\lambda(t-s)} L\|u(s) - u^n(s)\|_{L^2(\mathbb{R}^d)} ds\right)^2$$

$$+ 3\mathbb{E}\left(\int_{-n}^{t} \|S_\varepsilon(t-s)f(\cdot, u_\varepsilon^n(s)) - S(t-s)f(\cdot, u^n(s))\|_{L^2(\mathbb{R}^d)} ds\right)^2$$

$$:= 3(J_{11} + J_{12} + J_{13}).$$

By (3.4) and Cauchy–Schwartz, we have

$$J_{11} \leq \int_{-n}^{t} e^{\lambda(t-s)} ds \cdot \int_{-n}^{t} L^2 e^{\lambda(t-s)} \|u(s) - u^n(s)\|_{L^2(\mathbb{R}^d)}^2 ds$$

$$\leq -\frac{1}{\lambda} \int_{-n}^{t} e^{\lambda(t-s)} L^2 B_1 e^{2\lambda(s+n)} ds = -B_1 \frac{L^2}{\lambda} e^{\lambda(2n+t)} \int_{-n}^{t} e^{\lambda s} ds$$

$$\leq B_1 \frac{L^2}{\lambda^2} e^{\lambda(t+n)}.$$

Similarly,

$$J_{12} \leq B_1 \frac{L^2}{\lambda^2} e^{\lambda(t+n)}.$$

Finally,

$$J_{13} \leq \mathbb{E}\left[\int_{-n}^{t} \|S_\varepsilon(t-s)(f(\cdot, u_\varepsilon^n) - f(\cdot, u^n))\|_{L^2(\mathbb{R}^d)} ds\right.$$

$$\left. + \int_{-n}^{t} \|S_\varepsilon(t-s)(f(\cdot, u^n) - S(t-s)f(\cdot, u^n))\|_{L^2(\mathbb{R}^d)} ds\right]^2$$

$$\leq 2(J_{131} + J_{132})$$

We have

$$J_{131} \leq \mathbb{E}\left(\int_{-n}^{t} e^{\lambda(t-s)} L\|u_\varepsilon^n - u^n\|_{L^2(\mathbb{R}^d)} ds\right)^2$$

$$\leq -\frac{L^2}{\lambda} \int_{-n}^{t} e^{\lambda(t-s)} \mathbb{E}\|u_\varepsilon^n(s) - u^n(s)\|_{L^2(\mathbb{R}^d)}^2 ds.$$

Also, by Dominated Convergence Theorem and Lemma 3.1

$$J_{132} := \varphi_1(t, n, \varepsilon) \to 0, \quad \varepsilon \to 0.$$

Altogether,

$$J_1 \leq \frac{6}{\lambda^2} e^{\lambda(t+n)} - \frac{6L^2}{\lambda} \int_{-n}^{t} e^{\lambda(t-s)} \mathbb{E}\|u_\varepsilon^n(s) - u^n(s)\|_{L^2(\mathbb{R}^d)}^2 ds + 6\varphi_1(t, n, \varepsilon). \tag{3.6}$$

We now proceed with the estimate for J_2:

$$\begin{aligned} J_2 \leq\ & 3\mathbb{E}\left\|\int_{-n}^{t} S_\varepsilon(t-s)(\sigma(\cdot,u_\varepsilon(s))-\sigma(\cdot,u_\varepsilon^n(s)))dW(s)\right\|_{L^2(\mathbb{R}^d)}^2 \\ & + 3\mathbb{E}\left\|\int_{-n}^{t} S_\varepsilon(t-s)(\sigma(\cdot,u_\varepsilon^n(s)))-S(t-s)(\sigma(\cdot,u^n(s)))dW(s)\right\|_{L^2(\mathbb{R}^d)}^2 \\ & + 3\mathbb{E}\left\|\int_{-n}^{t} S(t-s)(\sigma(\cdot,u^n(s))-\sigma(\cdot,u(s)))dW(s)\right\|_{L^2(\mathbb{R}^d)}^2 \\ :=\ & 3(J_{21}+J_{22}+J_{23}). \end{aligned}$$

Using (3.4), we have

$$\begin{aligned} J_{21} & \leq L^2\sum_k a_k\int_{-n}^{t} e^{2\lambda(t-s)}\mathbb{E}\|u_\varepsilon(s)-u_\varepsilon^n(s)\|_{L^2(\mathbb{R}^d)}^2 ds \\ & \leq L^2 a\int_{-n}^{t} e^{2\lambda(t-s)}B_1 e^{2\lambda(s+n)}ds = aL^2B_1e^{2\lambda(t+n)}(t+n)\to 0, \quad n\to\infty. \end{aligned}$$

