Structure of Stable Solutions of a One-Dimensional Variational Problem

Nung Kwan Yip
Department of Mathematics, Purdue University
yip@math.purdue.edu

Abstract

We prove the periodicity of all H^2-local minimizers with low energy for a one-dimensional higher order variational problem. The results extend and complement an earlier work of Stefan Müller which concerns the structure of global minimizer. The energy functional studied in this work is motivated by the investigation of coherent solid phase transformations and the competition between the effects from regularization and formation of small scale structures. With a special choice of a bilinear double well potential function, we make use of explicit solution formulas to analyze the intricate interactions between the phase boundaries. Our analysis can provide insights for tackling the problem with general potential functions.

1 Introduction and Statements of Theorems

In this paper, we study the structure of H^2-local minimizers for the following functional:

$$\mathcal{E}(u) = \int_0^1 \epsilon^2 u_{xx}^2 + W(u_x) + u^2\,dx \quad \text{subject to } u_x(0) = u_x(1) = 0 \tag{1}$$

where W is some double-well potential function usually taken to be $W(p) = (p^2-1)^2$. In order to facilitate the use of explicit solution formulas, in the present paper, we consider the following form of W:

$$W(p) = (|p|-1)^2 = \begin{cases} (p-1)^2, & p \geq 0 \\ (p+1)^2, & p < 0 \end{cases} \tag{2}$$

so that $W'(p) = \begin{cases} 2(p-1), & p > 0 \\ 2(p+1), & p < 0 \end{cases}$, $W'(0^+) - W'(0^-) = -4$ and $W''(p) = 2 - 4\delta_0(p)$.

The reason for this choice is that the corresponding Euler-Lagrange equation for (1) is given by a linear differential equation with constant coefficients together with some linear jump conditions for the solutions.

Before presenting our results, we introduce for each $l > 0$, the function $P(x,l)$ which solves the following problem:

$$\begin{cases} \epsilon^2 P_{xxxx}(x,l) = P_{xx}(x,l) - P(x,l) \quad \text{and} \quad P_x(x,l) > 0 \quad \text{for } x \in (-\frac{l}{2}, \frac{l}{2}), \\ P_x(-\frac{l}{2}, l) \;=\; P_x(\frac{l}{2}, l) \;=\; 0, \\ \epsilon^2 P_{xxx}(-\frac{l}{2}, l) \;=\; \epsilon^2 P_{xxx}(\frac{l}{2}, l) \;=\; -1. \end{cases} \tag{3}$$

which represents the cell-problem for the Euler-Lagrange equation for $\mathcal{E}$. The existence and uniqueness of the solution will be given by **Proposition 3.3**. Now let $N = \frac{1}{l}$ be an integer and $Q^N(x)$ be the following periodic version of $P(\cdot, l)$:

$$Q^N(il + y) = P\left((-1)^i \left(y - \frac{l}{2}\right), l\right) \quad \text{for} \quad 0 \le y \le l \quad \text{and} \quad i = 0, 1, 2 \ldots N-1. \tag{4}$$

The following are our main results.

1.1 Theorem. *There exist constants $C_L < 10$, l_* and $\epsilon_* > 0$ such that for any $0 < \epsilon < \epsilon_*$, if $C_L \epsilon |\ln \epsilon| < l < l_*$ and $N = \frac{1}{l}$ is an integer, then the function $\pm Q^N(x)$ defined as in (4) is a* stable *stationary point of the functional $\mathcal{E}(\cdot)$.*

1.2 Theorem. *There exist constants $C_E > \frac{3}{200}$ and $\epsilon_* > 0$ such that for any $0 < \epsilon < \epsilon_*$, if u is a* weakly stable *stationary point of $\mathcal{E}(\cdot)$ satisfying*

$$\mathcal{E}(u) \le \frac{C_E}{|\ln \epsilon|}, \tag{5}$$

then $u(\cdot) = Q^N(\cdot)$ or $-Q^N(\cdot)$ for some positive integer N.

1.3 Theorem. *There exist constants $C_S < 10$ and $\epsilon_* > 0$ such that for all $0 < \epsilon < \epsilon_*$ and $0 < C < C_S$, if $l = C\epsilon|\ln \epsilon|$ and $N = \frac{1}{l}$ is an integer, then $Q^N(\cdot)$ is* unstable.

(Note that for the l in this **Theorem**, *$\mathcal{E}(Q^N(\cdot)) = O(\frac{1}{|\ln \epsilon|})$.)*

The notions of stationary points, their stability and relationship to local minimizers of $\mathcal{E}$ will be given in **Section 2**. All of the above results can be extended in a natural way to the Dirichlet $\{u(0) = u(1) = 0\}$ and periodic $\{u(0) = u(1);\ \ u_x(0) = u_x(1)\}$ boundary conditions.

Our results in essence establish the fact that if u is a stationary point of $\mathcal{E}(\cdot)$ of low enough energy, then it is stable if and only if it is periodic. **Theorem 1.3** states that we have captured the correct range of the energy values in terms of the stability properties of periodic structures. Our work hence extends and complements the following Theorem of S. Müller which studies the structure of global minimizer for the functional (1) in the case of $W(p) = (p^2-1)^2$. Let $A_0 = 2\int_{-1}^{1} W^{\frac{1}{2}}(p)\,dp$.

1.4 Theorem ([Mül]). *There exists an* $\epsilon_* > 0$ *such that for* $0 < \epsilon < \epsilon_*$, *if* u *is a* global minimizer *of* $\mathcal{E}(\cdot)$ *in the class of periodic functions:*

$$H^2_{\#}(0,1) = \left\{u \in H^2(0,1) : u(0) = u(1) \ \ \textit{and} \ \ u_x(0) = u_x(1)\right\},$$

then u *is* periodic *with minimal period* $T^\epsilon = 2(6A_0\epsilon)^{\frac{1}{3}} + O(\epsilon^{\frac{2}{3}})$. *Moreover,*

$$u\left(x + \frac{T^\epsilon}{2}\right) = -u(x) \ \ \textit{for} \ \ x \in (0,1) \ \ \textit{and} \ \ \mathcal{E}(u) = \frac{1}{4}(6A_0\epsilon)^{\frac{2}{3}} + O(\epsilon^{\frac{4}{3}}).$$

The motivations for the investigation of (1) are twofold. One comes from the study of coherent solid-solid transformations which very often give rise to some fine scale mixtures of different phases with characteristic length scales. The formulation of energy minimization in the modeling of these transformations can be found in [Kha, BJ1, BJ2]. In the one-dimensional setting, the phenomenology of the formation of mixtures of different phases can be captured to sufficient degree by (1) and related functionals. It is the combination of the *strain energy* $W(u_x)$ which favors u_x to be 1 or -1 and the *elastic foundation term* u^2 that leads to the formation of fine scale structures. However, definite length scales are determined by the incorporation of the *strain gradient* or *surface energy* term $\epsilon^2 u_{xx}^2$. We refer to [TZ] for a more detail account of (1). The works [RT, RW1, RW2] also consider some nonlocal versions of u^2. A two-dimensional model can be found in [KM1, KM2].

The second motivation is that (1) can be viewed as a regularization of some functional which is not lower-semicontinuous. For example, the following functional

$$\mathcal{E}_0(u) = \int_0^1 (W(u_x) + u^2)\,dx \tag{6}$$

does not have a minimizer. Instead it possesses a *minimizing sequence* $\{u^\delta\}_{\delta>0}$ of the form that u_x^δ oscillates between 1 and -1 with increasing frequencies such that $\|u^\delta\|_\infty \to 0$. The incorporation of the higher order term $\epsilon^2 u_{xx}^2$ in (6) penalizes the oscillations in u_x and hence limits their number. By simple compactness arguments, a minimizer for (1) exists. Furthermore, formal reasoning leads to that (1) can be approximated by

$$\mathcal{E}_*^1(u) = \int_0^1 \frac{1}{2}\epsilon A_0 |u_{xx}| + u^2\,dx \quad \text{subject to } |u_x| = 1 \text{ a.e.} \tag{7}$$

where $A_0 = 2\int_{-1}^1 \sqrt{W(p)}\,dp$, or even more explicitly, by

$$\mathcal{E}_*^2(u) = \int_0^1 u^2\,dx + \epsilon A_0 \times (\text{Number of times } u_x \text{ changes between 1 and } -1) \tag{8}$$

with u subjected to the same constraint. Assuming the validity of such an approximation, the minimizer u of (1) can be approximated by the minimizer of $\mathcal{E}_*^1$ or $\mathcal{E}_*^2$. Hence we infer that there is a collection of roughly equidistant points $\{c_i\}$'s at which u_x changes between 1 and -1. In between these points, $|u_x|$ stays very close to 1 so that u is very much like a sawtooth function. We call the transition of u_x between 1 and -1 an *interface* and the region between any two interfaces a *phase segment* or simply a *segment*. See the following figure.

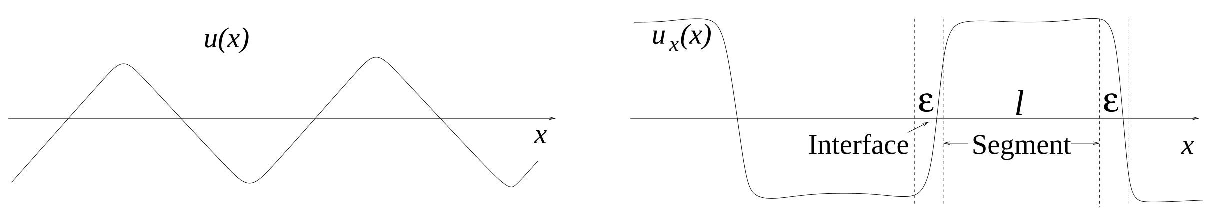

Figure 1.

Another interesting feature of the functional (1) is the competition between multiple — two — length scales. One is ϵ, the interfacial width and the other is l, the length of the segment. The result of this competition is that $l = O(\epsilon^{\frac{1}{3}})$ for the global minimizer of (1). Rigorous justification of the relationship between $\mathcal{E}$, $\mathcal{E}_*^1$ and $\mathcal{E}_*^2$ falls in the regime of Γ-convergence of functionals. We refer to [Mül, AM] for a discussion of this approach for the present problem. [RW1] also uses this method to study (1) but with u^2 replaced by ϵu^2 in which case, $l = O(1)$ and hence there is only one small length scale left in the problem.

In order to study dynamic problems such as gradient flows, the classification of critical points is also important as the ultimate observed patterns of the phase transformations are consequences of both *energetic* and *kinetic* effects. Our results state that (1) has many critical points but those having low enough energy and stability property are in fact periodic. Hence a time dependent solution can very likely fall into the basins of attraction of these local minimizers and stay there indefinitely. We refer to [ACJ] for some experimental investigations of such phenomena which is relevant to the current functional.

There are relatively few methods that can be used to describe the structure of critical points compared with those that can prove the existence of them. In addition, as (1) is a higher order variational problem, the usual techniques for second order equations are not easily at our disposal. Even though it is plausible to use the approach of Γ-convergence to deduce that there exist local-minimizers which are nearly periodic (see [KS] for a general framework), to conclude other statements such as the existence and characterization of other types of local minimizers or critical points, more precise estimates are needed.

Our approach, originated in [Mül], resembles that of asymptotic expansion. **Theorem 1.1** is proved by following quite closely the approach of [Mül, Thm 5.1]. However, the proof of **Theorem 1.2** is much more delicate. It requires careful analysis of the interactions between the interfaces. Our method can potentially be used to find all critical points which do not fall easily in the regime of Γ-convergence.

Our analysis is made possible by the special choice (2) of W. It allows the use of explicit solution formulas. Such a choice is also used in [TZ, VHRT] in which detailed analysis is performed for the case when the number of interfaces is small. We believe that our results can provide useful insights to tackle the case of general W's concerning the types of quantities and estimates to be looked upon.

It is instructive to compare (1) with the following Allen-Cahn functional:

$$\mathcal{F}(v) = \int_0^1 \epsilon^2 v_x^2 + W(v)\, dx \quad \text{subject to } v_x(0) = v_x(1) = 0 \tag{9}$$

for which the critical points are periodic and unstable except the global minimizer which has only one interface where v changes between 1 and -1 [CGS]. This model thus cannot capture any fine scale structures. Furthermore, the time dependent problem for the above functional demonstrates the existence of metastable states — interfaces move with exponentially small speeds [CP, BX]. This is due to the fact that each pair of adjacent interfaces gives rise to an *exponentially small eigenvalue*. However, for (1), we are dealing with *algebraically small eigenvalue*.

1.5 Outline of Proof

Using the analysis of [Mül], we can easily deduce that for any critical point of low enough energy, u has a sawtooth shape with $|u_x| \approx 1$ away from the zeros of u_x or the interfacial regions. This is seen by considering the Euler-Lagrange equation of (1) which for smooth W reads:

$$\epsilon^2 u_{xxxx} = \frac{W''(u_x)}{2} u_{xx} - u.$$

If the distance between the interfaces are long, then the behavior of u can be described by $\epsilon^2 u_{xxxx} = \frac{W''(u_x)}{2} u_{xx}$ for x near the interface and by $\frac{W''(u_x)}{2} u_{xx} = u$ for x far away from the interface. From these observations, sharp estimates can be deduced. **Theorem 1.1** is proved by showing that the minimum energy $E(l)$ of a monotonic function u over an segment of length l is a convex function of l. This is very similar to the approach of [Mül, Thm 5.1].

However, in order to deduce the periodicity of stable patterns, we need to consider the possibility of short segments. In this case, the above *separation of scales* is not quite useful. We need much more precise information about the interactions between the interfaces. The core idea of this paper is to capture these interactions by means of a *propagation map* which relates the behaviors of u at both ends of an segment. With this map, we can find out the relationship between the lengths of adjacent segments. As an illustrative example, consider the following configuration of u and u_x:

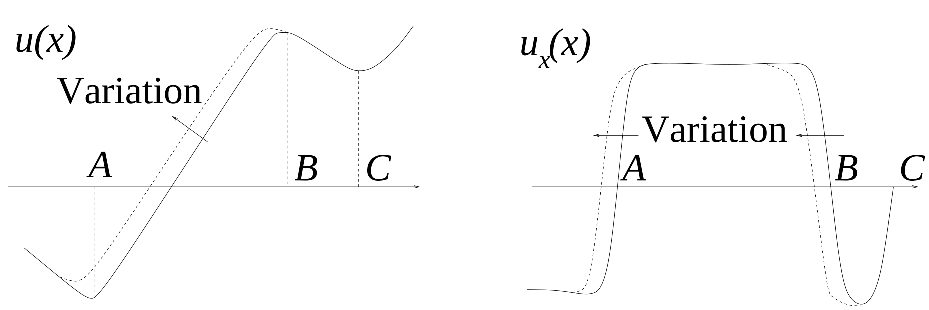

Figure 2.

In the above Figure, the lengths of $|AB|$ and $|BC|$ differ substantially from each other. We now vary u by moving the interfaces at A and B. The second variation of u is found to be:

$$\partial^2 \mathcal{E}(u) \approx -\frac{O(e^{-\frac{|BC|}{\epsilon}})}{\epsilon} + |AB|$$

In order to determine the sign of the second variation, we need to analyze the relationship between $|BC|$ and $|AB|$. This procedure falls into an unfortunately large number of cases which we will check one by one. From this analysis, we conclude that each *short* segment leads to an *unstable eigen-mode* and hence in order for u to be a stable critical point, it can only have long segments. Then a reasoning using first variation shows that u must be nearly periodic. A final step using an implicit function theorem type argument concludes the periodicity of u.

This paper is outlined as follows. **Section 2** gives the definitions of solutions and their stability for the Euler-Lagrange equation of (1). Some regularity property and preliminary estimates for u are also proved. **Section 3** introduces the propagation map which will be used throughout the paper. The proofs of the **Theorems** will be given in **Sections 4**, **5** and **6**. The **Appendix** gives some explicit examples of unstable solutions.

2 Euler-Lagrange Equation

The Euler-Lagrange Equation for the functional $\mathcal{E}$ is formally given by

$$\epsilon^2 u_{xxxx} = \frac{W''(u_x)}{2} u_{xx} - u \tag{10}$$

or, in the integrated form:

$$\epsilon^2 u_{xxx} = \frac{W'(u_x)}{2} - \int_0^x u(y)\, dy + C \quad \text{for some constant } C. \tag{11}$$

Note that there is an ambiguity about the meaning of $W'(0)$ which occurs at the points where $u_x = 0$. From **Theorem 2.8**, it is sufficient to consider those solutions u with only a *finite* number of zeros for u_x so that u is piecewise monotone. In this way, the ambiguity of $W'(0)$ can be handled by imposing appropriate jump conditions for u. However, we do not *a priori* limit the number of the zeros. The necessary formulation for the solutions of (10) will be given next.

2.1 Definition. *A function u is said to belong to class $\mathcal{Z}$ if $u \in C^2([0,1])$ and $[0,1]$ can be partitioned into a finite number of segments $\{[c_i, c_{i+1}] : i = 0, 1, \ldots N-1\}$ for some positive integer N and c_i's: $0 = c_0 < c_1 < \cdots < c_{N-1} < c_N = 1$ such that*

1. *u is monotone ($u_x \geq 0$ or $u_x \leq 0$) in each of the segment (c_i, c_{i+1}) and the sign of u_x alternates between adjacent segments, i.e. u_x changes sign across the c_i's.*

2. *the zeros of u_x are isolated. In particular, u_x is* not identically zero *in any interval.*

Remark. A priori, there can be two kinds of zeros for u_x. One is the *sign-changing* zero c_i which indicates the location of the interface. The other is the *interior* zero which lies in between any two adjacent c_i's. The sign of u_x does not change across these zeros. We are mainly concerned with the sign-changing zeros while the interior zeros can be shown not to occur for local minimizers or any stationary point of $\mathcal{E}$ with low energy. □

In the following, the notation $[f](x)$ refers to $f(x^+) - f(x^-)$.