A similar estimate holds for J_{23}. It remains to estimate J_{22}:

$$\begin{aligned} J_{22} =\ & \mathbb{E}\left\|\int_{-n}^{t}[S_\varepsilon(t-s)(\sigma(\cdot,u_\varepsilon^n)-\sigma(\cdot,u^n)) \right. \\ & \left. + (S_\varepsilon(t-s)-S(t-s))\sigma(\cdot,u^n)]dW(s)\right\|_{L^2(\mathbb{R}^d)}^2 \\ \leq\ & 2\mathbb{E}\left\|\int_{-n}^{t} S_\varepsilon(t-s)[\sigma(\cdot,u_\varepsilon^n)-\sigma(\cdot,u^n)]dW(s)\right\|_{L^2(\mathbb{R}^d)}^2 \\ & + 2\mathbb{E}\left\|\int_{-n}^{t}[S_\varepsilon(t-s)\sigma(\cdot,u^n(s))-S(t-s)\sigma(\cdot,u^n(s))]dW(s)\right\|_{L^2(\mathbb{R}^d)}^2 \\ :=\ & 2(J_{221}+J_{222}). \end{aligned}$$

The first term is estimated by Cauchy–Schwart

$$J_{221} \leq -\frac{aL^2}{\lambda}\int_{-n}^{t} e^{\lambda(t-s)}\mathbb{E}\|u_\varepsilon^n(s)-u^n(s)\|_{L^2(\mathbb{R}^d)}^2 ds.$$

On the other hand,

$$\begin{aligned} & \int_{-n}^{t}[S_\varepsilon(t-s)\sigma(\cdot,u^n(s))-S(t-s)\sigma(\cdot,u^n(s))]dW(s) \\ & \quad = \sum_k\sqrt{a_k}\int_{-n}^{t}[S_\varepsilon(t-s)\sigma(\cdot,u^n(s))e_k-S(t-s)\sigma(\cdot,u^n(s))e_k]d\beta_k(s), \end{aligned}$$

thus

$$\begin{aligned} J_{222} &= \sum_k a_k \int_{-n}^t \mathbb{E}\|S_\varepsilon(t-s)\sigma(\cdot, u^n(s))e_k - S(t-s)\sigma(\cdot, u^n(s))e_k\|^2_{L^2(\mathbb{R}^d)} ds \\ &:= \varphi_2(t,n,\varepsilon) \to 0 \end{aligned}$$

as $\varepsilon \to 0$ by the Dominated Convergence Theorem and Lemma 3.1. Altogether

$$J_2 \leq 3[B_2 e^{\lambda(t+n)} - \frac{2aL^2}{\lambda}\int_{-n}^t e^{\lambda(t-s)}\mathbb{E}\|u_\varepsilon^n(s) - u^n(s)\|^2_{L^2(\mathbb{R}^d)} ds + 2\varphi_2(t,n,\varepsilon)]. \tag{3.7}$$

Combining (3.5)–(3.7), we have

$$\begin{aligned} &\mathbb{E}\|u_\varepsilon^n(t) - u^n(t)\|^2_{L^2(\mathbb{R}^d)} \\ &\quad \leq B_3 e^{\lambda(t+n)} + \varphi_3(t,n,\varepsilon) - \frac{12L^2(1+a)}{\lambda}\int_{-n}^t e^{\lambda(t-s)}\mathbb{E}\|u_\varepsilon^n(s) - u^n(s)\|^2_{L^2(\mathbb{R}^d)} ds \end{aligned}$$

By Gronwall inequality, denoting $\gamma := -\frac{12L^2(1+a)}{\lambda} > 0$, we have

$$\begin{aligned} \mathbb{E}\|u_\varepsilon^n(t) - u^n(t)\|^2_{L^2(\mathbb{R}^d)} &\leq B_3 e^{\lambda(t+n)} + \varphi_3(t,n,\varepsilon) \\ &\quad + \gamma\int_{-n}^t e^{\gamma(t-s)} e^{\lambda(t-s)}(B_3 e^{\lambda(s+n)} + \varphi_3(s,n,\varepsilon)) ds. \end{aligned}$$

The condition (1.15) implies that $\lambda + \gamma < 0$, thus we have

$$\int_{-n}^t e^{\gamma(t-s)} e^{\lambda(t-s)} e^{\lambda(s+n)} ds = \frac{1}{\gamma}[e^{(\lambda+\gamma)(t+n)} - e^{\lambda(t+n)}] \to 0, \quad n \to \infty.$$

To complete the proof of Theorem 1.4, fix $\delta > 0$ and choose $n \geq 1$ such that

$$2B_1 e^{2\lambda(t+n)} + B_3 e^{\lambda(t+n)} + \frac{B_3}{\gamma}[e^{(\lambda+\gamma)(t+n)} - e^{\lambda(t+n)}] < \frac{\delta}{2}.$$

For such n we may now choose $\varepsilon > 0$ such that

$$\varphi_3(t,n,\varepsilon) + \gamma\int_{-n}^t e^{\gamma(t-s)} e^{\lambda(t-s)}\varphi_3(s,n,\varepsilon) ds < \frac{\delta}{2}$$

and therefore

$$\mathbb{E}\|u_\varepsilon(t) - u(t)\|^2_{L^2(\mathbb{R}^d)} \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

which completes the proof.