2.2 Definition. *A function $u \in \mathcal{Z}$ is called a solution of (10) if the following hold for all i:*

$$\epsilon^2 u_{xxxx} = u_{xx} - u, \quad x \in (c_i, c_{i+1}); \tag{12}$$

$$u_x(c_i) = 0 \ \textit{and} \ u_x \geq 0 \ \textit{(or } u_x \leq 0\textit{)} \quad \textit{for all } x \in (c_i, c_{i+1}); \tag{13}$$

$$[\epsilon^2 u_{xxx}](c_i) \left(= \left[\tfrac{1}{2} W'(u_x)\right](c_i)\right) = -2\text{sgn}^*(u_{xx}(c_i)). \tag{14}$$

$$u_x(0) = u_x(1) = 0, \quad \epsilon^2 u_{xxx}(0) = -\text{sgn}^*(u_{xx}(0)), \quad \epsilon^2 u_{xxx}(1) = \text{sgn}^*(u_{xx}(1)). \tag{15}$$

In (14), the symbol $2\text{sgn}^*(u_{xx}(c_i))$ refers to $\text{sgn}(u_x(c_i^+)) - \text{sgn}(u_x(c_i^-))$. In (15), $\text{sgn}^*(u_{xx}(0))$ and $\text{sgn}^*(u_{xx}(1))$ refer to $\text{sgn}(u_x(0^+))$ and $-\text{sgn}(u_x(1^-))$ respectively.

Remark. Since u satisfies (12) in (c_i, c_{i+1}), it is analytic. In addition, as u_x is assumed not to be identically zero, the zeros of u_x inside each segment are automatically isolated and also do not cluster at the c_i's. Hence the quantities $\text{sgn}(u_x(c_i^{\pm}))$'s are well defined. □

In fact, the following **Proposition** states that we can replace the sgn^* by the usual sgn.

2.3 Proposition. *If $u \in \mathcal{Z}$ and satisfies (14) and (15), then $u_{xx}(c_i) \neq 0$ for $i = 0, 1, \dots N$. Hence* $\text{sgn}^*(u_{xx}(c_i)) = \text{sgn}(u_{xx}(c_i))$.

Proof. The proof follows easily by contradiction. Suppose $u_{xx}(c_i) = 0$ and $u_x(c_i^+) > 0$ and $u_x(c_i^-) < 0$. Then we have $u_{xxx}(c_i^+) \geq 0$ and $u_{xxx}(c_i^-) \leq 0$ which contradicts (14). The other cases follow similarly. $\square$

The following **Proposition** motivates our definitions of solutions and stability.

2.4 Proposition. *Let $u \in \mathcal{Z}$ and $\varphi \in \mathcal{V}^\infty = C^\infty([0,1]) \bigcap \{\varphi_x(0) = \varphi_x(1) = 0\}$. Then the following statements hold.*

1. *The first variation of $\mathcal{E}(u)$ with respect to φ, defined as $\frac{1}{2}\frac{d}{dt}\mathcal{E}(u + t\varphi)\Big|_{t=0}$ equals:*

$$\sum_{i=1}^{N-1} \left\{[\epsilon^2 u_{xxx}](c_i) + 2\text{sgn}^*(u_{xx}(c_i))\right\} \varphi(c_i) + \left\{\epsilon^2 u_{xxx}(0) + \text{sgn}^*(u_{xx}(0))\right\} \varphi(0)$$
$$- \left\{\epsilon^2 u_{xxx}(1) - \text{sgn}^*(u_{xx}(1))\right\} \varphi(1) + \sum_{i=0}^{N-1} \int_{c_i(t)}^{c_{i+1}(t)} (\epsilon^2 u_{xxxx} - u_{xx} + u)\varphi \, dx \quad (16)$$

   In particular, $u \in \mathcal{Z}$ is a solution (in the sense of **Definition 2.2***) if and only if $\frac{d}{dt}\mathcal{E}(u + t\varphi)\Big|_{t=0} = 0$ for all $\varphi \in \mathcal{V}^\infty$.*

2. *Suppose $u_{xx}(c_i) \neq 0$ (which holds when u is a solution). Then there are C^1 functions $c_i(t)$ such that $c_i(0) = c_i$, and for small t, $u_x(c_i(t)) + t\varphi_x(c_i(t)) = 0$. Furthermore,*

$$\dot{c}_i(0) = -\frac{\varphi_x(c_i)}{u_{xx}(c_i)} \quad (17)$$

   (In essence, the $c_i(t)$'s are the sign-changing *zeros of $u_x + t\varphi_x$.)*

3. *Suppose $u_{xx}(c_i) \neq 0$ (which holds when u is a solution) and $u_x \neq 0$ for $x \in (c_i, c_{i+1})$, i.e. u_x has no interior zeros, then the second variation of $\mathcal{E}(u)$ with respect to φ, defined as $\frac{1}{2}\frac{d^2}{dt^2}\mathcal{E}(u + t\varphi)\Big|_{t=0}$ equals:*

$$\int_0^1 \epsilon^2 \varphi_{xx}^2 + \varphi_x^2 + \varphi^2 \, dx - 2\sum_{i=0}^{N-1} \frac{\varphi_x^2(c_i)}{|u_{xx}(c_i)|} \quad (18)$$

Proof of (16). To prove (16), we compute

$$\frac{1}{2}\frac{d}{dt}\mathcal{E}(u+t\varphi)\Big|_{t=0} = \frac{1}{2}\frac{d}{dt}\int_0^1 \epsilon^2(u_{xx}+t\varphi_{xx})^2 + (|u_x+t\varphi_x|-1)^2 + (u+t\varphi)^2\,dx\Big|_{t=0}$$
$$= \int_0^1 \epsilon^2 u_{xx}\varphi_{xx} + u_x\varphi_x + u\varphi\,dx - \lim_{t\to 0}\int_0^1 \frac{|u_x+t\varphi_x|-|u_x|}{t}\,dx$$

The last term of the above equals

$$\lim_{t\to 0}\int_{\{u_x\neq 0\}} \frac{|u_x+t\varphi_x|-|u_x|}{t}\,dx + \lim_{t\to 0}\int_{\{u_x=0\}}\frac{|t\varphi_x|}{t}\,dx \tag{19}$$

As u_x is assumed to have isolated zeros, we can ignore the second integral of (19). To simplify the first integral, note that

$$\left|\frac{|a+tb|-|a|}{t}\right| \leq |b| \text{ for all } a,b,t \text{ and } \lim_{t\to 0}\frac{|a+tb|-|a|}{t} = \mathrm{sgn}(a)b \text{ if } a\neq 0.$$

By the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{t\to 0}\int_{\{u_x\neq 0\}} \frac{|u_x+t\varphi_x|-|u_x|}{t}\,dx = \int_0^1 \mathrm{sgn}(u_x)\varphi_x\,dx \tag{20}$$

The above steps thus lead to

$$\frac{1}{2}\frac{d}{dt}\mathcal{E}(u+t\varphi)\Big|_{t=0} = \int_0^1 \epsilon^2 u_{xx}\varphi_{xx} + (|u_x|-1)\mathrm{sgn}(u_x)\varphi_x + u\varphi\,dx \tag{21}$$

Performing integration by parts twice on each segment (c_i, c_{i+1}) and using the facts that $[u+t\varphi] = [u_x+t\varphi_x] = [u_{xx}+t\varphi_{xx}] = 0$ at $x = c_i$, (16) follows.

It is then easy to infer the equivalence of u being a solution in the sense of **Definition 2.2** and the vanishing of its first variation for all $\varphi\in\mathcal{V}^\infty$.

Proof of (17). It follows from elementary computations. Since u_x and φ_x are both C^1 functions and $u_{xx}(c_i)\neq 0$, we can use the implicit function to show the existence of C^1 functions $c_i(t)$ (for small enough t) satisfying $u_x(c_i(t)) + t\varphi_x(c_i(t)) = 0$. Furthermore,

$$u_{xx}(c_i(t))\dot{c}_i(t) + t\varphi_{xx}(c_i(t))\dot{c}_i(t) + \varphi_x(c_i(t)) = 0$$

which leads to (17).

Proof of (18). We start from the first variation, $\frac{1}{2}\frac{d}{dt}\mathcal{E}(u+t\varphi)$ which equals

$$\frac{1}{2}\frac{d}{dt}\sum_{i=0}^{N-1}\int_{c_i(t)}^{c_{i+1}(t)} \epsilon^2(u_{xx}+t\varphi_{xx})^2 + (u_x+t\varphi_x-\mathrm{sgn}(u_x+t\varphi_x))^2 + (u+t\varphi)^2\,dx$$
$$= \sum_{i=0}^{N-1}\int_{c_i(t)}^{c_{i+1}(t)} \epsilon^2(u_{xx}+t\varphi_{xx})\varphi_{xx} + (u_x+t\varphi_x-\mathrm{sgn}(u_x+t\varphi_x))\varphi_x + (u+t\varphi)\varphi\,dx \tag{22}$$

The condition of no interior zeros for u_x is to ensure that for t small enough, there is no new zeros for $u_x + t\varphi_x$ other than the sign-changing ones, $c_i(t)$'s. Now (18) follows by differentiating (22):

$$\frac{1}{2}\frac{d^2}{dt^2}\mathcal{E}(u+t\varphi) = \sum_{i=0}^{N-1} 2\mathrm{sgn}^*(u_{xx}(c_i(t)))\varphi_x(c_i(t))\dot{c}_i(t) + \int_{c_i(t)}^{c_{i+1}(t)} \epsilon^2\varphi_{xx}^2 + \varphi_x^2 + \varphi^2\,dx.$$

and then utilizing (17). □

We now introduce the following definitions for all $u \in \mathcal{Z}$ such that $u_{xx}(c_i) \neq 0$.

$$\mathcal{V} = H^2(0,1) \cap \{\varphi : \varphi_x(0) = \varphi_x(1) = 0\}, \tag{23}$$

$$\mathcal{D}(u,\varphi) = \int_0^1 \epsilon^2\varphi_{xx}^2 + \varphi_x^2\,dx - 2\sum_{i=0}^{N-1}\frac{\varphi_x^2(c_i)}{|u_{xx}(c_i)|} \text{ for } u \in \mathcal{Z} \text{ and } \varphi \in \mathcal{V} \tag{24}$$

$$\text{and } \partial^2\mathcal{E}(u,\varphi) = \mathcal{D}(u,\varphi) + \int_0^1 \varphi^2\,dx \tag{25}$$

In general, if u_x has interior zeros, $\partial^2\mathcal{E}(u,\varphi)$ will not be equal to $\left.\frac{1}{2}\frac{d^2}{dt^2}\mathcal{E}(u+t\varphi)\right|_{t=0}$. But this is not an issue in this work due to **Theorem 2.8**(3) and (41) of **Proposition 2.9** below. Also, all our main **Theorems** are formulated in terms of $\partial^2\mathcal{E}(u,\cdot)$.

Again, the condition of $u_{xx}(c_i) \neq 0$ follows from **Proposition 2.3** if u is a solution.

2.5 Definition. *A function $u \in \mathcal{V}$ is called a* stationary point *of $\mathcal{E}$ if for all $\varphi \in \mathcal{V}$,*

$$\left.\frac{d}{dt}\mathcal{E}(u+t\varphi)\right|_{t=0} \text{ exists and equals to } 0.$$

2.6 Definition. *A solution $u \in \mathcal{Z}$ of (10) is called* (weakly) stable *if*

$$\text{for all } \varphi \neq 0, \in \mathcal{V},\ \partial^2\mathcal{E}(u,\varphi)\ (\geq)\ > 0.$$

It is called unstable *if*

$$\text{there exists a } \varphi \in \mathcal{V} \text{ such that } \partial^2\mathcal{E}(u,\varphi) < 0.$$

2.7 Definition. *A function $u \in \mathcal{V}$ is called a (strict) H^2-local minimizer of $\mathcal{E}$ if there is a $\delta > 0$ such that*

$$\mathcal{E}(v)\,(>)\ \geq \mathcal{E}(u) \text{ for all } v \in \mathcal{V} \text{ with } 0 < \|v-u\|_{H^2} \leq \delta.$$

(In the literature, the above definition is frequently called a *weak*-local minimizer in contrast to *strong*-local minimizer in which the H^2-norm is replaced by the L^∞-norm.)

Equipped with the above definitions, the following **Theorem** addresses the regularity property of stationary points and the relationship between the notions of stationarity, stability and local minimizers with respect to $\mathcal{E}$.

2.8 Theorem. *Let $u \in \mathcal{V}$.*

1. *If u is a stationary point, then the one-dimensional Lebesgue measure of the zeros of u_x vanishes, i.e. $\mathcal{L}^1\{x : u_x(x) = 0\} = 0$. In addition, $u \in H^3([0,1])$ (and hence $u_{xx} \in C^{\frac{1}{2}}([0,1])$) and $\int_0^1 u(x)\,dx = 0$. Let $V(x) = \int_0^x u(y)\,dy$. Then the following identities also hold:*

$$\epsilon^2 u_{xxx} - (|u_x| - 1)\mathrm{sgn}(u_x) + V(x) = 0, \quad a.e.\ x \in [0,1]; \tag{26}$$

$$\epsilon^2 u_{xx}^2(x) - (|u_x| - 1)^2 + 2V(x)u_x(x) - u^2(x) = \epsilon^2 u_{xx}(0) - 1 - u^2(0), \quad for\ all\ x \in [0,1]. \tag{27}$$

2. *There exists a $\delta_* > 0$ (independent of ϵ) such that any stationary point u with $\mathcal{E}(u) < \delta_*$ belongs to $\mathcal{Z}$ and hence is a solution of (10) in the sense of* **Definition 2.2**. *In particular, u_{xxx} satisfies the conditions (14) and (15) at the* sign-changing *zeros c_i of u_x and hence $u_{xx}(c_i) \neq 0$ by the* **Proposition 2.3**.

3. *Any H^2-local minimizer u is a stationary point. Hence if further $\mathcal{E}(u) < \delta_*$ (same as in the previous statement), then $u \in \mathcal{Z}$. In addition, $u_x(x) \neq 0$ for $x \neq c_i$, i.e. u_x does not have any interior zeros.*

4. *If $u \in \mathcal{Z}$ is an H^2-local minimizer, then it is weakly-stable.*

5. *If $u \in \mathcal{Z}$ is a stable stationary point such that $u_x(x) \neq 0$ for $x \neq c_i$, i.e. u_x does not have any interior zeros, then u is a strict H^2-local minimizer.*

This **Theorem** indicates that in order to study the properties of low energy H^2-local minimizers, it is sufficient to consider the function space $\mathcal{Z}$ and the notions of solutions and stabilities given in **Definitions 2.2** and **2.6**. It also follows from **Theorems 1.1** and **1.2** that all H^2-local minimizers of $\mathcal{E}$ with low enough energy are actually *stable*. For simplicity, after the proof of this **Theorem**, we will omit the prefix H^2 for the rest of this paper.

Proof of 1. We can basically follow the same computation leading to (21) which also applies for any $\varphi \in \mathcal{V}$. It now takes the form of:

$$\frac{1}{2}\frac{d}{dt}\mathcal{E}(u+t\varphi)\bigg|_{t=0} = \int_0^1 \epsilon^2 u_{xx}\varphi_{xx} + (|u_x|-1)\mathrm{sgn}(u_x)\varphi_x + u\varphi\,dx + \lim_{t\to 0}\int_{\{u_x=0\}}\frac{|t\varphi_x|}{t}\,dx$$

Since *by assumption*, $\frac{d}{dt}\mathcal{E}(u+t\varphi)\Big|_{t=0}$ exists for all $\varphi \in \mathcal{V}$, it must be that $\mathcal{L}^1\{x : u_x(x) = 0\} = 0$. Hence (21) also holds.

Choosing $\varphi \equiv 1$ gives $\int_0^1 u\,dx = 0$. Thus, without loss of generality, we can assume $\int_0^1 \varphi\,dx = 0$. Then

$$\begin{aligned} \int_0^1 \epsilon^2 u_{xx}\varphi_{xx}\,dx &= -\int_0^1 (|u_x|-1)\mathrm{sgn}(u_x)\varphi_x + u\varphi\,dx \\ \text{so}\quad \left|\int_0^1 \epsilon^2 u_{xx}\varphi_{xx}\,dx\right| &\le \||u_x|-1\|_{L^2}\|\varphi_x\|_{L^2} + \|u\|_{L^2}\|\varphi\|_{L^2} \le C\|\varphi_x\|_{L^2} \end{aligned}$$

for some constant C. Hence $u \in H^3([0,1])$ and $u_{xx} \in C^{\frac{1}{2}}([0,1])$. Now the use of integration by parts in (21) is justified and (26) follows.

Identity (27) follows by simply multiplying u_{xx} to (26) and then integrating from 0 to x.