4. Homogenization

4.1. *Notation and preliminaries*

Assume that $A(x)$ is uniformly elliptic

$$\nu|\xi|^2 \leq A(x)\xi\cdot\xi \leq \frac{1}{\nu}|\xi|^2, \quad 0 < \nu \leq 1, \ \ \xi \in \mathbb{R}^d. \tag{4.1}$$

Denote $A_0(x)$ to be the homogenized matrix (see [25, Sec. 1.2] for the precise definition). It follows from linear homogenization theory that A_0 satisfies (4.1) given that $A(x)$ satisfies it. Next, let

- $K(x, y, t)$ be the fundamental solution of
$$u_t = \mathrm{div}(A(x)\nabla u);$$
- $K_\varepsilon(x, y, t)$ be the fundamental solution of
$$u_t = \mathrm{div}(A(x/\varepsilon)\nabla u);$$
- $K_0(x, y, t)$ be the fundamental solution of
$$u_t = \mathrm{div}(A_0(x)\nabla u).$$
- $$S_\varepsilon(t)u := \int_{\mathbb{R}^d} K_\varepsilon(x, y, t)u(y)dy; \quad S_0(t)u := \int_{\mathbb{R}^d} K_0(x, y, t)u(y)dy.$$

A simple rescaling yields
$$K_\varepsilon(x, y, t) = \varepsilon^{-d} K\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{t}{\varepsilon^2}\right).$$

We also assume that $B_{f,\sigma} > 0$ is such that
$$|f(x, u)| \le B_{f,\sigma} \quad \text{and} \quad |\sigma(x, u)| \le B_{f,\sigma}.$$

Let $\Omega = B_R(0)$, $Y = \Pi = [0, 1]^d$. Following [1], denote $L^1[\Omega, C_\#(Y)]$ to be the space of functions, measurable and integrable in $x \in \Omega$, with values in the Banach space of continuous functions, Y-periodic in y. We are going to make use of the following lemmas to characterize the space $L^1[\Omega, C_\#(Y)]$.

Lemma 4.1. ([1, Lemma 5.3]) *A function $\psi(x, y)$ belongs to $L^1[\Omega, C_\#(Y)]$ if and only if there exists a subset E (independent of y) of measure zero in Ω such that*

(i) *For any $x \in \Omega\backslash E$, the function $y \to \psi(x, y)$ is continuous and Y-periodic;*
(ii) *For any $y \in Y$, the function $x \to \psi(x, y)$ is measurable on Ω.*
(iii) *The function $x \to \sup_y |\psi(x, y)|$ has a finite $L^1(\Omega)$ norm.*

Lemma 4.2. ([1, Lemma 5.2]) *If $\psi(x, y) \in L^1[\Omega, C_\#(Y)]$, then*
$$\int_\Omega \psi\left(x, \frac{x}{\varepsilon}\right) dx \to \int_\Pi \int_\Omega \psi(x, y)dxdy, \quad \varepsilon \to 0.$$

4.2. *Nash–Aranson estimates. Semigroup estimates*

Lemma 4.3. ([25, Chap. 2]) *There exists $C_N = C(d, \nu)$ such that*
$$0 \le K(x, y, t) \le C_N t^{-d/2} e^{-\frac{\nu|x-y|^2}{8t}};$$
$$0 \le K_0(x, y, t) \le C_N t^{-d/2} e^{-\frac{\nu|x-y|^2}{8t}}.$$

If $S^{(m)}(t)u$ is the semigroup, generated by $\Delta u - mu$ in $L^2(\mathbb{R}^d)$, then $S^{(m)}(t)u = e^{-mt}S^{(0)}(t)u$ and thus

$$\|S^{(m)}(t)u\|_{L^2(\mathbb{R}^d)} \leq e^{-mt}\|u\|_{L^2(\mathbb{R}^d)}. \tag{4.2}$$

An analog of this estimate holds in weighted spaces.

Lemma 4.4. ([11, Proposition 9.4.5]) *Let* $u \in L^2_\rho(\mathbb{R}^d)$ *with* $\rho(x) = e^{-\kappa|x|}$. *Then*

$$\|S^{(m)}(t)u\|_\rho \leq e^{(\kappa^2/2-m)t}\|u\|_\rho. \tag{4.3}$$

A similar estimate holds for $\rho(x) = \frac{1}{1+\kappa|x|^r}, \kappa > 0, r > d$.

Rescaling time $s = tm$, one can see that the estimates (4.2) and (4.3) are valid for the semigroup, generated by $\frac{1}{m}\Delta u - u$ in $L^2(\mathbb{R}^d)$ and $L^2_\rho(\mathbb{R}^d)$, respectively. Combining Lemmas 4.3 and 4.4, for any $\varphi \in L^2_\rho(\mathbb{R}^d)$ we have

$$\begin{aligned}
\|S_\varepsilon(t)\varphi\|_\rho^2 &= \int_{\mathbb{R}^d} e^{-2t}\left|\int_{\mathbb{R}^d} K_\varepsilon(x,y,t)\varphi(y)dy\right|^2 \rho(x)dx \\
&\leq \int_{\mathbb{R}^d} e^{-2t}\left(\int_{\mathbb{R}^d} C_N t^{-d/2} e^{-\frac{|x-y|^2}{8t/\nu}}|\varphi(y)|dy\right)^2 \rho(x)dx \\
&= \left(\frac{8\pi}{\nu}\right)^d C_N^2 \int_{\mathbb{R}^d} e^{-2t}\left(\int_{\mathbb{R}^d} \frac{1}{(8\pi t/\nu)^{d/2}} e^{\frac{-|x-y|^2}{8t/\nu}}|\varphi|dy\right)^2 \rho(x)dx \\
&= \left(\frac{8\pi}{\nu}\right)^d C_N^2 \|S^{(\nu)}(t/\nu)|\varphi(\cdot)|\|_\rho^2 \leq \left(\frac{8\pi}{\nu}\right)^d C_N^2 e^{2(\kappa^2/2-\nu)t/\nu}\|\varphi\|_\rho^2.
\end{aligned}$$

In other words, if

$$\kappa^2/2 < \nu, \tag{4.4}$$

we showed that

$$\|S_\varepsilon(t)\varphi\|_\rho \leq C_\nu e^{-\gamma t}\|\varphi\|_\rho, \tag{4.5}$$

with $\gamma := 1 - \frac{\kappa^2}{2\nu} > 0$ and $C_\nu^2 = (\frac{8\pi}{\nu})^d C_N^2$. Similarly,

$$\|S_0(t)\varphi\|_\rho \leq C_\nu e^{-\gamma t}\|\varphi\|_\rho.$$

We may now apply Theorem 1.1 to conclude that, the Lipschits constant L satisfies the smallness conditions [18]:

$$C_\nu^2 L^2 \frac{4+2\gamma a}{\gamma^2} < 1 \tag{4.6}$$

and

$$C_\nu^2 L^2 \frac{1+\gamma a}{\gamma} < \frac{2}{3}, \tag{4.7}$$

then there exist unique stationary solutions u_ε and u_0 of (1.16) and (1.17), respectively, defined for $t \in \mathbb{R}$, such that

$$\sup_{t\in\mathbb{R}} \mathbb{E}\|u_\varepsilon(t)\|_\rho^2 < C \quad \text{and} \quad \sup_{t\in\mathbb{R}} \mathbb{E}\|u_0(t)\|_\rho^2 < C.$$

Finally, assume that L satisfies one more smallness condition

$$12\left(\frac{1}{\gamma}+a\right)C_\nu^2 L^2 < \gamma. \tag{4.8}$$

4.3. *Proof of Theorem* 1.5

For fixed $t > 0$ we have

$$u_\varepsilon(t) = \int_{-\infty}^t S_\varepsilon(t-s) f\left(\frac{\cdot}{\varepsilon}, u_\varepsilon(s,\cdot)\right) ds + \int_{-\infty}^t S_\varepsilon(t-s)\sigma\left(\frac{\cdot}{\varepsilon}, u_\varepsilon(s,\cdot)\right) dW(s)$$

and

$$u_0(t) = \int_{-\infty}^t S_0(t-s) f_0(u_0(s,\cdot))ds + \int_{-\infty}^t S_0(t-s)\sigma_0(u_0(s,\cdot))dW(s).$$

For $n \geq 1$ denote

$$u_\varepsilon^n(t) := \int_{-n}^t S_\varepsilon(t-s) f\left(\frac{\cdot}{\varepsilon}, u_\varepsilon(s,\cdot)\right) ds + \int_{-n}^t S_\varepsilon(t-s)\sigma\left(\frac{\cdot}{\varepsilon}, u_\varepsilon(s,\cdot)\right) dW(s)$$

and

$$u_0^n(t) := \int_{-n}^t S_0(t-s) f_0(u_0(s,\cdot))ds + \int_{-n}^t S_0(t-s)\sigma_0(u_0(s,\cdot))dW(s).$$

Then

$$\begin{aligned}\mathbb{E}\|u_\varepsilon^n(t) - u_\varepsilon(t)\|_\rho^2 &\leq 2\mathbb{E}\left\|\int_{-\infty}^{-n} S_\varepsilon(t-s) f\left(\frac{\cdot}{\varepsilon}, u_\varepsilon(s,\cdot)\right) ds\right\|_\rho^2 \\ &\quad + 2\mathbb{E}\left\|\int_{-\infty}^{-n} S_\varepsilon(t-s)\sigma\left(\frac{\cdot}{\varepsilon}, u_\varepsilon(s,\cdot)\right) dW(s)\right\|_\rho^2 \\ &\leq B_1 e^{-2\gamma(t+n)} \to 0, \quad n\to\infty, \end{aligned} \tag{4.9}$$

where B_1 is independent of n, t and ε. Similarly,

$$\mathbb{E}\|u_0^n(t) - u_0(t)\|_\rho^2 \leq B_1 e^{-2\gamma(t+n)} \to 0, \quad n\to\infty.$$