Proof of 2. We need to show that the zeros of u_x are isolated. Assume on the contrary that there exist x_i's such that $x_i \to b \in [0,1]$ and $u_x(x_i) = 0$. Then $u_x(b) = 0$ and $u_{xx}(b) = 0$. From (27), we have

$$\epsilon^2 u_{xx}^2(x) - (|u_x|-1)^2 + 2V(x)u_x(x) - u^2(x) = -1 - u^2(b). \tag{28}$$

Let $\max\{|u_x(x)| : x \in [b,x_i]\cup[x_i,b]\} = |u_x(\bar{x}_i)|$. By the assumption, the $\bar{x}_i$'s exist and $\bar{x}_i \to b$. Furthermore, $u_{xx}(\bar{x}_i) = 0$. Substituting $\bar{x}_i$ into (28) and rearranging give

$$\begin{aligned} |u_x|\left[2 - |u_x| + 2V(x)\mathrm{sgn}(u_x)\right]\big|_{x=\bar{x}_i} &\le 2\left|\int_b^{\bar{x}_i} u(y)u_y(y)\,dy\right| \\ &\le 2|u_x(\bar{x}_i)|\int_b^{\bar{x}_i}|u(y)|\,dy \le c|u_x(\bar{x}_i)| \end{aligned}$$

where the constant c can be chosen as small as possible if $\bar{x}_i$ is close enough to b. Now by Cauchy-Schwarz Inequality, if $\mathcal{E}(u)$ is small enough, so is $\|V\|_\infty$. Hence $|u_x(\bar{x}_i)| = 0$ and

$u_x(x) \equiv 0$ for $x \in [b, \bar{x}_i]$ contradicting the fact that $\mathcal{L}^1\{x : u_x(x) = 0\} = 0$. Thus, the zeros of u_x are isolated and so $u \in \mathcal{Z}$. Now it is easy to check that u is a solution of (10) in the sense of **Definition 2.2**.

Proof of 3. Let u be a H^2-local minimizer, then for all $\varphi \in \mathcal{V}$, there exists $s > 0$ such that $\mathcal{E}(u + t\varphi) \geq \mathcal{E}(u)$ if $|t| \leq s$, i.e.

$$2t \int_0^1 \left(\epsilon^2 u_{xx}\varphi_{xx} + u_x\varphi_x + u\varphi - \frac{|u_x + t\varphi_x| - |u_x|}{t} \right) dx \geq -t^2 \int_0^1 \epsilon^2 \varphi_{xx}^2 + \varphi_x^2 + \varphi^2 \, dx$$

Upon dividing t in the above and following (19) and (20) give

$$\begin{aligned} \int_0^1 \epsilon^2 u_{xx}\varphi_{xx} + u_x\varphi_x + u\varphi \, dx - \int_{\{u_x \neq 0\}} \operatorname{sgn}(u_x)\varphi_x \, dx - \int_{\{u_x = 0\}} |\varphi_x| \, dx &\geq 0 \\ \text{and} \quad \int_0^1 \epsilon^2 u_{xx}\varphi_{xx} + u_x\varphi_x + u\varphi \, dx - \int_{\{u_x \neq 0\}} \operatorname{sgn}(u_x)\varphi_x \, dx + \int_{\{u_x = 0\}} |\varphi_x| \, dx &\leq 0. \end{aligned}$$

Hence $\mathcal{L}^1\{x : u_x(x) = 0\} = 0$. It then follows easily that u is a stationary point of $\mathcal{E}$.

We now proceed to prove directly the non-existence of interior zeros for local minimizers. (This fact can also follows from (41) in **Proposition 2.9** below.) Suppose there is a $p \in (c_i, c_{i+1})$ such that $u_x(p) = 0$ and $u_x \geq 0$ for $x \in (c_i, c_{i+1})$. Then $u_{xx}(p) = 0$ and $u_{xxx}(p) \geq 0$. In addition, p is an isolated zero of u_x as u is an analytic function which satisfies (12). Let $0 < A \ll 1$. Consider the test function $\varphi \in C_0^\infty([0,1])$ such that $\varphi_x(x) = 1$ for $|x - p| \leq A$, $\varphi_x(x) = 0$ for $|x - p| \geq 2A$ and $\int_{p-2A}^{p+2A} \varphi_x \, dx = 0$. For $0 < t \ll 1$, the difference of the energies $\mathcal{E}(u - t\varphi) - \mathcal{E}(u)$ equals

$$\int_{x \in [p-A, p+A]} + \int_{x \notin [p-A, p+A]} \epsilon^2 (u_{xx} - t\varphi_{xx})^2 + (|u_x - t\varphi_x| - 1)^2 + (u - t\varphi)^2 - \epsilon^2 u_{xx}^2 - (|u_x| - 1)^2 - u^2 \, dx$$

All the terms of the above are smooth functions of t except:

$$-\int_{p-A}^{p+A} |u_x - t\varphi_x| \, dx.$$

If $u_{xxx}(p) = 0$, then $u_{xxxx}(p)$ must be zero in order for $u_x(x) \geq 0$ near p. Then we would have $u_x = u_{xx} = u_{xxx} = u_{xxxx} = 0$ at p so that $u_x \equiv 0$ in (c_i, c_{i+1}) contradicting the fact that the zeros of u_x have zero Lebesgue measure. Thus $u_{xxx}(p) = 2c > 0$. Now for A small enough, the above integral can be approximated by

$$-2\int_0^A |cx^2 - t|\,dx = -\frac{8}{3\sqrt{c}}t^{\frac{3}{2}} + 2tA - \frac{2cA^3}{3}.$$

Since u is a stationary point, we can ignore the constant and first order term of t. The first term will give a definite negative contribution for $0 < t \ll 1$. Other negative contributions can only come from some $O(t^2)$ terms which is of much smaller magnitude than $t^{\frac{3}{2}}$. Hence u cannot be a local minimizer. This concludes that u cannot have zeros in the interior of any segment (c_i, c_{i+1}).

Proof of 4. If $u \in \mathcal{Z}$ is a H^2-local minimizer, from the previous statement, u is a stationary point and a solution of (10). In addition, u_x has no interior zeros in any segment (c_i, c_{i+1}). (The assumption of low energy is not used in the proof.) By following the same computation leading to (18), we have $\partial^2\mathcal{E}(u,\varphi) \geq 0$ for all $\varphi \in \mathcal{V}^\infty$. Simple approximation extends the conclusion to all $\varphi \in \mathcal{V}$.

Proof of 5. The goal here is to show that if $u \in \mathcal{Z}$ is a *stable stationary* point of $\mathcal{E}$, then there exists a $\delta > 0$ such that

$$\text{if } 0 < \|v - u\|_{H^2} \leq \delta, \text{ then } \mathcal{E}(v) > \mathcal{E}(u).$$

The proof is elementary but a little delicate due to the non-differentiability of $W(p)$ at $p = 0$. It is achieved by modifying v in several stages. Each stage improves the regularity of v and simultaneously lowers its energy. Throughout the proof, we will make use of four **Claims** whose proof will be provided right afterwards.

Recall that the c_i's denote the zeros of $u_x(x)$ (with $c_0 = 0$ and $c_N = 1$). Let further $d_i = u(c_i)$. By the Remark after **Definition 2.2** and the finiteness of the number of zeros of u_x, there exists an $m > 0$ such that $|u_{xx}(c_i)| \geq m$ for all i. In the following, δ is some small positive number. The notation $o(1)$ denotes some quantity (independent of ϵ, c_i and d_i's) which goes to zero as $\delta \longrightarrow 0$.

Now the fact that $\|v - u\|_{H^2} \leq \delta$ leads to $v_x \in C^{\frac{1}{2}}([0,1])$ and $\|v_x - u_x\|_{L^\infty} \leq o(1)$. Then v "approximately" belongs to $\mathcal{Z}$ in the following sense: there exist $\{c_i^1 \leq c_i^2 : i = 0, 1, \ldots N\}$

such that for all i, $c_i^2 < c_{i+1}^1$ and

$$|c_i^1 - c_i|, |c_i^2 - c_i|, |v(c_i^1) - d_i|, |v(c_i^2) - d_i| < o(1),$$
$$v_x(c_i^1) = v_x(c_i^2) = 0 \text{ and } \mathrm{sgn}(v_x)|_{x\in(c_i^2, c_{i+1}^1)} = \mathrm{sgn}(u_x)|_{x\in(c_i, c_{i+1})}.$$

Consider the following problem:

$$\epsilon^2 V^i_{xxxx} - V^i_{xx} + V^i = 0, \ \ x \in (c_i^2, c_{i+1}^1);$$
$$V^i(c_i^2) = v(c_i^2), \ \ V^i(c_{i+1}^1) = v(c_{i+1}^1), \ \ V^i_x(c_i^2) = 0, \ \ V^i_x(c_{i+1}^1) = 0.$$

By **Claim (ii)**, it has a unique solution. We now replace v by V^i for x in each of the segments (c_i^2, c_{i+1}^1) and call the resultant function $\tilde{v}$. **Claim (i)** gives $\mathcal{E}(v) \geq \mathcal{E}(\tilde{v})$. Since δ can be chosen as small as possible, we can further assume that $|\tilde{v}_{xx}(c_i^2)|$ and $|\tilde{v}_{xx}(c_i^1)| > \frac{m}{2}$. Next we eliminate the segments (c_i^1, c_i^2) by modifying $\tilde{v}$ to be ψ such that $\mathrm{sgn}(\psi_x)|_{x\in(c_i^1, c_{i+1}^1)} = \mathrm{sgn}(u_x)|_{x\in(c_i, c_{i+1})}$ and $\mathcal{E}(\tilde{v}) > \mathcal{E}(\psi)$. By **Claim (iii)**, this can be done. Using the similar construction for $\tilde{v}$ from v leads to a $\tilde{\psi}$ such that $\tilde{\psi}_x(c_i^1) = 0$, $\tilde{\psi}(c_i^1) = \psi(c_i^1)$ and $\mathcal{E}(\psi) > \mathcal{E}(\tilde{\psi})$. Thus we have $\mathcal{E}(v) \geq \mathcal{E}(\tilde{v}) \geq \mathcal{E}(\psi) \geq \mathcal{E}(\tilde{\psi})$. Finally, **Claim (iv)** gives that $\mathcal{E}(\tilde{\psi}) > \mathcal{E}(u)$. This completes the proof of statement (5) of the **Theorem**. □

Now we provide the statements and the proofs of the four **Claims** used in the proof of the above **Theorem**.

Claim (i). *Let w be the solution of the following boundary value problem:*

$$\begin{cases} \epsilon^2 w_{xxxx} - w_{xx} + w = 0 \ \ \textit{with} \ \ w_x > 0 \ \ \textit{for} \ \ 0 < x < l, \\ \textit{and} \ \ w(0) = d_1, \ \ w(l) = d_2, \ \ w_x(0) = 0, \ \ w_x(l) = 0. \end{cases} \tag{29}$$

Then $\int_0^l \epsilon^2 v_{xx}^2 + (v_x - 1)^2 + v^2\, dx \geq \int_0^l \epsilon^2 w_{xx}^2 + (w_x - 1)^2 + w^2\, dx$ *for all v such that $v(0) = w(0)$, $v(l) = w(l)$, $v_x(0) = 0$, $v_x(l) = 0$ and $v_x > 0$ for $0 < x < l$.*

Proof. The proof is simply given by the following consideration. Let $\varphi = v - w$. Then

$$\int_0^l \epsilon^2 v_{xx}^2 + (v_x - 1)^2 + v^2\, dx - \int_0^l \epsilon^2 w_{xx}^2 + (w_x - 1)^2 + w^2\, dx$$
$$= 2\int_0^l \epsilon^2 w_{xx}\varphi_{xx} + (w_x - 1)\varphi_x + w\varphi\, dx + \int_0^l \epsilon^2 \varphi_{xx}^2 + \varphi_x^2 + \varphi^2\, dx.$$

The first integral is easily seen to vanish by using integration by parts. The **Claim** follows.

□

Claim (ii). *Let w be the solution of (29) with the additional property that $w_{xx}(0) > 0$ and $w_{xx}(l) < 0$, then for all d_1', d_2' and l' such that $|d_1' - d_1|$, $|d_2' - d_2|$ and $|l' - l|$ are small enough, (29) has a unique solution w' with $w'(0) = d_1'$, $w'(l') = d_2'$ and $w_x'(0) = w_x'(l') = 0$.*

Proof. This is proved by observing that the solution v can be written as $v = Ae^{\Lambda x} + Be^{-\Lambda x} + Ce^{\lambda x} + De^{-\lambda x}$ (see (47) below) where A, B, C and D satisfy a 4×4 linear system (similar to (48)). It is easily verified that the determinant of the system is a smooth non-zero function of $l \neq 0$. It then follows that a unique solution exists and it varies smoothly with respect to d_1, d_2 and l. □

Claim (iii). *For any $m, n, p > 0$, there exist $\delta, A, B > 0$ such that if*

$$\int_0^\delta \varphi_{xx}^2 \, dx < n, \;\; \varphi_x(0) = \varphi_x(\delta) = 0, \;\; \varphi_{xx}(0) = m, \;\; |\varphi_{xxx}| \leq p \;\; \textit{for } x \geq \delta, \tag{30}$$

then there exists a $\psi(x)$ with the properties that: ($L = \delta + A + 2B$)

$$\psi(y) = \varphi(y), \;\; \psi_x(y) = \varphi_x(y), \;\; \textit{for } y = 0, L; \;\; \psi_x(x) > 0 \textit{ for } 0 < x < L; \tag{31}$$

$$\textstyle\int_0^L \epsilon^2 \varphi_{xx}^2 + (|\varphi_x| - 1)^2 + \varphi^2 \, dx > \int_0^L \epsilon^2 \psi_{xx}^2 + (|\psi_x| - 1)^2 + \psi^2 \, dx. \tag{32}$$

The statement of this **Claim** is reminiscent to the fact that if φ is a function with discontinuous φ_{xx}, the value of $\mathcal{E}(\varphi)$ can always be reduced by small H^2-perturbations.

Proof. Let δ, A, B be some positive numbers with their values to be determined. For simplicity, first assume that in addition to (30), $\varphi_x(x) \equiv 0$ for $0 < x < \delta$ and $\varphi_{xx}(x) \equiv m$ for $x \geq \delta$. We define ψ as follows:

$$\begin{aligned} \psi(0) &= \varphi(0) \\ \psi_x(x) &= \begin{cases} \left(\frac{mA}{A+\delta}\right) x & 0 \leq x \leq \delta + A \\ m(\delta + A) + m\left(1 - \frac{A\delta}{2B}\right)(x - \delta - A) & \delta + A \leq x \leq \delta + A + B \\ mL + m\left(1 + \frac{A\delta}{2B}\right)(x - L) & \delta + A + B \leq x \leq L \end{cases} \end{aligned}$$

Figure 3 describes the shapes of φ and ψ. Its properties to be used in the proof are:

1. it satisfies the conditions of (31);
2. $\|\psi_x - \varphi_x\|_{L^\infty[0,L]} \leq m\delta\left(1 + \frac{A}{2B}\right)$
3. $\|\psi - \varphi\|_{L^\infty[0,L]} = \frac{mA\delta}{2}$.

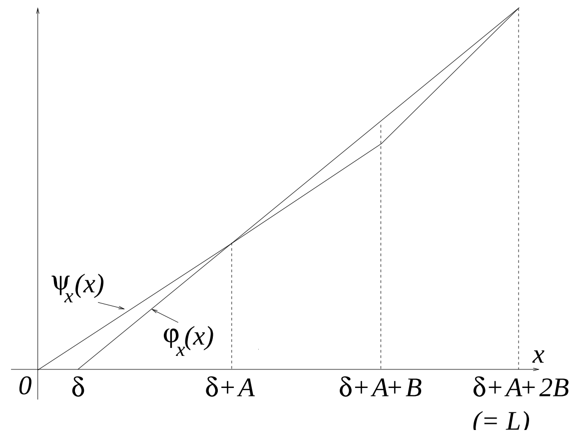

Figure 3

Now compute the difference between the energies of φ and ψ:

$$\int_0^L \epsilon^2\varphi_{xx}^2 + (|\varphi_x| - 1)^2 + \varphi^2\,dx - \int_0^L \epsilon^2\psi_{xx}^2 + (|\psi_x| - 1)^2 + \psi^2\,dx$$
$$= \int_0^L \epsilon^2\varphi_{xx}^2 - \epsilon^2\psi_{xx}^2\,dx + \int_0^L (|\varphi_x| - 1)^2 - (|\psi_x| - 1)^2 + \varphi^2 - \psi^2\,dx$$

The first integral of the above is equal to:

$$\epsilon^2 m^2(A + 2B) - \epsilon^2\left\{\left(\frac{mA}{\delta + A}\right)^2(\delta + A) + m^2\left(1 - \frac{A\delta}{2B}\right)^2 B + m^2\left(1 + \frac{A\delta}{2B}\right)^2 B\right\}$$
$$= \epsilon^2 m^2\delta\left(1 - \frac{\delta}{A + \delta} - \frac{A^2\delta}{2B}\right) \tag{33}$$

while the second integral can be bounded by

$$C\left(\|\varphi_x - \psi_x\|_{L^\infty[0,L]} + \|\varphi - \psi\|_{L^\infty[0,L]}\right)L \leq Cm\delta\left(1 + \frac{A}{2B} + \frac{A}{2}\right)L \tag{34}$$

where C depends on the Lipshitz constants of the function $(|x| - 1)^2$ and x^2 for bounded range of x. Comparing (33) and (34), it can be seen that upon choosing

$$0 \ll \delta \ll A \ll B \ll \epsilon^2, \tag{35}$$

the energy of ψ in $[0, L]$ will be lower than that of φ by at least an amount of $C\delta$. (The constant C depends on ϵ and m).

The extension to the more general φ is elementary.