Then

$$\|u_\varepsilon(t) - u_0(t)\|_\rho^2 \leq 3(\|u_\varepsilon(t) - u_\varepsilon^n(t)\|_\rho^2 + \|u_0(t) - u_0^n(t)\|_\rho^2 + \|u_0^n(t) - u_\varepsilon^n(t)\|_\rho^2). \tag{4.10}$$

We now estimate the last term in (4.10).

$$\begin{aligned}&\mathbb{E}\|u_0^n(t) - u_\varepsilon^n(t)\|_\rho^2 \\ &\quad\leq 2\mathbb{E}\left\|\int_{-n}^t \left(S_\varepsilon(t-s) f\left(\frac{\cdot}{\varepsilon}, u_\varepsilon(s,\cdot)\right) - S_0(t-s) f(u_0(s,\cdot))\right) ds\right\|_\rho^2 \\ &\qquad + 2\mathbb{E}\left\|\int_{-n}^t \left(S_\varepsilon(t-s)\sigma\left(\frac{\cdot}{\varepsilon}, u_\varepsilon(s,\cdot)\right) - S_0(t-s)\sigma(u_0(s,\cdot))dW(s)\right)\right\|_\rho^2 \\ &\quad := J_1 + J_2. \end{aligned} \tag{4.11}$$

Once we add and subtract $S_\varepsilon(t-s)f(\cdot/\varepsilon, u_n^\varepsilon)$ and $S_0(t-s)f(u_0)$, the term J_1 may be estimated as

$$J_1 \le 3(J_{11} + J_{12} + J_{13})$$

Using (4.9) and the Lipschitz property, one may obtain the bounds for J_{11}

$$J_{11} := C_1 L^2 \mathbb{E}\left(\int_{-n}^{t} e^{-\gamma(t-s)}\|u_\varepsilon(s) - u_\varepsilon^n(s)\|_\rho ds\right)^2 \le B_2 e^{-\gamma(t+n)}$$

and

$$J_{12} := C_1 L^2 \mathbb{E}\left(\int_{-n}^{t} e^{-\gamma(t-s)}\|u_0(s) - u_0^n(s)\|_\rho ds\right)^2 \le B_2 e^{-\gamma(t+n)}.$$

We now proceed with the estimate for J_{13}. Using (4.5), we have

$$\begin{aligned} J_{13} \le\ & 2\mathbb{E}\left(\int_{-n}^{t} \sqrt{C_1} e^{-\gamma(t-s)} L\|u_\varepsilon^n(s) - u_0^n(s)\|_\rho ds\right)^2 \\ & + 2\mathbb{E}\left(\int_{-n}^{t} \left\|S_\varepsilon(t-s) f\left(\frac{\cdot}{\varepsilon}, u_0^n(s)\right) - S_0(t-s) f_0(u_0^n(s))\right\|_\rho ds\right)^2. \end{aligned}$$

The last term may be rewritten as

$$\begin{aligned} & \mathbb{E}\left(\int_{-n}^{t} \left\|S_\varepsilon(t-s) f\left(\frac{\cdot}{\varepsilon}, u_0^n(s)\right) - S_0(t-s) f_0(u_0^n(s))\right\|_\rho ds\right)^2 \\ & \quad \le \int_{-n}^{t} e^{-(t-s)} \varphi_1(t, s, \varepsilon, n) ds, \end{aligned}$$

where

$$\begin{aligned} \varphi_1(t, s, \varepsilon, n) := \mathbb{E}\int_{\mathbb{R}^d} \Bigg(\int_{\mathbb{R}^d} & K_\varepsilon(x, y, t-s) f\left(\frac{y}{\varepsilon}, u_0^n(s)\right) \\ & - K_0(x, y, t-s) f_0(u_0^n(s))\Bigg)^2 \rho(x) dx. \end{aligned} \tag{4.12}$$

Lemma 4.5. *For fixed t, s and n, $\varphi_1(t, s, n, \varepsilon)$ converges to 0 as $\varepsilon \to 0$.*

Proof. For $R > 0$ we can write (4.12) as

$$\varphi_1(t, s, n, \varepsilon) = \int_{|x|\ge R} \{\ldots\}\rho(x)dx + \int_{|x|<R} \{\ldots\}\rho(x)dx := J_{131} + J_{132}.$$

Next, we are going to use the facts that $|f(x, u)| \le B_{f,\sigma}$ is uniformly bounded, and that $\int_{\mathbb{R}^d} K_\varepsilon(x, y, t)dy = \int_{\mathbb{R}^d} K_0(x, y, t)dy = 1$, in order to estimate J_{131}. We have

$$J_{131} \le 4B_{f,\sigma}^2 \int_{|x|\ge R} \rho(x)dx.$$

Therefore, for any $\eta > 0$ there exists $R > 0$ such that

$$J_{131} \le \frac{\eta}{2}.$$

For such R we proceed with the estimate for

$$J_{132} := \mathbb{E}\int_{|x|<R}\left(\int_{\mathbb{R}^d}(K_\varepsilon(x,y,t-s)f\left(\frac{y}{\varepsilon},u_0^n(s)\right)\right.$$
$$\left.-K_0(x,y,t-s)f_0(u_0^n(s)))dy\right)^2\rho(x)dx$$

Let $h(x)\in L^2(\Omega)$ be a fixed function. By Lemmas 4.1 and 4.2

$$\psi(x,y) := f(y,u_0^n(x,s,\omega))h(x)\in L^1[B_R(0),C_\#(Y)].$$

thus for a.e. ω

$$\int_{|x|<R}\psi\left(x,\frac{x}{\varepsilon}\right)dx \to \int_\Pi\int_{|x|<R}h(x)f(y,u_0^n(s,x,\omega))dxdy,\quad \varepsilon\to 0.$$

This, in turn, implies that

$$f\left(\frac{x}{\varepsilon},u_0^n(s,x,\omega)\right)\rightharpoonup f_0(u_0^n(s,x,\omega))\quad\text{in } L^2_{\text{loc}}(\mathbb{R}^d),\quad \varepsilon\to 0.$$

We are now in position to apply the convergence result [25, Chap. 2, Theorem 1] to conclude that

$$\int_{\mathbb{R}^d}\left(K_\varepsilon(x,y,t-s)f\left(\frac{y}{\varepsilon},u_0^n(s)\right)-K_0(x,y,t-s)f_0(u_0^n(s))\right)dy\to 0,\quad \varepsilon\to 0$$

in $L^2_{\text{loc}}(\mathbb{R}^d)$. By Dominated Convergence Theorem, choosing $\varepsilon>0$ small enough, we have

$$J_{132}<\frac{\eta}{2},$$

or $\varphi_1(t,s,n,\varepsilon)<\eta$, which concludes the proof of Lemma 4.5. □

Therefore,

$$J_{13}\le\frac{2C_\nu^2}{\gamma}L^2\int_{-n}^t e^{-\gamma(t-s)}\mathbb{E}\|u_\varepsilon^n(s)-u_0^n(s)\|_\rho^2ds+\int_{-n}^t e^{-(t-s)}\varphi_1(t,s,\varepsilon,n)ds,$$

and combining the bounds for J_{11}, J_{12} and J_{13} we arrive at

$$J_1\le B_1e^{-\gamma(t+n)}+\frac{6C_\nu^2L^2}{\gamma}\int_{-n}^t e^{-\gamma(t-s)}\mathbb{E}\|u_\varepsilon^n(s)-u_0^n(s)\|_\rho^2ds$$
$$+3\int_{-n}^t e^{-(t-s)}\varphi_1(t,s,\varepsilon,n)ds. \tag{4.13}$$

We now estimate the stochastic contribution J_2. In a similar manner, we write

$$J_2\le 3(J_{21}+J_{22}+J_{23}),$$

where

$$J_{21} := \mathbb{E}\left\|\int_{-n}^{t} S_\varepsilon(t-s)(\sigma(\cdot/\varepsilon, u_\varepsilon) - \sigma(\cdot/\varepsilon, u_\varepsilon^n))dW(s)\right\|_\rho^2$$
$$\leq C_\nu a \int_{-n}^{t} e^{-2\gamma(t-s)} L^2 \mathbb{E}\|u_\varepsilon(s) - u_\varepsilon^n(s)\|_\rho^2 ds \leq B_3 e^{-\gamma(t+n)}.$$

Analogously,

$$J_{22} := \mathbb{E}\left\|\int_{-n}^{t} S_0(t-s)(\sigma_0(u_0) - \sigma_0(u_0^n))dW(s)\right\|_\rho^2 \leq B_3 e^{-\gamma(t+n)}.$$

We proceed with the estimate for J_{23}:

$$J_{23} = \mathbb{E}\left\|\int_{-n}^{t} S_\varepsilon(t-s)(\sigma(\cdot/\varepsilon, u_\varepsilon^n) - \sigma_0(u_0^n))dW(s)\right\|_\rho^2$$
$$= \sum_{k=1}^{\infty} a_k \int_{-n}^{t} \mathbb{E}\|S_\varepsilon(t-s)\sigma(\cdot/\varepsilon, u_\varepsilon^n)e_k(\cdot) - S_0(t-s)\sigma_0(u_0^n)e_k(\cdot)\|_\rho^2 ds.$$