1. Suppose $\varphi_x(x) \not\equiv 0$ for $0 < x < \delta$. By the hypothesis that $\int_0^\delta \varphi_{xx}^2\,dx < n$ and $\varphi_x(0) = 0$, we have $\varphi_x \in C^{\frac{1}{2}}([0, \delta])$. Hence $\|\varphi_x\|_{L^\infty[0,\delta]} \leq O(\delta^{\frac{1}{2}})$ and $\int_0^\delta |\varphi_x|\,dx \leq O(\delta^{\frac{3}{2}})$. Thus

modifying φ_x to be zero in $[0,\delta]$ will cause an increase in its energy of at most $O(\delta^{1+\alpha})$ for some $\alpha > 0$. If δ is small enough, this change can be absorbed by the term of (33).

2. Suppose $\varphi_{xx}(x) \not\equiv m$ for $x \geq \delta$. We can replace it by $\bar{\varphi}$ which does satisfy $\bar{\varphi}_{xx} \equiv m$ for $x \geq \delta$. Since $|u_{xxx}| \leq p$ by assumption, we have $\|\varphi_{xx} - \bar{\varphi}_{xx}\|_{L^\infty[\delta,L]} \leq O(L)$ for $0 < L \ll 1$ which leads to $|\mathcal{E}(\varphi) - \mathcal{E}(\bar{\varphi})| \leq O(L^2)$. On the other hand, from the previous estimate, we have $\mathcal{E}(\psi) - \mathcal{E}(\bar{\varphi}) \leq -C\delta$. Hence if L is further chosen to be $0 < L \ll O(\delta^{\frac{1}{2}})$ – still admissible to (35) – $\mathcal{E}(\psi)$ will have a lower value than $\mathcal{E}(\varphi)$.

The whole **Claim** is thus proved. □

Claim (iv). *Let $\partial^2\mathcal{E}(u,\varphi) > 0$ for all $\varphi \in V$. Then there is a $\eta > 0$ such that if v is a function satisfying for all $i = 0, 1 \ldots N-1$ the following conditions:*

$$\epsilon^2 v_{xxxx} - v_{xx} + v = 0 \;\; \textit{for} \;\; x \in (a_i, a_{i+1}),$$

$$v_x(a_i) = 0, \;\; v(a_i) = b_i, \;\; \mathrm{sgn}(v_x)|_{x\in(a_i,a_{i+1})} = \mathrm{sgn}(u_x)|_{x\in(c_i,c_{i+1})},$$

$$\textit{and} \;\; |a_i - c_i|, |b_i - d_i| < \eta,$$

then $\mathcal{E}(v) > \mathcal{E}(u)$.

Proof. This **Claim** is essentially a *finite dimensional* local-minimization problem. Let $c_i(t) = c_i + t(a_i - c_i)$. Consider the collection of functions $\{U(x,t) : x \in [0,1]\}_{0\leq t\leq 1}$ such that: $U_x(c_i(t),t) = 0$, $U(c_i(t),t) = d_i + t(b_i - d_i)$, $\mathrm{sgn}(U_x(x,t))|_{x\in(c_i(t),c_{i+1}(t))} = \mathrm{sgn}(u_x)|_{x\in(c_i,c_{i+1})}$ and $U(\cdot,t)$ satisfies (12) for $x \in (c_i(t), c_{i+1}(t))$. Let $\mathcal{E}(t) = \mathcal{E}(U(x,t))$. Following the same derivation as for (16) and (18), it is easy to verify that $\mathcal{E}'(0) = 0$ and $\mathcal{E}''(0) > 0$ for all a_i and b_i's close to c_i and d_i's. Then $\mathcal{E}(u)$ is a *strict* local minimizer among all small deviations of the a_i and b_i's from the c_i and d_i's. □

Next, we prove some preliminary estimates for any stationary point of the functional (1) having small enough energy. The results are crude but important to prepare for our proof later. They are similar to the results of [Mül, Section 2].

2.9 Proposition. *Let $u \in \mathcal{Z}$ be a solution of (10) and E be the energy of u:*

$$E = \mathcal{E}(u) = \int_0^1 \epsilon^2 u_{xx}^2 + W(u_x) + u^2\,dx.$$

Then there exist constants $E_* < 1$, $\epsilon_*, C_1, C_2 > 0$ such that for all $\epsilon < \epsilon_*$ and u with $E < E_*$, the following statements hold for all $i = 0, 1, \dots N-1$ and $x \in [0,1]$:

$$\left|\int_a^b u(y)\,dy\right| \leq C_1 E^{\frac{1}{2}}; \tag{36}$$

$$\left||\epsilon^2 u_{xxx}(c_i^{\pm})| - 1\right| \leq C_1 E^{\frac{1}{2}}, \tag{37}$$

$$|u_x(x)| \leq 1 + C_1 E^{\frac{1}{2}}; \tag{38}$$

$$|u(x)| \leq C_1 E^{\frac{1}{3}}; \tag{39}$$

$$|\epsilon^2 u_{xx}^2(c_i) - 1| \leq C_1 E^{\frac{1}{2}} \tag{40}$$

$$||u_x(\bar{x})| - 1| \leq C_1 E^{\frac{1}{4}} \ \text{ at all } \bar{x} \text{ such that } u_{xx}(\bar{x}) = 0. \tag{41}$$

$$C_2 \epsilon |\ln E| \leq (c_{i+1} - c_i) \leq C_1 E^{\frac{1}{3}}. \tag{42}$$

In particular, if E is small enough, (41) states that within any segment (c_i, c_{i+1}), u_x does not have any interior zero *and also there must be a point $\bar{x}$ such that $|u_x(\bar{x})| > \frac{1}{2}$. Furthermore, (42) states that the length of any segment cannot be too long but is much longer than ϵ.*

The proof relies on the following integrated version of (12) is used often in the analysis:

$$\epsilon^2 u_{xxx} - \epsilon^2 u_{xxx}(c_i^+) = u_x(x) - \int_{c_i}^x u(y)\,dy. \tag{43}$$

Multiplying the above equation by u_{xx} and integrating from c_i^+ to x gives:

$$\begin{aligned} \epsilon^2 u_{xx}^2(x) - (u_x(x) - \operatorname{sgn}(u_x))^2 = \epsilon^2 u_{xx}^2(c_i) - 1 + 2(\epsilon^2 u_{xxx}(c_i^+) + \operatorname{sgn}(u_x))u_x \\ - 2\left(\int_{c_i}^x u(y)\,dy\right) u_x(x) + u^2(x) - u^2(c_i). \end{aligned} \tag{44}$$

In the following, we will use C to denote some generic constant independent of ϵ and u. Its values can change from line to line. The symbol $O(a)$ refers to some quantity bounded by Ca. Without loss of generality, we also assume that $\operatorname{sgn}(u_x)|_{(c_i, c_{i+1})} = (-1)^i$.

Proof of (36). This is done by using the Cauchy-Schwartz Inequality:

$$\left|\int_a^b u(x)\,dx\right| \leq \left(\int_a^b u^2(x)\,dx\right)^{\frac{1}{2}} (b-a)^{\frac{1}{2}} \leq O(E^{\frac{1}{2}}).$$

Proof of (37). Letting $x = c_{i+1}^-$ in (43) gives:

$$\epsilon^2 u_{xxx}(c_{i+1}^-) - \epsilon^2 u_{xxx}(c_i^+) = -\int_{c_i}^{c_{i+1}} u\,dx.$$

Upon summing over the i's, we have

$$\epsilon^2 u_{xxx}(c_j^-) - \sum_{i=1}^{j-1} \left(\epsilon^2 u_{xxx}(c_i^+) - \epsilon^2 u_{xxx}(c_i^-)\right) - \epsilon^2 u_{xxx}(c_0^+) = -\int_{c_0}^{c_j} u\, dx.$$

Without loss of generality, assume $c_0 = 0$ and $u_x > 0$ on (c_0, c_1). Then by (14) and (15), we have

$$\epsilon^2 u_{xxx}(c_j^-) = -1 - 2\sum_{i=1}^{j}(-1)^i - \int_{c_0}^{c_j} u(x)\, dx.$$

In view of (36), the above leads to $\left||\epsilon^2 u_{xxx}(c_j^-)| - 1\right| < O(E^{\frac{1}{2}})$ which by (14) also gives $\left||\epsilon^2 u_{xxx}(c_j^+)| - 1\right| < O(E^{\frac{1}{2}})$.

Proof of (38). Consider (43). Let $u_x > 0$ in (c_i, c_{i+1}) and attain its maximum at $\bar{x}$. Then $u_{xxx}(\bar{x}) < 0$ and hence

$$0 < u_x(\bar{x}) \leq -\epsilon^2 u_{xxx}(c_i^+) + \int_{c_i}^{\bar{x}} u(y)\, dy \leq 1 + O(E^{\frac{1}{2}}).$$

The lower bound $-1 - O(E^{\frac{1}{2}}) < u_x$ can be proved similarly.

Proof of (39). Let $u(x) = D > 0$. By the (38) just proved, we must have

$$\text{either } \int_x^{x+A} u^2(y)\, dy \ \text{ or } \int_{x-A}^{x} u^2(y)\, dy > \frac{D^3}{3(1 + O(E^{\frac{1}{2}}))}$$

for some constant $A = \dfrac{D}{1 + O(E^{\frac{1}{2}})}$. This leads to $D \leq O(E^{\frac{1}{3}})$.

Proof of (40). Consider (44) and assume $u_x > 0$ for $x \in (c_i, c_{i+1})$. Let $\bar{x}$ be the point where u_x attains its maximum. Then $u_{xx}(\bar{x}) = 0$. Furthermore, by (36) – (39), we have $-(u_x(\bar{x}) - \text{sgn}(u_x(\bar{x}))^2 = \epsilon^2 u_{xx}^2(c_i) - 1 + O(E^{\frac{1}{2}})$ so that $\epsilon^2 u_{xx}^2(c_i) - 1 < O(E^{\frac{1}{2}})$.

On the other hand, integrating (44) from $x = 0$ to $x = 1$, we obtain:

$$\epsilon^2 u_{xx}^2(c_i) - 1 + O(E^{\frac{1}{2}}) = \int_0^1 \epsilon^2 u_{xx}^2 - (u_x - \text{sgn}(u_x))^2\, dx \geq -\int_0^1 W(u_x)\, dx \geq -E.$$

Hence we also have the lower bound $\epsilon^2 u_{xx}^2(c_i) - 1 > -O(E^{\frac{1}{2}})$

Proof of (41). This follows immediately from (44) and all the previous estimates.

Proof of (42). Without loss of generality, assume that $u_x > 0$ on (c_i, c_{i+1}). First we consider the upper bound. Let

$$m_+ = \text{meas}\left\{x \in (c_i, c_{i+1}) : u_x \geq \frac{1}{2}\right\} \ \text{ and } \ m_- = \text{meas}\left\{x \in (c_i, c_{i+1}) : u_x < \frac{1}{2}\right\}.$$

Then obviously,

$$\int_{c_i}^{c_{i+1}} W(u_x)\,dx \geq W\left(\frac{1}{2}\right) m_-.$$

Moreover, simple rearrangement argument leads to:

$$\int_{c_i}^{c_{i+1}} u^2(x)\,dx \geq \int_{-\frac{m_+}{2}}^{\frac{m_+}{2}} \left(\frac{x}{2}\right)^2\,dx = \frac{m_+^3}{48}$$

Then (42) follows by the fact that: $(c_{i+1} - c_i) = m_+ + m_- \leq O(E^{\frac{1}{3}}) + O(E^{\frac{1}{2}})$.

For the lower bound, using the previous estimates, we can write (44) as:

$$\epsilon^2 u_{xx}^2(x) = (u_x - 1)^2 \pm O(E^{\frac{1}{2}}) \quad \text{i.e.} \quad dx = \frac{\epsilon du_x}{\sqrt{(u_x - 1)^2 \pm O(E^{\frac{1}{2}})}}$$

Upon integrating the above from $u_x = 0$ to $u_x = 1 - CE^{\frac{1}{4}}$ for some large enough constant C, the following estimates hold:

$$(c_{i+1} - c_i) \geq \frac{\epsilon}{\sqrt{2}} \int_0^{1-CE^{\frac{1}{4}}} \frac{dv}{1-v} = O(\epsilon|\ln E|)$$

concluding (42). □

We also have the following lower bound for the energy inside any segment (c_i, c_{i+1}):

2.10 Lemma. *Under the same hypothesis as* **Proposition 2.9**, *then for all* $\epsilon < \epsilon_*$ *and* u *such that* $E \leq E_*$, *it holds that*

$$\int_{c_i}^{c_{i+1}} \epsilon^2 u_{xx}^2 + W(u_x)\,dx \geq \frac{3}{4}\epsilon \textit{ for all } i.$$

Proof.

$$\begin{aligned}\int_{c_i}^{c_{i+1}} \epsilon^2 u_{xx}^2 + W(u_x)\,dx &\geq \epsilon \int_{c_i}^{c_{i+1}} \epsilon u_{xx}^2 + \frac{1}{\epsilon} W(u_x)\,dx \\ &\geq 2\epsilon \int_0^{\frac{1}{2}} \sqrt{W(u_x)}|u_{xx}|\,dx = \frac{3}{4}\epsilon\end{aligned}$$

since from (41), there must be a point $\bar{x} \in (c_i, c_{i+1})$ such that $|u_x(\bar{x})| \geq \frac{1}{2}$. □

3 Propagation Map for Euler-Lagrange Equation

Here we introduce the *propagation map* which relates the boundary values at the end points of a segment over which the solution u is monotone. Precisely, we consider the following boundary values problem:

$$\epsilon^2 u_{xxxx} - u_{xx} + u = 0 \text{ such that } u_x(0) = u_x(l) = 0 \text{ and } u_x \neq 0 \text{ for } 0 \leq x \leq l. \tag{45}$$

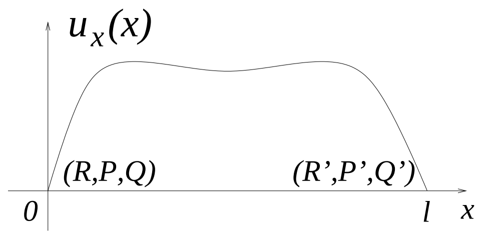

Figure 4.

For the case $u_x > 0$, the *initial* and *final values* are defined as:

$$u(0) = R, \;\; \epsilon u_{xx}(0) = P, \;\; \epsilon^2 u_{xxx}(0) = -Q$$
$$u(l) = R', \;\; \epsilon u_{xx}(l) = -P', \;\; \epsilon^2 u_{xxx}(l) = -Q'.$$

The existence of a unique solution to the above problem is proved in **Proposition 3.3**. In the actual application of the result, the segment $(0, l)$ will be one of the (c_i, c_{i+1})'s. As we are interested in the case of $\epsilon \to 0$ and $\mathcal{E}(u) \ll 1$, by **Proposition 2.9**, we only need to consider the regime of $l, R, R' = o(1)$ and $P, P', Q, Q' = 1 + o(1)$.

Our goal is to compute the following *propagation* map as a function of l and ϵ:

$$(R, P, Q) \longrightarrow_{l,\epsilon} (R', P', Q').$$

This requires solving (45). Its characteristic polynomial is $\epsilon^2 r^4 - r^2 + 1 = 0$ which has four roots $\pm\Lambda$ and $\pm\lambda$:

$$\Lambda = \sqrt{\frac{1 + \sqrt{1 - 4\epsilon^2}}{2\epsilon^2}} \quad \text{and} \quad \lambda = \sqrt{\frac{1 - \sqrt{1 - 4\epsilon^2}}{2\epsilon^2}}. \tag{46}$$

As $\epsilon \to 0$, they have the following asymptotic forms:

$$\Lambda = \frac{1 - \frac{1}{2}\epsilon^2 + O(\epsilon^4)}{\epsilon} \text{ and } \lambda = 1 + O(\epsilon^2).$$

The solution u of (45) is given by:

$$u = Ae^{\Lambda x} + Be^{-\Lambda x} + Ce^{\lambda x} + De^{-\lambda x} \tag{47}$$

where A, B, C, D satisfy:

$$\left\{\begin{array}{rcl} A+B+C+D & = & R \\ \Lambda(A-B)+\lambda(C-D) & = & 0 \\ \Lambda^2(A+B)+\lambda^2(C+D) & = & \frac{P}{\epsilon} \\ \Lambda^3(A-B)+\lambda^3(C-D) & = & -\frac{Q}{\epsilon^2} \end{array}\right. \tag{48}$$

The exact solutions and asymptotic approximations for these constants are:

$$\left\{\begin{array}{rcccccl} A & = & \dfrac{\epsilon P-\Lambda^{-1}Q-\epsilon^2\lambda^2 R}{2\sqrt{1-4\epsilon^2}} & = & \dfrac{\epsilon(P-Q-\epsilon R)}{2}+O(\epsilon^3) & = & o(\epsilon) \\ B & = & \dfrac{\epsilon P+\Lambda^{-1}Q-\epsilon^2\lambda^2 R}{2\sqrt{1-4\epsilon^2}} & = & \dfrac{\epsilon(P+Q-\epsilon R)}{2}+O(\epsilon^3) & = & \epsilon+o(\epsilon) \\ C & = & \dfrac{\epsilon^2\left(\Lambda^2 R-\frac{P}{\epsilon}+\frac{Q}{\lambda\epsilon^2}\right)}{2\sqrt{1-4\epsilon^2}} & = & \dfrac{-\epsilon P+Q+R}{2}+O(\epsilon^2) & = & \dfrac{1}{2}+o(1) \\ D & = & \dfrac{\epsilon^2\left(\Lambda^2 R-\frac{P}{\epsilon}-\frac{Q}{\lambda\epsilon^2}\right)}{2\sqrt{1-4\epsilon^2}} & = & \dfrac{-\epsilon P-Q+R}{2}+O(\epsilon^2) & = & -\dfrac{1}{2}+o(1) \end{array}\right. \tag{49}$$

Now, if $u_x(l)=0$, we have:

$$\Lambda\left(Ae^{\Lambda l}-Be^{-\Lambda l}\right)+\lambda(Ce^{\lambda l}-De^{-\lambda l})=0 \quad \text{or} \quad Ae^{\Lambda l}=Be^{-\Lambda l}-\lambda\Lambda^{-1}(Ce^{\lambda l}-De^{-\lambda l}).$$

Then the solution form (47) for u becomes:

$$u(x)=\left(Be^{-\Lambda l}-\lambda\Lambda^{-1}\left(Ce^{\lambda l}-De^{-\lambda l}\right)\right)e^{-\Lambda l}e^{\Lambda x}+Be^{-\Lambda x}+Ce^{\lambda x}+De^{-\lambda x}. \tag{50}$$

which leads to the following expressions:

$$\left\{\begin{array}{rcccl} R' & = & u(l) & = & 2Be^{-\Lambda l}+(Ce^{\lambda l}+De^{-\lambda l})-\epsilon(Ce^{\lambda l}-De^{-\lambda l})+O(\epsilon^2) \\ P' & = & -\epsilon u_{xx}(l) & = & -\frac{2Be^{-\Lambda l}}{\epsilon}+(Ce^{\lambda l}-De^{-\lambda l})-\epsilon(Ce^{\lambda l}+De^{-\lambda l})+O(\epsilon^2) \\ Q' & = & -\epsilon^2 u_{xxx}(l) & = & (Ce^{\lambda l}-De^{-\lambda l})-\epsilon^2(Ce^{\lambda l}-De^{-\lambda l})+O(\epsilon^2) \end{array}\right. \tag{51}$$

Similar results can also be derived for the case $u_x<0$. Then we would write

$$P=-\epsilon u_{xx}(0),\;\; Q=\epsilon^2 u_{xxx}(0) \quad \text{and} \quad P'=\epsilon u_{xx}(l),\;\; Q'=\epsilon^2 u_{xxx}(0)$$

so that P, P', Q, Q' are all positive numbers.