Adding and subtracting $S_\varepsilon(t-s)\sigma(\cdot/\varepsilon, u_0^n)$, we have

$$J_{23} \leq 2\sum_{k=1}^{\infty} a_k \int_{-n}^{t} \mathbb{E}\|S_\varepsilon(t-s)[\sigma(\cdot/\varepsilon, u_\varepsilon^n) - \sigma(\cdot/\varepsilon, u_0^n)]e_k(\cdot)\|_\rho^2 ds$$
$$+ 2\sum_{k=1}^{\infty} a_k \int_{-n}^{t} \mathbb{E}\|S_\varepsilon(t-s)\sigma(\cdot/\varepsilon, u_0^n)e_k(\cdot) - S_0(t-s)\sigma_0(u_0^n)e_k(\cdot)\|_\rho^2 ds$$
$$\leq 2aC_\nu^2 L^2 \int_{-n}^{t} e^{-\gamma(t-s)} \mathbb{E}\|u_\varepsilon^n(s) - u_0^n(s)\|_\rho^2 ds$$
$$+ 2\sum_{k=1}^{\infty} a_k \int_{-n}^{t} e^{-\gamma(t-s)} \varphi_2(t,s,\varepsilon,n,k) ds,$$

where

$$\varphi_2(t,s,\varepsilon,n,k) := \mathbb{E}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} [K_\varepsilon(x,y,t-s)\sigma(y/\varepsilon, u_0^n)e_k(y) - K_0(x,y,t-s)\sigma_0(u_0^n)e_k(y)dy\right)^2 \rho(x)dx.$$

By an argument, identical to Lemma 4.5, $\varphi_2(t,s,\varepsilon,n,k)$ is uniformly bounded by an absolute constant, and $\varphi_2(t,s,\varepsilon,n,k) \to 0, \varepsilon \to 0$. Consequently,

$$\varphi_3(t,s,\varepsilon,n) := \sum_{k=1}^{\infty} a_k \varphi_2(t,s,\varepsilon,n,k) \to 0, \quad \varepsilon \to 0.$$

Altogether, we have

$$J_2 \leq B_3 e^{-\gamma(t+n)} + 6aC_\nu^2 L^2 \int_{-n}^{t} e^{-\gamma(t-s)} \mathbb{E}\|u_\varepsilon^n(s) - u_0^n(s)\|_\rho^2 ds$$

$$+ 6 \int_{-n}^{t} e^{-(t-s)} \varphi_3(t, s, \varepsilon, n) ds]. \tag{4.14}$$

Now, combining (4.14) with (4.13) and (4.11), we have

$$\mathbb{E}\|u_\varepsilon^n(t) - u_0^n(t)\|_\rho^2 \leq B_4 e^{-\gamma(t+n)} + \int_{-n}^{t} e^{-\gamma(t-s)} \varphi_4(t, s, \varepsilon, n) ds \tag{4.15}$$

$$+ 12 \left(\frac{1}{\gamma} + a\right) C_\nu^2 L^2 \int_{-n}^{t} e^{-\gamma(t-s)} \mathbb{E}\|u_\varepsilon^n(s) - u_0^n(s)\|_\rho^2 ds, \tag{4.16}$$

where $\varphi_4(t, \varepsilon, n) = 12 \int_{-n}^{t} (\varphi_1 + \varphi_3)(t, s, \varepsilon, n) ds \to 0, \varepsilon \to 0$. Denoting

$$z(t) := e^{\gamma t} \mathbb{E}\|u_\varepsilon^n(t) - u_0^n(t)\|_\rho^2,$$

and

$$H := 12 \left(\frac{1}{\gamma} + a\right) C_\nu L^2$$

(4.16) may be rewritten as

$$z(t) \leq B_4 e^{-\gamma n} + \varphi_4(t, \varepsilon, n) + H \int_{-n}^{t} z(s) ds,$$

and hence, by Gronwall inequality

$$z(t) \leq B_4 e^{-\gamma n} + \varphi_4(t, \varepsilon, n) + \int_{-n}^{t} e^{H(t-s)} (B_4 e^{-\gamma n} + \varphi_4(s, \varepsilon, n)) ds.$$

If (4.8) holds, for fixed $t \geq 0$ and any $\eta > 0$ we can choose $n \geq 1$ such that

$$B_4 e^{-\gamma n} + B_4 e^{-\gamma n} \int_{-n}^{t} e^{H(t-s)} ds < \frac{\eta}{2}.$$

For such n we may now choose $\varepsilon > 0$ such that

$$\varphi_4(t, \varepsilon, n) + \int_{-n}^{t} e^{H(t-s)} \hat{\varphi}(s, \varepsilon, n) ds < \frac{\eta}{2}$$

and hence

$$z(t) = e^{\gamma t} \mathbb{E}\|u_\varepsilon^n(t) - u_0^n(t)\|_\rho^2 < \eta.$$

This completes the proof of Theorem 1.5.