The following classification of segments is crucial to the approach we use.

3.1 Definition. *A segment* (c_i, c_{i+1}) *is called* long *if* $(c_{i+1}-c_i)\geq 10\epsilon|\ln\epsilon|$. *It is called* short *otherwise.*

By means of (49), we now expand the formulas (51) so as to relate (R', P', Q') to (R, P, Q). The results are grouped into four categories.

3.2 Proposition. *Let $Q = 1 + \alpha$ and $\nu^2 = e^{-\frac{l}{\epsilon}}$. For any $c_1 > 0$, there exist ϵ_* and $l_* > 0$ such that the following expressions hold for all $\epsilon < \epsilon_*$:*

Long Positive Segment. *If $u_x > 0$ and $10\epsilon|\ln \epsilon| \le l = L \le l_*$, then*

$$P = Q + \epsilon R + O(\epsilon^2) \tag{52}$$

$$P' = Q' - \epsilon R' + O(\epsilon^2) \tag{53}$$

$$R' = R + L - 2\epsilon + \alpha L - 2\epsilon\alpha + \frac{L^3}{6} + \frac{RL^2}{2} - \epsilon RL + \frac{\alpha L^3}{6} + o|L^3| + O(\epsilon^2) \tag{54}$$

$$P' = Q + \left(R + \frac{L}{2} - \epsilon\right) L + \frac{\alpha L^2}{2} - \epsilon\alpha L - \epsilon R' + o|L^3| + O(\epsilon^2) \tag{55}$$

$$Q' = Q + \left(R + \frac{L}{2} - \epsilon\right) L + \frac{\alpha L^2}{2} - \epsilon\alpha L + o|L^3| + O(\epsilon^2) \tag{56}$$

Long Negative Segment. *If $u_x < 0$ and $10\epsilon|\ln \epsilon| \le l = L \le l_*$, then*

$$P = Q - \epsilon R + O(\epsilon^2), \tag{57}$$

$$P' = Q' + \epsilon R' + O(\epsilon^2) \tag{58}$$

$$R' = R - L + 2\epsilon - \alpha L + 2\epsilon\alpha - \frac{L^3}{6} + \frac{RL^2}{2} - \epsilon RL - \frac{\alpha L^3}{6} + o|L^3| + O(\epsilon^2) \tag{59}$$

$$P' = Q - \left(R - \frac{L}{2} + \epsilon\right) L + \frac{\alpha L^2}{2} - \epsilon\alpha L + \epsilon R' + o|L^3| + O(\epsilon^2) \tag{60}$$

$$Q' = Q - \left(R - \frac{L}{2} + \epsilon\right) L + \frac{\alpha L^2}{2} - \epsilon\alpha L + o|L^3| + O(\epsilon^2) \tag{61}$$

Short Positive Segment. *If $u_x > 0$ and $l \le 10\epsilon|\ln \epsilon|$, then*

$$P = Q - 2Q\nu^2 + \epsilon R + O(\epsilon^2) + o(\nu^2) \tag{62}$$

$$P' = Q' - 2Q\nu^2 - \epsilon R' + O(\epsilon^2) + o(\nu^2) \tag{63}$$

$$R' = R + l - 2\epsilon + \alpha l - 2\epsilon\alpha + \frac{Rl^2}{2} - \epsilon Rl + 2\epsilon\nu^2 + o(\epsilon\nu^2) + O(\epsilon^2) + o(\nu^2) \tag{64}$$

$$P' = Q - 2Q\nu^2 + \left(R + \frac{l}{2} - \epsilon\right) l + \frac{\alpha l^2}{2} - \epsilon\alpha l - \epsilon R' + O(\epsilon^2) + o(\nu^2) \tag{65}$$

$$Q' = Q + \left(R + \frac{l}{2} - \epsilon\right) l + \frac{\alpha l^2}{2} - \epsilon\alpha l + O(\epsilon^2) + o(\nu^2) \tag{66}$$

Short Negative Segment. *If $u_x < 0$ and $l \leq 10\epsilon|\ln\epsilon|$, then*

$$
\begin{aligned}
P &= Q - 2Q\nu^2 - \epsilon R + O(\epsilon^2) + o(\nu^2) && (67)\\
P' &= Q' - 2Q\nu^2 + \epsilon R' + O(\epsilon^2) + o(\nu^2) && (68)\\
R' &= R - l + 2\epsilon - \alpha l + 2\epsilon\alpha + \frac{Rl^2}{2} - \epsilon Rl - 2\epsilon\nu^2 + o(\epsilon\nu^2) + O(\epsilon^2) + o(\nu^2) && (69)\\
P' &= Q - 2Q\nu^2 - \left(R - \frac{l}{2} + \epsilon\right) l + \frac{\alpha l^2}{2} - \epsilon\alpha l + \epsilon R' + O(\epsilon^2) + o(\nu^2) && (70)\\
Q' &= Q - \left(R - \frac{l}{2} + \epsilon\right) l + \frac{\alpha l^2}{2} - \epsilon\alpha l + O(\epsilon^2) + o(\nu^2) && (71)
\end{aligned}
$$

We make the following remarks about the above expansions:

1. If $l \geq 10\epsilon|\ln\epsilon|$, then $e^{-\frac{l}{\epsilon}} \ll O(\epsilon^2)$;

2. For all δ, if $\mathcal{E}(u)$ is small enough, then $\nu^2 = e^{-\frac{l}{\epsilon}} < \delta$ (see (42));

3. $e^{-\Lambda l} = \nu^2 + o(\nu^2)$.

4. The expressions (53), (58), (63), and (68) come directly from (51) and are consistent with the formulas for R', P', and Q' (such as (54), (55), (56) and so forth).

5. Not all the terms are relevant in the actual analysis. In fact, we just need to keep the terms of order up to L^2, l^2, and ν^2.

As a first application of the use of the explicit solution formula, we prove the existence and uniqueness of the initial-final value problem (45) and also the functions $P(x,l)$'s defined in (3).

3.3 Proposition. *There exist $l_*, \alpha, \beta > 0$ such that for all $0 < l \leq l_*$ and Q, Q' satisfying $|Q - Q'| \leq \alpha l$ and $|Q + 1| \leq \beta$ the following boundary value problem*

$$
\begin{aligned}
\epsilon^2 u_{xxxx} - u_{xx} + u = 0 &\qquad \text{for } x \in (0,l) && (72)\\
u_x(0) = 0; &\qquad u_x(l) = 0;\\
\epsilon^2 u_{xxx}(l) = Q, &\qquad \epsilon^2 u_{xxx}(0) = Q'
\end{aligned}
$$

has a unique solution u. In addition, $u_x(x) > 0$ for $x \in (0,l)$.

Proof. The existence and uniqueness follows from the explicit solution formula similar to (49). In the present case, the expressions become

$$\begin{cases} A &= \dfrac{Qe^{-\Lambda l} - Q'}{\epsilon^2\Lambda(\Lambda^2-\lambda^2)(e^{\Lambda l} - e^{-\Lambda l})} \\ B &= \dfrac{Qe^{\Lambda l} - Q'}{\epsilon^2\Lambda(\Lambda^2-\lambda^2)(e^{\Lambda l} - e^{-\Lambda l})} \\ C &= \dfrac{-Qe^{-\lambda l} + Q'}{\epsilon^2\lambda(\Lambda^2-\lambda^2)(e^{\lambda l} - e^{-\lambda l})} \\ D &= \dfrac{-Qe^{\lambda l} + Q'}{\epsilon^2\lambda(\Lambda^2-\lambda^2)(e^{\lambda l} - e^{-\lambda l})} \end{cases} \tag{73}$$

(Uniqueness can also be proved from an energy type estimate.)

To prove the positivity of u_x, using the integrated version (43) of (72), we have

$$u_x(x) = \epsilon^2 u_{xxx}(x) - \int_0^x u(y)\,dy - \epsilon^2 u_{xxx}(0).$$

Suppose $u_x \le 0$ somewhere in $(0,l)$, then u_x attains a non-positive local minimum at a point $\bar{x} \in (0,l)$. Then $u_{xxx}(\bar{x}) \ge 0$. Hence,

$$u_x(\bar{x}) \ge -\left|\int_0^{\bar{x}} u(y)\,dy\right| - \left|\epsilon^2 u_{xxx}(0) + 1\right| + 1$$

If it can be shown that $\int_0^{\bar{x}} u(y)dy$ is small enough, then by the assumptions of this **Proposition**, we would have $u_x(\bar{x}) > 0$ which leads to a contradiction. To achieve this, we use (43) to get:

$$\left|\int_0^l u(y)\,dy\right| = \left|\epsilon^2 u_{xxx}(0) - \epsilon^2 u_{xxx}(l)\right| \le \alpha l. \tag{74}$$

Multiplying (72) by u and then integrating by parts give

$$\begin{aligned} &\int_0^l \epsilon^2 u_{xx}^2 + u_x^2 + u^2\,dx \\ = \;& -\epsilon^2 u_{xxx}(l)u(l) + \epsilon^2 u_{xxx}(0)u(0) \\ \le \;& |\epsilon^2 u_{xxx}(0) - \epsilon u_{xxx}(l)||u(l)| + |\epsilon^2 u_{xxx}(0)||u(l) - u(0)| \\ \le \;& \alpha\left|\int_0^l (u(l) - u(y))\,dy\right| + \alpha\left|\int_0^l u(y)\,dy\right| + \left|\epsilon^2 u_{xxx}(0)\right|\left|\int_0^l u_y(y)\,dy\right| \end{aligned}$$

By $\left|\int_0^l u_y\,dy\right| \le l^{\frac{1}{2}}\left(\int_0^l u_y^2\,dy\right)^{\frac{1}{2}}$ and $\left|\int_0^l u\,dy\right| \le l^{\frac{1}{2}}\left(\int_0^l u^2\,dy\right)^{\frac{1}{2}}$, the above leads to

$$\int_0^l \epsilon^2 u_{xx}^2 + u_x^2 + u^2\,dx \le \delta$$

in which δ can be made as small as possible if l_*, α, β are chosen small enough. A simple application of Cauchy-Schwarz inequality gives for all x,

$$\left|\int_0^x u(y)\,dy\right| \le \int_0^l |u(y)|\,dy \le l^{\frac{1}{2}}\delta^{\frac{1}{2}}$$

which finishes the proof. □

Notation. Before leaving this section, we introduce some notations to be used in all of the following analysis and Figures. The function u always refers to a function from $\mathcal{Z}$ which solves (10) in the sense of **Definition 2.2**. The triples $(R_i, \pm P_i, \pm Q_i)$ and $(R_i', \pm P_i', \pm Q_i')$ denote the values of $(u, \epsilon u_{xx}, \epsilon^2 u_{xxx})$ at c_i^+ and c_{i+1}^-. The $\pm$ are chosen according to the sign of u_x in (c_i, c_{i+1}). With this notation, we then have

$$R_i' = R_{i+1}, \;\; P_i' = P_{i+1} \;\text{ and }\; Q_i' + Q_{i+1} = 2.$$

4 Proof of Theorem 1.1

In this section, we will prove the stability of periodic structures with long periods.

By approximation, we can assume that φ is C^∞. In this case, we have the expression (18) for the second variable of $\mathcal{E}$ with respect to φ. In addition, the zeros $\{c_i(t)\}_{i=1}^{N-1}$ of $u_x(x) + t\varphi_x(x)$ are C^1 functions satisfying (17).

Let $E(l)$ be the quantity defined in **Proposition 4.1**. Then for small enough t,

$$\mathcal{E}(u + t\varphi) \ge \sum_{i=0}^{N-1} E(l_i(t)) \;\text{ where } l_i(t) = c_{i+1}(t) - c_i(t).$$

The above together with $\mathcal{E}(u) = \sum_i E(l_i(0))$ lead to

$$\frac{d}{dt}\mathcal{E}(u+t\varphi)\Big|_{t=0} = \sum_i \frac{d}{dt}E(l_i(t))\Big|_{t=0} \quad\text{and}\quad \frac{d^2}{dt^2}\mathcal{E}(u+t\varphi)\Big|_{t=0} \ge \sum_i \frac{d^2}{dt^2}E(l_i(t))\Big|_{t=0}$$

Since

$$\frac{d}{dt}E(l_i(t)) = E'(l_i(t))\dot{l}_i(t) \;\text{ and }\; \frac{d^2}{dt^2}E(l_i(t)) = E''(l_i(t))(\dot{l}_i(t))^2 + E'(l_i(t))\ddot{l}_i(t).$$

we have

$$
\begin{aligned}
\frac{d^2}{dt^2}\mathcal{E}(u+t\varphi)\bigg|_{t=0} &\geq \sum_i E''(l_i)(\dot{l}_i(0))^2 + E'(l_i)\ddot{l}_i(0) \\
&= E''(l)\sum_i(\dot{l}_i(0))^2 + E'(l)\sum_i(\ddot{l}_i(0)) \quad \text{(since } l_i(0)=l\text{)} \\
&= E''(l)\sum_i(\dot{l}_i(0))^2 \quad \text{(since } \textstyle\sum_i l_i = 1\text{)} \\
&\geq Ml\sum_i(\dot{l}_i(0))^2 \quad \text{(by (77) in } \textbf{Proposition 4.1}\text{)}
\end{aligned}
$$

where the constant M can be chosen to be close to $\frac{1}{2}$ (independent of ϵ and l). The desired result follows by using the fact that $\dot{l}_i(0) = \dot{c}_{i+1}(0) - \dot{c}_i(0)$ and also (17).

4.1 Proposition. *Let $P(x,l)$ be defined as in (3) and*

$$E(l) = \int_{-\frac{l}{2}}^{\frac{l}{2}} \epsilon^2 P_{xx}^2 + (P_x - 1)^2 + P^2\,dx$$

Then

1. $\displaystyle\int_{-\frac{l}{2}}^{\frac{l}{2}} \epsilon^2 v_{xx}^2 + (v_x-1)^2 + v^2\,dx \geq E(l)$ *where* $v_x > 0$ *for* $x \in (-\frac{l}{2}, \frac{l}{2})$ *and* $v_x(\pm\frac{l}{2}) = 0$.
2. *There exist constants $C_L < 10$, l_* and $\epsilon_* > 0$ such that for any $0 < \epsilon < \epsilon_*$, if $C_L\epsilon|\ln\epsilon| < l < l_*$, then for some constant C,*

$$
\begin{aligned}
\left|E(l) - \left(2\epsilon + \frac{l^3}{12}\right)\right| &\leq Cl^3\left(l^2 + \frac{\epsilon}{l}\right) &\quad (75)\\
\left|E'(l) - \frac{l^2}{4}\right| &\leq Cl^2\left(l^2 + \frac{\epsilon}{l}\right) &\quad (76)\\
\left|E''(l) - \frac{l}{2}\right| &\leq Cl\left(l^2 + \frac{\epsilon}{l}\right) &\quad (77)
\end{aligned}
$$

(This **Proposition** is basically the same as [Mül, Thm 4.2(i), 5.1]. The proof here is much simplified due to the use of explicit solution formulas.)

Proof of 1. (This statement is similar to **Claim (i)**, page 16.) Let $\varphi = v - P$. Then

$$
\begin{aligned}
&\int_{-\frac{l}{2}}^{\frac{l}{2}} \epsilon^2 v_{xx}^2 + (v_x-1)^2 + v^2\,dx - E(l) \\
&\qquad = 2\int_{-\frac{l}{2}}^{\frac{l}{2}} \epsilon^2 P_{xx}\varphi_{xx} + (P_x - 1)\varphi_x + P\varphi\,dx + \int_{-\frac{l}{2}}^{\frac{l}{2}} \epsilon^2\varphi_{xx}^2 + \varphi_x^2 + \varphi^2\,dx
\end{aligned}
$$

The first integral of the right hand side of the above equals

$$2\left(\epsilon^2 P_{xx}\varphi_x - \epsilon^2 P_{xxx}\varphi - \varphi\right)\Big|_{-\frac{l}{2}}^{\frac{l}{2}} + 2\int_{-\frac{l}{2}}^{\frac{l}{2}} (\epsilon^2 P_{xxxx} - P_{xx} + P)\varphi\, dx$$

which vanishes by the property (3) of P and the fact that $\varphi_x(\pm\frac{l}{2}) = 0$.

Proof of 2. We first simplify the expressions for $E(l)$, $E'(l)$ and $E''(l)$.

$$\begin{aligned} E(l) &= \int_{-\frac{l}{2}}^{\frac{l}{2}} \epsilon^2 P_{xx}^2 + (P_x - 1)^2 + P^2\, dx = \int_{-\frac{l}{2}}^{\frac{l}{2}} \epsilon^2 P_{xx}^2 + P_x^2 + P^2 - 2P_x + 1\, dx \\ &= \epsilon^2 P_{xx} P_x\Big|_{-\frac{l}{2}}^{\frac{l}{2}} + \int_{-\frac{l}{2}}^{\frac{l}{2}} (-\epsilon^2 P_{xxx} + P_x)P_x + P^2\, dx - 4P(\frac{l}{2}, l) + l \\ &= -\epsilon^2 P_{xxx} P\Big|_{-\frac{l}{2}}^{\frac{l}{2}} - 4P(\frac{l}{2}, l) + l = l - 2P(\frac{l}{2}, l). \end{aligned}$$

Similarly,

$$\begin{aligned} E'(l) &= \int_{-\frac{l}{2}}^{\frac{l}{2}} 2\epsilon^2 P_{xx} P_{lxx} + 2(P_x - 1)P_{lx} + 2PP_l\, dx + \epsilon^2 P_{xx}^2(\frac{l}{2}, l) + 1 + P^2(\frac{l}{2}, l) \\ &= 2\epsilon^2 P_{xx} P_{lx}\Big|_{-\frac{l}{2}}^{\frac{l}{2}} + \epsilon^2 P_{xx}^2(\frac{l}{2}, l) + 1 + P^2(\frac{l}{2}, l) \\ &= 1 - \epsilon^2 P_{xx}^2(\frac{l}{2}, l) + P^2(\frac{l}{2}, l) \end{aligned}$$

where we have used the fact that $P_x(\pm\frac{l}{2}, l) = 0$ and hence $P_{xl}(\pm\frac{l}{2}, l) = \mp\frac{1}{2}P_{xx}(\pm\frac{l}{2}, l)$. Finally

$$E''(l) = -2\epsilon^2 P_{xx}(\frac{l}{2}, l)\frac{d}{dl}P_{xx}(\frac{l}{2}, l) + 2P(\frac{l}{2}, l)\frac{d}{dl}P(\frac{l}{2}, l)$$

Next we express $P(x, l)$ by the explicit solution formula (47) which now takes the following form:

$$P(x, l) = A(e^{\Lambda x} - e^{-\Lambda x}) + C(e^{\lambda x} - e^{-\lambda x}).$$

From the boundary conditions for P: $P_x(\frac{l}{2}, l) = 0$ and $\epsilon^2 P_{xxx}(\frac{l}{2}, l) = -1$, we have:

$$A = -\frac{1}{\epsilon^2\Lambda(\Lambda^2 - \lambda^2)}\left(\frac{1}{e^{\Lambda\frac{l}{2}} + e^{-\Lambda\frac{l}{2}}}\right) \text{ and } C = \frac{1}{\epsilon^2\lambda(\Lambda^2 - \lambda^2)}\left(\frac{1}{e^{\lambda\frac{l}{2}} + e^{-\lambda\frac{l}{2}}}\right).$$

Then the following formula hold:

$$\begin{aligned}
P(\frac{l}{2},l) &= \frac{1}{\epsilon^2(\Lambda^2-\lambda^2)}\left[\frac{1}{\lambda}\left(\frac{e^{\lambda\frac{l}{2}}-e^{-\lambda\frac{l}{2}}}{e^{\lambda\frac{l}{2}}+e^{-\lambda\frac{l}{2}}}\right)-\frac{1}{\Lambda}\left(\frac{e^{\Lambda\frac{l}{2}}-e^{-\Lambda\frac{l}{2}}}{e^{\Lambda\frac{l}{2}}+e^{-\Lambda\frac{l}{2}}}\right)\right]\\
\frac{d}{dl}P(\frac{l}{2},l) &= \frac{1}{\epsilon^2(\Lambda^2-\lambda^2)}\left[\frac{2}{(e^{\lambda\frac{l}{2}}+e^{-\lambda\frac{l}{2}})^2}-\frac{2}{(e^{\Lambda\frac{l}{2}}+e^{-\Lambda\frac{l}{2}})^2}\right]\\
P_{xx}(\frac{l}{2},l) &= \frac{1}{\epsilon^2(\Lambda^2-\lambda^2)}\left[\lambda\left(\frac{e^{\lambda\frac{l}{2}}-e^{-\lambda\frac{l}{2}}}{e^{\lambda\frac{l}{2}}+e^{-\lambda\frac{l}{2}}}\right)-\Lambda\left(\frac{e^{\Lambda\frac{l}{2}}-e^{-\Lambda\frac{l}{2}}}{e^{\Lambda\frac{l}{2}}+e^{-\Lambda\frac{l}{2}}}\right)\right]\\
\frac{d}{dl}P_{xx}(\frac{l}{2},l) &= \frac{1}{\epsilon^2(\Lambda^2-\lambda^2)}\left[\frac{2\lambda^2}{(e^{\lambda\frac{l}{2}}+e^{-\lambda\frac{l}{2}})^2}-\frac{2\Lambda^2}{(e^{\Lambda\frac{l}{2}}+e^{-\Lambda\frac{l}{2}})^2}\right]
\end{aligned}$$

which give

$$\left|P(\frac{l}{2},l)-\left(\frac{l}{2}-\frac{l^3}{24}-\epsilon\right)\right|\leq Cl^3\left(l^2+\frac{\epsilon}{l}\right),\qquad \left|\frac{d}{dl}P(\frac{l}{2},l)-\frac{1}{2}\right|\leq C\left(l^2+\frac{\epsilon}{l}\right),$$
$$\left|\epsilon P_{xx}(\frac{l}{2},l)+1\right|\leq Cl^2\left(l^2+\frac{\epsilon}{l}\right),\qquad \left|\epsilon\frac{d}{dl}P_{xx}(\frac{l}{2},l)-\frac{\epsilon}{2}\right|\leq C\epsilon\left(l^2+\frac{\epsilon}{l}\right)$$

The above expressions immediately lead to the desired results. □

5 Proof of Theorem 1.2

The essence of the proof is to analyze the patterns consisting of both long and short segments and show that they have negative second variations. This is achieved by careful consideration of the interactions and matchings between long and short segments. We rely heavily on the formulas derived in **Section 3**.

To prepare for the proof, we first show that there must be at least one segment of length longer than $5\bar{L}$ where $\bar{L}=10\epsilon|\ln\epsilon|$ is the length of a long segment (see **Definition 3.1**). For if otherwise, by **Lemma 2.10**, we would have

$$\begin{aligned}
\mathcal{E}(u) &\geq \text{(No. of segments)} \times \text{(Lower bound for the energy in each segment)}\\
&> \frac{1}{5\bar{L}}\frac{3}{4}\epsilon=\frac{3}{200|\ln\epsilon|}
\end{aligned}$$

contradicting the smallness assumption on the energy (5).

Now let (c_i, c_{i+1}) be any one of the segments with length $\geq 5\bar{L}$. Assume for the moment this segment is not at the boundary. Consider the two adjacent segments (c_{i-1}, c_i) and (c_{i+1}, c_{i+2}). The following three cases can happen:

1. If both of them are short, we call the collection of segments $\{(c_j, c_{j+1}) : j = i-1, i, i+1\}$ **SLS**.

2. If only one of them is short, we call the same collection of segments **SLL** or **LLS**.

3. If both of them are long, without loss of generality, we consider the segments to the right of (c_{i+1}, c_{i+2}). We keep searching to the right until we find a short segment (c_k, c_{k+1}). By **Proposition 5.3** (proved later), the segments $\{(c_j, c_{j+1}) : j = i, \ldots k-2\}$ have roughly the same length. Now if $c_k - c_{k-1} \geq \frac{1}{2}(c_{k-1} - c_{k-2}) \geq 2\bar{L}$, then we consider the collection of segments $\{(c_j, c_{j+1}) : j = k-2, k-1, k\}$ and call it **LLS**. If otherwise, we then consider the collection $\{(c_j, c_{j+1}) : j = k-3, k-2, k-1\}$ and still regard it as **LLS**. (Note that in this case, the segment (c_{k-1}, c_k) is not yet known to be short according to **Definition 3.1**.)

If the initial long segment (c_i, c_{i+1}) is at the boundary, we can reflect the function u cross the boundary point (which is admissible due to the boundary conditions). Now the segment will have at least one long segments as a neighbor. Hence only the cases 2 or 3 above will arise.

In view of the above discussion, if u has both long and short segments, then it must contain a pattern of **SLL**, **LLS** or **SLS**. Hence **Theorem 1.2** will be proved once we establish the following claims:

Claim I. If u has a chain of adjacent long segments, then a sub-chain of them must have roughly the same length.

Claim II. If u contains any of the patterns **LLS**, **SLL** or **SLS**, then it is unstable.

Claim III. If u is a (nearly periodic) pattern with low enough energy with only long segments, then it is actually periodic.

We now proceed to prove the above claims. Their precise statements will be given along the proof.

5.1 Proof of Claim I

5.2 Lemma. *If both* (c_{i-1}, c_i) *and* (c_i, c_{i+1}) *are long segments, then*

$$Q'_{i-1} = 1 + O(\epsilon^2) \quad \textit{and} \quad Q_i = 1 + O(\epsilon^2) \tag{78}$$

Proof. Consider the point c_i. By (53) and (57), we have

$$P'_{i-1} = Q'_{i-1} - \epsilon R'_{i-1} + O(\epsilon^2) \quad \text{and} \quad P_i = Q_i - \epsilon R_i + O(\epsilon^2).$$

Since $P'_{i-1} = P_i$ and $R'_{i-1} = R_i$, it follows that $Q'_{i-1} - Q_i = O(\epsilon^2)$. This, together with the continuity condition (14) — $Q'_{i-1} + Q_i = 2$ — gives the desired statement. □

5.3 Proposition. *If* $\{(c_i, c_{i+1}),\ \ i = m, m+1, \ldots, n\}$ *are all long segments, then there exists an* $L_\epsilon > 0$ *such that*

$$\begin{aligned} \left| |R_i| - \left(\frac{L_\epsilon}{2} - \epsilon\right)\right| &= o(\epsilon) \quad \textit{for} \quad i = m+1, \ldots n; & (79) \\ \textit{and} \quad L_i = c_{i+1} - c_i &= L_\epsilon + o(L_\epsilon^2) + o(\epsilon) \quad \textit{for } i = m+1, \ldots n-1. & (80) \end{aligned}$$

Proof. First, by the previous **Lemma**, we have $Q_i = 1 + O(\epsilon^2)$ for $i = m+1, \ldots, n$. Then (56) and (61) lead to

$$R_i = \pm \frac{L_i}{2} \mp \epsilon + o(L_i^2) + o(\epsilon), \quad i = m+1, \ldots, n. \tag{81}$$

Hence we must have $|R_i| \gg O(\epsilon)$ for the same range of the i's.

Next, by setting $x = c_{i+1}$ into (44), we obtain

$$\epsilon^2 u_{xx}^2(c_{i+1}) - \epsilon^2 u_{xx}^2(c_i) = u^2(c_{i+1}) - u^2(c_i) \quad \text{or} \quad P_{i+1}^2 - P_i^2 = R_{i+1}^2 - R_i^2.$$

Upon summing over the i's, it holds that

$$P_j^2 - P_i^2 = R_j^2 - R_i^2.$$

By restricting $m+1 \leq i, j \leq n$, we can again invoke **Lemma 5.2** which together with (52), (53), (57) and (58) gives

$$(1 \pm \epsilon R_j)^2 - (1 \pm \epsilon R_i)^2 = R_j^2 - R_i^2 + O(\epsilon^2).$$

If $R_i R_j > 0$, then

$$\begin{aligned} (1+\epsilon R_j)^2 - (1+\epsilon R_i)^2 &= R_j^2 - R_i^2 + O(\epsilon^2) \\ 2\epsilon(R_j - R_i) &= (R_j - R_i)(R_j + R_i) + O(\epsilon^2) \\ R_j - R_i &= \frac{O(\epsilon^2)}{R_i + R_j + 2\epsilon} = o(\epsilon) \quad (\text{since } |R_i|, |R_j| \gg \epsilon.) \end{aligned}$$

Similarly, if $R_i R_j < 0$, then

$$R_j + R_i = \frac{O(\epsilon^2)}{R_i - R_j + 2\epsilon} = o(\epsilon).$$

To conclude (79), we can just take L_ϵ to be any of the $2(|R_i| + \epsilon)$'s. Statement (80) would also follow from (81) which now says $L_i = L_\epsilon + o(L_i^2) + o(\epsilon)$. □

The following **Corollary** is interesting in its own right even though it is not used in the later parts of our proof. It demonstrates that just the consideration of the first variation can already lead to some strong conclusion.

5.4 Corollary (Nearly Periodic Structures). *If $u \in \mathcal{Z}$ has only long segments, then it is* nearly periodic *in the sense that there exists R_ϵ and L_ϵ such that (79) and (80) hold for all $i = 0, 1, \dots N-1$.*

Proof. We can extend the conclusion to the boundary segments because it automatically holds that $\epsilon^2 u_{xxx} = \pm 1 + O(\epsilon^2)$ at the boundary points. This is what is actually needed in the proof. □

5.5 Proof of Claim II

Without loss of generality, it suffices to consider the following categories of patterns for u_x:

LLS (Long-Long-Short): (Figure 5.) The lengths of the segments satisfy:

$$|EA| = L_1 \geq 10\epsilon|\ln\epsilon|; \quad |AB| = L \geq 10\epsilon|\ln\epsilon|; \quad |BC| = l \leq \tfrac{1}{2}|AB|.$$

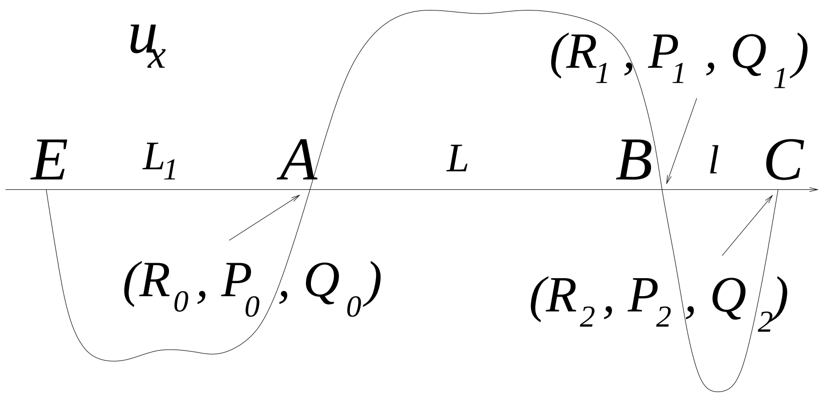

Figure 5.

SLS (Short-Long-Short): (Figure 6.) The lengths of the segments satisfy:

$$|EA| = m; \;\; |AB| = L \geq 10\epsilon|\ln\epsilon|; \;\; |BC| = l; \;\; |EA|, \; |BC| \;\; \leq \frac{|AB|}{2}$$

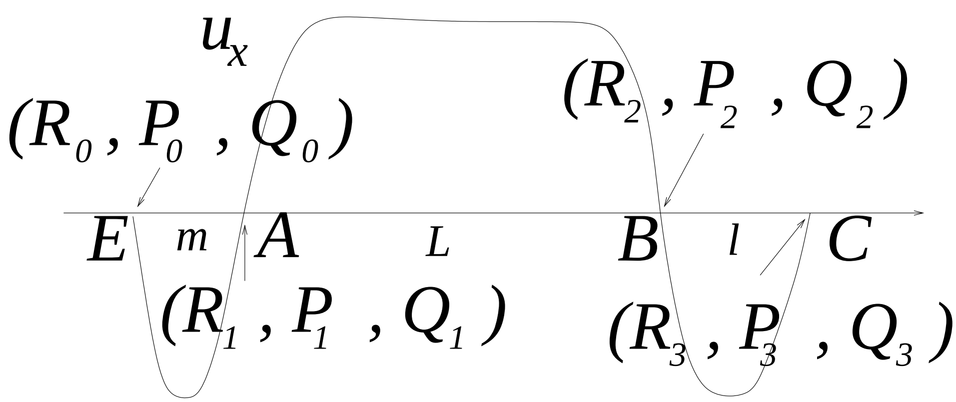

Figure 6.