Proof of Theorem 1.6. Let $S_\varepsilon(t)$ and $S_0(t)$ be the semigroups generated by A_ε and A_0 with Dirichlet boundary conditions $u = 0$ on ∂G. Then

$$\|S_\varepsilon \varphi\|_{L^2(G)} \leq \tilde{C}_\nu e^{-\tilde{\gamma} t} \|\varphi\|_{L^2(G)}$$

and

$$\|S_0\varphi\|_{L^2(G)} \leq \tilde{C}_\nu e^{-\tilde{\gamma}t}\|\varphi\|_{L^2(G)},$$

where $\tilde{\gamma} > 0$ depends on ν and on the first eigenvalue of the Laplace operator in G. Provided (4.6)–(4.8) hold with C_ν and γ replaced with $\tilde{C}_\nu$ and $\tilde{\gamma}$ correspondingly, the rest of the proof of Theorem 1.6 follows the lines of Theorem 1.5. □

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References

1. G. Allaire, Homogenization and two-scale convergence, *SIAM J. Math. Anal.* **6** (1991) 1482–1518.
2. S. Assing and R. Manthey, Invariant measures for stochastic heat equations with unbounded coefficients, *Stochastic Process. Appl.* **103** (2003) 237–256.
3. A. Bensoussan, Homogenization of a class of stochastic partial differential equations, *Prog. Nonlinear Differential Equations Appl.* **5** (1991) 47–65.
4. H. Bessaih, Y. Efendiev and F. Maris, Homogenization of Brinkman flows in heterogeneous dynamic media, *Stoch Partial Differential Equations*: *Anal. Comput.* **3** (2015) 479–505.
5. V. Bogachev and M. Rockner, Elliptic equations for measures on infinite-dimensional spaces and applications, *Probab. Theory Related Fields* **120** (2001) 445–496.
6. Z. Brzezniak and D. Gatarek, Martingale solutions and invariant measures for stochastic evolution equations in Banach spaces, *Stochastic Process. Appl.* **84** (1999) 87–225.
7. S. Cerrai, *Second Order PDE's in Finite and Infinite Dimension* (Springer, 2001).
8. S. Cerrai, Stochastic reaction–diffusion systems with multiplicative noise and non-Lipschitz reaction term, *Probab. Theory Related Fields* **125** (2003) 271–304.
9. S. Cerrai, Asymptotic behavior of systems of stochastic partial differential equations with multiplicative noise, in *Stochastic Partial Differential Equations and Applications VII*, Lecture Notes in Pure and Applied Mathematics, Vol. 245 (CRC Press, 2006), pp. 61–75.
10. G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions* (Cambridges University Press, 1992).
11. G. Da Prato and J. Zabczyk, *Ergodicity for Infinite-Dimensional Systems* (Cambridge University Press, 1996).
12. J. Eckmann and M. Hairer, Invariant measures for stochastic partial differential equations in unbounded domains, *Nonlinearity* **14** (2001) 133–151.
13. B. Goldys and B. Maslowski, Uniform exponential ergodicity of stochastic dissipative systems *Czechoslovak Math. J.* **51** (2001) 745–762.
14. N. Ichihara, Homogenization for stochastic partial differential equations derived from nonlinear filtering with feedback, *J. Math. Soc. Japan* **57** (2005) 593–603.
15. N. Kryloff and N. Bogoliouboff, La théorie générale de la mesure dans son application à l'étude des systèmes dynamiques de la mécanique non linéaire, *Ann. of Math.* (2) **38** (1937) 65–113.

16. B. Maslowski and J. Seidler, On sequentially weakly Feller solutions to SPDEs, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei Mat. Appl.* **10** (1999) 69–78.
17. J. Mattingly and E. Pardoux, Invariant measure selection by noise, an example, *Discrete Contin. Dynam. Syst. A* **34** (2014) 237–256.
18. O. Misiats, O. Stanzhytskyi and N. K. Yip, Existence and uniqueness of invariant measures for stochastic reaction–diffusion equations in unbounded domains, *J. Theor. Probab.* **29** (2016) 996–1026.
19. O. Misiats, O. Stanzhytskyi and N. K. Yip, Ito's formula in infinite dimension and its application to the existence of invariant measures, preprint.
20. C. Mueller, Coupling and invariant measures for the heat equation with noise, *Ann. Probab.* **21** (1993) 2189–2199.
21. G. Tessitore and J. Zabczyk, Invariant measures for stochastic heat equations, *Probab. Math. Statist.* **2** (1998) 271–287.
22. E. Titchmarsh, *Eigenfunction Expansions Associated With Second-Order Differential Equations*, *Part 2* (Clarendon Press, 1958).
23. V. Volpert, *Elliptic Partial Differential Equations II*: *Reaction–Diffusion Equations* (Springer, 2014).
24. W. Wang, D. Cao and J. Duan, Effective macroscopic dynamics of stochastic partial differential equations in perforated domains, *SIAM J. Math. Anal.* **38** (2007) 1508–1527.
25. V. Zhikov, S. Kozlov and O. Oleinik, *Homogenization of Differential Operators and Integral Functionals* (Springer, 1994).