We will produce test function φ's such that $\partial^2\mathcal{E}(u,\varphi) < 0$ for the above patterns. They are constructed by moving the interfaces and are given precisely in the next section.

5.5.1 Construction of Test Functions

Consider the following second order (Allen-Cahn) equation:

$$\epsilon^2 U_{xxx} = \frac{W'(U_x)}{2} \text{ or equivalently, } \epsilon^2 U_{xxx} = U_x - 1$$
$$\text{such that } U_x(\pm l) = 0 \text{ and } U_x > 0 \text{ for } |x| \leq l.$$

The solution, considered as a function of x and l, is given by:

$$U_x(x,l) = 1 - \frac{e^{\frac{x}{\epsilon}} + e^{-\frac{x}{\epsilon}}}{e^{\frac{l}{\epsilon}} + e^{-\frac{l}{\epsilon}}}.$$

Let $\nu^2 = e^{-\frac{2l}{\epsilon}}$ — note that $2l$ is the length of the segment. Associated with the above function are the following quantities:

$$\begin{aligned}
U_x(-l,l) &= 0, & U_x(l,l) &= 0\\
U_{xx}(-l,l) &= \frac{1}{\epsilon}\left(\frac{1-\nu^2}{1+\nu^2}\right), & U_{xx}(l,l) &= -\frac{1}{\epsilon}\left(\frac{1-\nu^2}{1+\nu^2}\right)\\
U_{xl}(-l,l) &= \frac{1}{\epsilon}\left(\frac{1-\nu^2}{1+\nu^2}\right), & U_{xl}(l,l) &= \frac{1}{\epsilon}\left(\frac{1-\nu^2}{1+\nu^2}\right)\\
U_{xxx}(-l,l) &= -\frac{1}{\epsilon^2}, & U_{xxx}(l,l) &= -\frac{1}{\epsilon^2}\\
U_{xxl}(-l,l) &= -\frac{1}{\epsilon^2}\left(\frac{1-\nu^2}{1+\nu^2}\right)^2, & U_{xxl}(l,l) &= \frac{1}{\epsilon^2}\left(\frac{1-\nu^2}{1+\nu^2}\right)^2
\end{aligned}$$

which will be used to compute the second variations.

Making use of the above $U_x(x,l)$, we introduce two types of test functions which mimic the movements of the interfaces:

Type One – $F_x(x,l)$ – **Movement of One Interface** (Figure 7.)

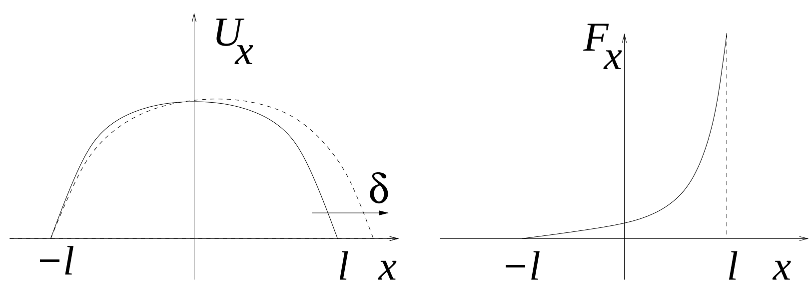

Figure 7.

$$F_x(x,l) = \lim_{\delta\to 0}\frac{U_x(x-\frac{\delta}{2}, l+\frac{\delta}{2}) - U_x(x,l)}{\delta} = -\frac{U_{xx}(x,l)}{2} + \frac{U_{xl}(x,l)}{2}.$$

Then F satisfies the following estimates:

$$F_x(-l,l) = 0, \qquad F_x(l,l) = \frac{1}{\epsilon}\left(\frac{1-\nu^2}{1+\nu^2}\right)$$

$$\int_{-l}^{l} \epsilon^2 F_{xx}^2 + F_x^2\,dx = \frac{1}{\epsilon}(1 - 4\nu^2 + o(\nu^2)) \tag{82}$$

$$F(l,l) - F(-l,l) = 1 - 4\nu^2 + o(\nu^2) \tag{83}$$

Type Two – $G_x(x,l)$ – **Translation of Two Adjacent Interfaces** (Figure 8.)

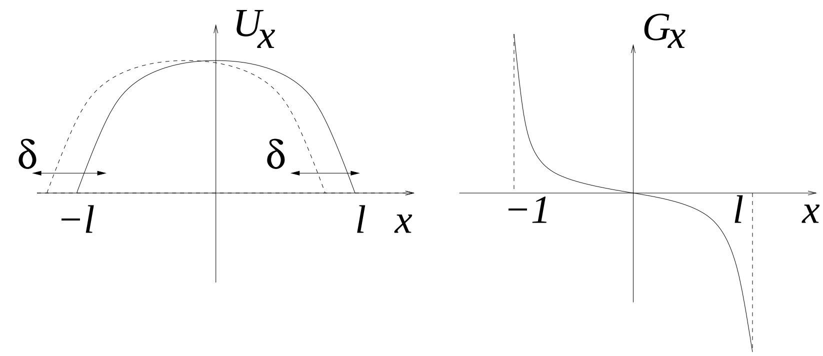

Figure 8.

$$G_x(x,l) = \lim_{\delta\to 0} \frac{U_x(x\pm\delta,l) - U_x(x,l)}{\delta} = \pm U_{xx}(x,l)$$

Similarly, G satisfies:

$$G_x(-l,l) = \pm\frac{1}{\epsilon}\left(\frac{1-\nu^2}{1+\nu^2}\right), \qquad G_x(l,l) = \mp\frac{1}{\epsilon}\left(\frac{1-\nu^2}{1+\nu^2}\right)$$

$$\int_{-l}^{l} \epsilon^2 G_{xx}^2 + G_x^2\, dx = \frac{2}{\epsilon}(1-2\nu^2+o(\nu^2)) \tag{84}$$

$$G(l,l) - G(-l,l) = 0 \tag{85}$$

A bit of motivation for the following analysis is in place. From (82) and (84), the negativity of the second variations comes from $-\nu^2$. So our goal is to characterize ν^2 as accurately as possible. It turns out that if a short segment is adjacent to a long segment, then its length cannot be arbitrary. This is due to some intricate interactions between the long and short segments. We will use the results of **Proposition 3.2** to provide some lower bounds for the values of ν^2. It is unfortunate that the present approach requires the consideration of a large number of cases.

We now proceed to analyze the patterns **LLS** and **SLS**.

5.5.2 Instability of LLS

We refer to Figure 5. Consider the point A. By (52) and (78), we have

$$P_0 = 1 + \epsilon R_0 + O(\epsilon^2) \quad \text{and} \quad Q_0 = 1 + O(\epsilon^2). \tag{86}$$

Using (55) and (56) for the propagation map $(R_0, P_0, Q_0) \longrightarrow_L (R'_0, P'_0, Q'_0)$, at B, we have

$$P'_0 = 1 + (R_0 + \frac{L}{2} - \epsilon)L - \epsilon R_1 + O(L^3) + O(\epsilon^2) \tag{87}$$
$$Q'_0 = 1 + (R_0 + \frac{L}{2} - \epsilon)L + O(L^3) + O(\epsilon^2). \tag{88}$$

Next consider the triples (R'_0, P'_0, Q'_0) and (R_1, P_1, Q_1) associated with the point B. Let $\nu^2 = e^{-\frac{l}{\epsilon}}$. By (53) and (67), the following hold:

$$P'_0 = Q'_0 - \epsilon R_1 + O(\epsilon^2) \tag{89}$$
$$P_1 = Q_1 - 2Q_1\nu^2 - \epsilon R_1 + O(\epsilon\nu^2) + O(\epsilon^2) \tag{90}$$

which together with $P'_0 = P_1$ and $Q'_0 + Q_1 = 2$ lead to

$$Q'_0 = 1 - \nu^2 + o(\nu^2) + O(\epsilon^2) \tag{91}$$
$$Q_1 = 1 + \nu^2 + o(\nu^2) + O(\epsilon^2). \tag{92}$$

By substituting (92) into (90) and comparing (91) and (88), we have

$$P_1 = 1 - \nu^2 - \epsilon R_1 + o(\nu^2) + O(\epsilon^2) \tag{93}$$
$$\nu^2 = -\left(R_0 + \frac{L}{2} - \epsilon\right) L + O(L^3) + o(\nu^2) + O(\epsilon^2) \tag{94}$$

We now consider the following cases.

Case of $\mathbf{LLS_1}$: assume $R_0 \le -\frac{2}{3}L$. By (94), this assumption leads to

$$\nu^2 \ge \left(\frac{2}{3} - \frac{1}{2}\right) L^2 + o(L^2) \ge \frac{L^2}{7}. \tag{95}$$

Now vary the pattern by moving the interfaces at A and B as shown in Figure 9.

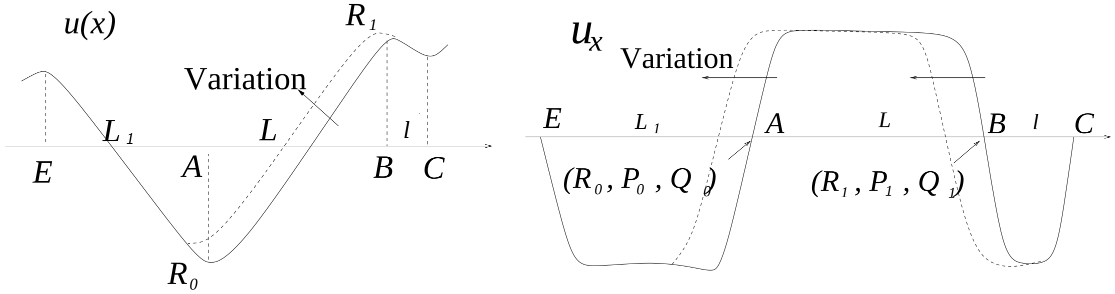

Figure 9.

This corresponds to the test function:

$$\Psi_x = \begin{cases} F_x\left(x - \frac{E+A}{2}, \frac{A-E}{2}\right) & x \in (E, A) \\ G_x\left(x - \frac{A+B}{2}, \frac{B-A}{2}\right) & x \in (A, B) \\ -\left(\frac{1+\nu^2}{1-\nu^2}\right) F_x\left(-x + \frac{B+C}{2}, \frac{C-B}{2}\right) & x \in (B, C) \end{cases} \tag{96}$$

(See the remark at the end of this case.) Now the quantity $\mathcal{D}(u, \Psi)$ equals:

$$\begin{aligned} & \int_E^C (\epsilon^2 \Psi_{xx}^2 + \Psi_x^2)\, dx - 2\frac{\Psi_x^2(A)}{|u_{xx}(A)|} - 2\frac{\Psi_x^2(B)}{|u_{xx}(B)|} \\ = & \int_E^A (\epsilon^2 F_{xx}^2 + F_x^2)\, dx + \int_A^B (\epsilon^2 G_{xx}^2 + G_x^2)\, dx + \left(\frac{1+\nu^2}{1-\nu^2}\right)^2 \int_B^C (\epsilon^2 F_{xx}^2 + F_x^2)\, dx \\ & -2\frac{1}{\epsilon}\frac{1}{P_0} - 2\frac{1}{\epsilon}\frac{1}{P_1} \\ = & \frac{1}{\epsilon} + \frac{2}{\epsilon} + \left(\frac{1+\nu^2}{1-\nu^2}\right)^2 \frac{1}{\epsilon}(1 - 4\nu^2) \\ & -\frac{2}{\epsilon}\frac{1}{1 + \epsilon R_0 + O(\epsilon^2)} - \frac{2}{\epsilon}\frac{1}{1 - \nu^2 - \epsilon R_1 + o(\nu^2) + O(\epsilon^2)} \\ = & \frac{-2\nu^2 + 2\epsilon R_0 - 2\epsilon R_1 + o(\nu^2) + O(\epsilon^2)}{\epsilon} \end{aligned} \tag{97}$$

where in the above computation, we have used (86) and (93). By the lower bound (95) for ν^2 and the fact that $|R_1 - R_0| = L + o(L)$, we have

$$\mathcal{D}(u, \Psi) \le -\frac{L^2}{7\epsilon}. \tag{98}$$

Now $\Psi(x) \approx 1$ for $x \in (A, B)$ and ≈ 0 for $x \in (E, A) \cup (B, C)$. So $\int_E^C \Psi^2 dx = L + o(L)$. Hence we have

$$\partial^2 \mathcal{E}(u, \Psi) = \mathcal{D}(u, \Psi) + \int_E^C \Psi^2\, dx = -\frac{L^2}{7\epsilon} + L + o(L) < 0\ !$$

proving that the pattern $\mathbf{LLS_1}$ is unstable.

Remark about the test function Ψ (96).

1. The extra multiplicative factor for F_x in the segment of (B, C) is to make Ψ_x continuous with a common value of $\frac{1}{\epsilon}$ at A and B. In this way, the continuity of Ψ holds up to error terms consisting of $e^{-\frac{L}{\epsilon}}$ which are transcendentally small compared with ν^2 and hence can be ignored.

2. By (83), $\Psi(B) - \Psi(C) = 1 - O(\nu^2) \neq 1$, so an extra piece of perturbation needs to be added to Ψ_x in order to make $\Psi(E) = 0$ and $\Psi(C) = 0$. This can be done by choosing an appropriate function $g_x(x)$ satisfying:

$$\|g_x\|_\infty \leq \frac{O(\nu^2)}{L}, \quad \|g_{xx}\|_\infty \leq \frac{O(\nu^2)}{L^2} \quad \text{and } \operatorname{spt}(g_x) \subset (A, B).$$

Note that $\|\Psi_x\|_\infty$ is transcendentally small near the center region of the *long* segment (A, B), the error introduced by g_x can be bounded by:

$$\int_A^B \epsilon^2 g_{xx}^2 + g_x^2 \, dx \leq \epsilon^2 \times \frac{\nu^4}{L^4} \times L + \frac{\nu^4}{L^2} \times L = \frac{1}{\epsilon} o(\nu^2).$$

Compared with the expression (97) for $\mathcal{D}(u, \Psi)$, the above is within the range of acceptable error for our analysis. Hence we can in effect ignore g_x.

We will not repeat the above remarks for the remaining analysis. □

Case of $\mathbf{LLS_2}$: assume $R_0 \geq -\frac{2}{3}L$ and $Q_2 = 1 + O(\epsilon^2)$. (The reason for introducing this and the following cases is that we might not have (95) — this can actually happen (see **Section A**). Thus we cannot directly deduce the same negativity for $\mathcal{D}(u, \Psi)$ as in (98). To tackle this, we further consider the pattern to the right of LLS.)

First, just the assumption on R_0 and the fact that $l \leq \frac{L}{2}$ imply:

$$\left(R_1 - \frac{l}{2} + \epsilon\right) l = \left(R_0 + L + o(L) - \frac{l}{2} + \epsilon\right) l \geq \left(\frac{L}{3} - \frac{L}{4} + o(L)\right) l \geq \frac{Ll}{13} \tag{99}$$

Consider the point C. In view of (71) and (92), it holds that:

$$Q_1' = 1 + \nu^2 - \left(R_1 - \frac{l}{2} + \epsilon\right) l + o(\nu^2) + O(\epsilon^2). \tag{100}$$

Since $Q_1' = 2 - Q_2$ which equals $1 + O(\epsilon^2)$ by the assumption of this case, the above leads to the following lower bound:

$$\nu^2 = \left(R_1 - \frac{l}{2} + \epsilon\right) l + o(\nu^2) + O(\epsilon^2) \geq \frac{Ll}{13} \tag{101}$$

Now using the same test function (96) and reasoning as in $\mathbf{LLS_1}$, we have:

$$\mathcal{D}(u, \Psi) \quad = \quad \frac{-2\nu^2 + 2\epsilon R_0 - 2\epsilon R_1 + o(\nu^2) + O(\epsilon^2)}{\epsilon} \leq -\frac{Ll}{7\epsilon} \quad (\text{note: } l \gg \epsilon)$$

Thus it holds that

$$\partial^2 \mathcal{E}(u, \Psi) \leq -\frac{Ll}{7\epsilon} + L + o(L) < 0$$

so that this case is also unstable.

Case of LLSS: assume $R_0 \geq -\frac{2L}{3}$ and there is a short segment (C, D) to the right of (B, C). The configurations of u and u_x are shown in Figure 10.

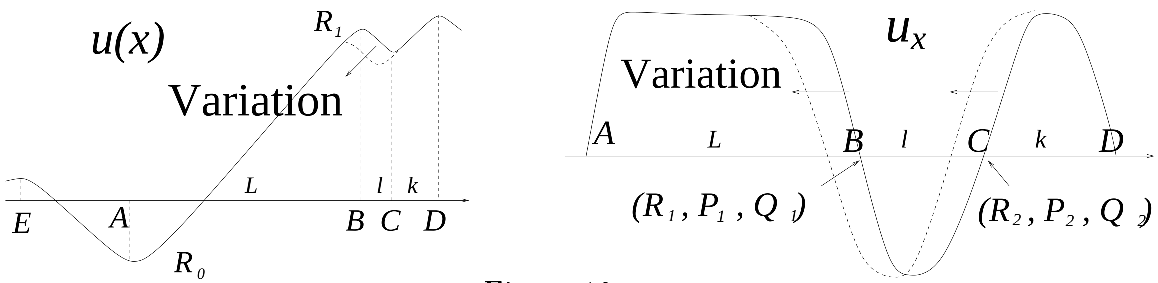

Figure 10.

Let $|CD| = k$ and $\mu^2 = e^{-\frac{k}{\epsilon}}$. Consider the point C again. Using the (70) and (92) associated with the propagation map from B to C, we have

$$\begin{aligned} P_1' &= Q_1 - 2Q_1\nu^2 - \left(R_1 - \frac{l}{2} + \epsilon\right) l + \epsilon R_2 + o(\nu^2) + O(\epsilon^2) \\ &= 1 - \nu^2 - \left(R_1 - \frac{l}{2} + \epsilon\right) l + \epsilon R_2 + o(\nu^2) + O(\epsilon^2) \end{aligned} \tag{102}$$

In addition, applying the map from C to D and by (62), we have:

$$P_2 = Q_2 - 2Q_2\mu^2 + \epsilon R_2 + o(\mu^2) \tag{103}$$

Comparing (102) and (103) and using (100) and $Q_1' + Q_2 = 2$ give:

$$Q_2 = 1 - \nu^2 + \left(R_1 - \frac{l}{2} + \epsilon\right) l + o(\nu^2) + O(\epsilon^2). \tag{104}$$

$$\mu^2 = \left(R_1 - \frac{l}{2} + \epsilon\right) l + o(\nu^2) + o(\mu^2) + O(\epsilon^2) \tag{105}$$

which lead to

$$P_2 = P_1' = 1 - \nu^2 - \mu^2 + \epsilon R_2 + o(\nu^2) + o(\mu^2) + O(\epsilon^2). \tag{106}$$

Now vary the pattern by moving the interfaces at B and C as shown in Figure 10. The test function Ψ used is thus:

$$\Psi_x = \begin{cases} -F_x\left(x - \frac{A+B}{2}, \frac{B-A}{2}\right) & x \in (A, B) \\ \left(\frac{1+\nu^2}{1-\nu^2}\right) G_x\left(x - \frac{B+C}{2}, \frac{C-B}{2}\right) & x \in (B, C) \\ \left(\frac{1+\mu^2}{1-\mu^2}\right) F_x\left(-x + \frac{C+D}{2}, \frac{D-C}{2}\right) & x \in (C, D) \end{cases} \tag{107}$$

In this case,

$$\begin{aligned} \mathcal{D}(u, \Psi) &= \int_A^D (\epsilon^2 \Psi_{xx}^2 + \Psi_x^2)\, dx - 2\frac{\Psi_x^2(B)}{|u_{xx}(B)|} - 2\frac{\Psi_x^2(C)}{|u_{xx}(C)|} \\ &= \frac{1}{\epsilon} + \left(\frac{1+\nu^2}{1-\nu^2}\right)^2 \frac{2}{\epsilon}(1 - 2\nu^2) + \left(\frac{1+\mu^2}{1-\mu^2}\right)^2 \frac{1}{\epsilon}(1 - 4\mu^2) - \frac{2}{\epsilon}\frac{1}{P_1} - \frac{2}{\epsilon}\frac{1}{P_2} \\ &= \frac{1}{\epsilon} + \frac{2}{\epsilon}(1 + 2\nu^2) + \frac{(1 + o(\mu^2))}{\epsilon} - \frac{2}{\epsilon}\frac{1}{1 - \nu^2 - \epsilon R_1} - \frac{2}{\epsilon}\frac{1}{1 - \nu^2 - \mu^2 + \epsilon R_2} \\ &= \frac{-2\mu^2 - 2\epsilon R_1 + 2\epsilon R_2 + o(\mu^2) + o(\nu^2)}{\epsilon} \\ &\leq \frac{-\frac{3}{2}\mu^2 - 2\epsilon R_1 + 2\epsilon R_2 + o(\nu^2)}{\epsilon} \end{aligned}$$

In view of (99), (105) and the fact that $|R_2 - R_1| = l + o(l)$, we have

$$\mu^2 \geq \frac{Ll}{13\epsilon} + o(\nu^2) \quad \text{and hence} \quad \mathcal{D}(u, \Psi) \leq -\frac{Ll + o(\nu^2)}{14\epsilon} \tag{108}$$

Consider two situations. If $o(\nu^2) \leq \frac{Ll}{2}$, then $\mathcal{D}(u, \Psi) \leq -\frac{Ll}{28\epsilon}$. For Ψ, it is approximately equal to 0 in the segments $(A, B) \cup (C, D)$ and 1 in the segment (B, C). So $\int_A^D \Psi^2\, dx = O(L + l + k)$. Again we have

$$\partial^2 \mathcal{E}(u, \Psi) = \mathcal{D}(u, \Psi) + \int_A^D \Psi^2\, dx = -\frac{Ll}{28\epsilon} + O(L + l + k) < 0\ !$$

If $o(\nu^2) \geq \frac{Ll}{2}$, then (101) holds and the same reasoning and test function (96) as in $\mathbf{LLS_1}$ can be used to conclude that this case is also unstable.

Case of LLSL: assume $R_0 \geq -\frac{2L}{3}$ and there is a long segment (C, D) to the right of (B, C). The configurations of u and u_x are illustrated in Figure 11.

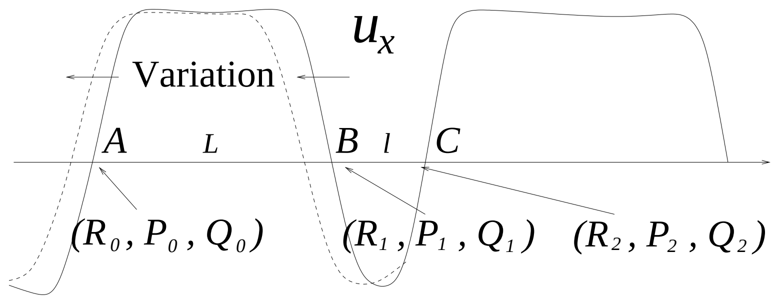

Figure 11.

In this case, at the point C, by (52), we have $P_2 = Q_2 + \epsilon R_2 + O(\epsilon^2)$. In view of (102), (104) and (99), the following hold:

$$o(\nu^2) + O(\epsilon^2) = \left(R_1 - \frac{l}{2} + \epsilon\right) l + O(\epsilon^2) \geq \frac{Ll}{13} + O(\epsilon^2)$$

so that $\nu^2 \geq o(\nu^2) \geq \frac{Ll}{13}$. Using the same test function as in $\mathbf{LLS_1}$ also concludes that this pattern has a negative second variation.

Thus all the examples of the category **LLS** are unstable.

5.5.3 Instability of SLS

We refer to Figure 6. Let $\mu^2 = e^{-\frac{m}{\epsilon}}$ and $\nu^2 = e^{-\frac{l}{\epsilon}}$. First, the relations (67), (70) and (71) lead to $P_1' = Q_1' - \epsilon R_2 + O(\epsilon^2)$ and $P_2 = Q_2 - 2Q_2\nu^2 - \epsilon R_2 + o(\nu^2) + O(\epsilon^2)$. Their comparison gives

$$P_2 = 1 - \nu^2 - \epsilon R_2 + o(\nu^2) + O(\epsilon^2) \tag{109}$$

$$Q_2 = 1 + \nu^2 + o(\nu^2) + O(\epsilon^2) \tag{110}$$

Similarly

$$P_1 = 1 - \mu^2 + \epsilon R_1 + o(\mu^2) + O(\epsilon^2) \tag{111}$$

Using the map $(R_2, P_2, Q_2) \longrightarrow_l (R_2', P_2', Q_2')$, we have:

$$P_2' = 1 - \nu^2 - \left(R_2 - \frac{l}{2} + \epsilon\right) l + \epsilon R_3 + o(\nu^2) + O(\epsilon^2) \tag{112}$$

$$Q_2' = Q_2 - \left(R_2 - \frac{l}{2} + \epsilon\right) l + O(\epsilon^2) \tag{113}$$

Without loss of generality, we can assume $R_2 > \frac{L}{3}$. In addition, by the hypothesis of this case — $l \leq \frac{L}{2}$, we have the following estimate:

$$\left(R_2 - \frac{l}{2} + \epsilon\right) l \geq \frac{Ll}{13}. \tag{114}$$

Now we divide this category into the following two cases each of which will be shown to have negative second variations.

Case of SLSS: assume that there is a short segment (C, D) to the right of (B, C). Figure 12 shows the configuration of u_x.

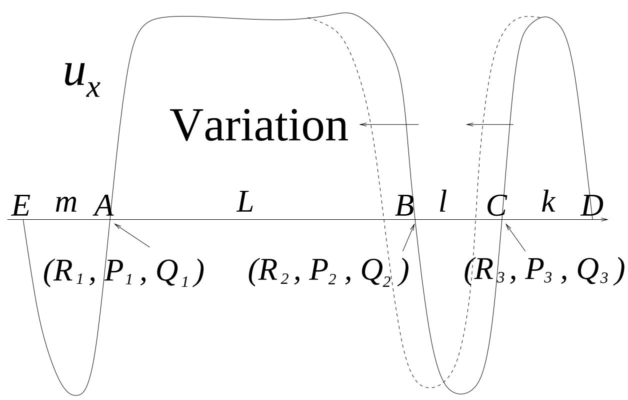

Figure 12.

Let $k = |CD| \leq 10\epsilon|\ln\epsilon|$ and $\delta^2 = e^{-\frac{k}{\epsilon}}$. Consider the point C. Using (62), we have:

$$P_3 = Q_3 - 2Q_3\delta^2 + \epsilon R_3 + o(\epsilon\delta^2).$$

Since $P_2' = P_3$ and $Q_3 + Q_2' = 2$, (112) gives:

$$\delta^2 = \left(R_2 - \frac{l}{2} + \epsilon\right) l + o(\nu^2) + o(\delta^2) + O(\epsilon^2) \tag{115}$$

$$P_3 = 1 - \nu^2 - \delta^2 + \epsilon R_3 + o(\nu^2) + o(\delta^2) + O(\epsilon^2) \tag{116}$$

Consider the following test function Ψ which mimics the movement of interfaces at B and C:

$$\Psi_x = \begin{cases} -F_x\left(x - \frac{A+B}{2}, \frac{B-A}{2}\right) & x \in (A, B) \\ \left(\frac{1+\nu^2}{1-\nu^2}\right) G_x\left(x - \frac{B+C}{2}, \frac{C-B}{2}\right) & x \in (B, C) \\ \left(\frac{1+\delta^2}{1-\delta^2}\right) F_x\left(-x + \frac{C+D}{2}, \frac{D-C}{2}\right) & x \in (C, D) \end{cases} \tag{117}$$

Then $\mathcal{D}(u, \Psi)$ equals:

$$
\begin{aligned}
& \int_A^D (\epsilon^2 \Psi_{xx}^2 + \Psi_x^2)\, dx - 2\frac{\Psi_x^2(B)}{|u_{xx}(B)|} - 2\frac{\Psi_x^2(C)}{|u_{xx}(C)|} \\
= \;& \frac{1}{\epsilon} + \left(\frac{1+\nu^2}{1-\nu^2}\right)^2 \frac{2(1-2\nu^2)}{\epsilon} + \left(\frac{1+\delta^2}{1-\delta^2}\right)^2 \frac{(1-4\delta^2)}{\epsilon} - \frac{2}{\epsilon}\frac{1}{P_2} - \frac{2}{\epsilon}\frac{1}{P_3} \\
= \;& \frac{1}{\epsilon} + \frac{2+4\nu^2}{\epsilon} + \frac{1+o(\delta^2)}{\epsilon} \\
& -\frac{2}{\epsilon}\frac{1}{1-\nu^2-\epsilon R_2 + o(\nu^2) + O(\epsilon^2)} - \frac{2}{\epsilon}\frac{1}{1-\nu^2-\delta^2+\epsilon R_3 + +o(\nu^2)+o(\delta^2)+O(\epsilon^2)} \\
= \;& \frac{-2\delta^2 - 2\epsilon(R_2-R_3) + o(\nu^2) + o(\delta^2)}{\epsilon}
\end{aligned}
$$

Hence, by (114), (115) and the fact that $|R_2 - R_3| = l + o(l)$, we have

$$\mathcal{D}(u, \Psi) \leq -\frac{Ll + o(\nu^2)}{7\epsilon}$$

Again, we consider two cases. If $o(\nu^2) \leq \frac{Ll}{2}$, then

$$\partial^2 \mathcal{E}(u, \Phi) = -\frac{Ll}{14\epsilon} + l + o(l) < 0.$$

If $o(\nu^2) \geq \frac{Ll}{2}$, then we move the interfaces at A and B as shown in Figure 13:

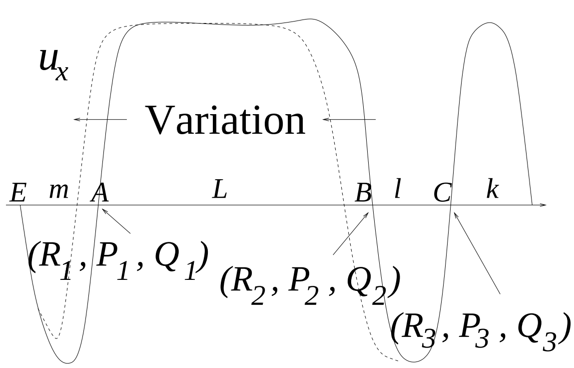

Figure 13.

The corresponding test function is:

$$\Psi_x = \begin{cases} \left(\frac{1+\mu^2}{1-\mu^2}\right) F_x\left(x - \frac{E+A}{2}, \frac{A-E}{2}\right) & x \in (E, A) \\ G_x\left(x - \frac{A+B}{2}, \frac{B-A}{2}\right) & x \in (A, B) \\ -\left(\frac{1+\nu^2}{1-\nu^2}\right) F_x\left(-x + \frac{B+C}{2}, \frac{C-B}{2}\right) & x \in (B, C) \end{cases} \tag{118}$$

Similarly we have:

$$\begin{aligned}\mathcal{D}(u,\Psi) &= \int_E^C (\epsilon^2\Psi_x^2+\Psi_x^2)\,dx - 2\frac{\Psi_x^2(A)}{|u_{xx}(A)|} - 2\frac{\Psi_x^2(B)}{|u_{xx}(B)|}\\ &= \left(\frac{1+\mu^2}{1-\mu^2}\right)^2\frac{1-4\mu^2}{\epsilon} + \frac{2}{\epsilon} + \left(\frac{1+\nu^2}{1-\nu^2}\right)^2\frac{1-4\nu^2}{\epsilon} - \frac{2}{\epsilon}\frac{1}{P_1} - \frac{2}{\epsilon}\frac{1}{P_2}\\ &= \frac{-2\mu^2-2\nu^2+2\epsilon R_1-2\epsilon R_2+o(\mu^2)+o(\nu^2)}{\epsilon}\end{aligned}$$

The assumption on $o(\nu^2)$ now gives:

$$\partial^2\mathcal{E}(u,\Phi) = -\frac{Ll}{\epsilon} + L + o(L) < 0\ !$$

Case of SLSL: assume that there is a long segment (C, D) to the right of (B, C). In this case, we have $P_3 = Q_3 + \epsilon R_3$. Combining the fact $Q_2' + Q_3 = 2$ together with (112) and (113) gives

$$o(\nu^2) + O(\epsilon^2) = \left(R_2 - \frac{l}{2} + \epsilon\right) l \geq \frac{Ll}{13}\ \text{(by (114))}$$

Hence $\nu^2 \geq o(\nu^2) \geq \left(R_2 - \frac{l}{2} + \epsilon\right) l \geq \frac{Ll}{13}$. and we can then use the same Ψ as in (118) to deduce that the second variation is negative.

Hence, we have also shown that all the cases of **SLS** are unstable. **Claim II** is thus proved.

5.6 Proof of Claim III

In this section, we assume that u is a solution of (10) with only long segments. We will prove that $l_i = l_{i+1}$ for all i and hence u is periodic by **Proposition 3.3**. The approach taken here resembles the use of implicit function theorem. We will exploit again the solution formula given by the propagation map defined in **Section 3**. We use the same notation as in that section.

Consider the boundary value problem as stated in **Proposition 3.3**. Given Q, Q', using

(47) and (73), we can express the solution as follows:

$$
\begin{aligned}
R\ (= u(0)) &= \frac{1}{\epsilon^2(\Lambda^2-\lambda^2)}\left\{\frac{Q(e^{\Lambda l}+e^{-\Lambda l})-2Q'}{\Lambda(e^{\Lambda l}-e^{-\Lambda l})}-\frac{Q(e^{\lambda l}-e^{-\lambda l})-2Q'}{\lambda(e^{\lambda l}-e^{-\lambda l})}\right\}\\
R'\ (= u(l)) &= \frac{1}{\epsilon^2(\Lambda^2-\lambda^2)}\left\{\frac{2Q-Q'(e^{\Lambda l}+e^{-\Lambda l})}{\Lambda(e^{\Lambda l}-e^{-\Lambda l})}-\frac{2Q-Q'(e^{\lambda l}+e^{-\lambda l})}{\lambda(e^{\lambda l}-e^{-\lambda l})}\right\}\\
P\ (= \epsilon u_{xx}(0)) &= \frac{1}{\epsilon(\Lambda^2-\lambda^2)}\left\{\frac{\Lambda\left[Q(e^{\Lambda l}+e^{-\Lambda l})-2Q'\right]}{(e^{\Lambda l}-e^{-\Lambda l})}-\frac{\lambda\left[Q(e^{\lambda l}-e^{-\lambda l})-2Q'\right]}{(e^{\lambda l}-e^{-\lambda l})}\right\}\\
P'\ (= \epsilon u_{xx}(l)) &= \frac{1}{\epsilon(\Lambda^2-\lambda^2)}\left\{-\frac{\Lambda\left[2Q-Q'(e^{\Lambda l}+e^{-\Lambda l})\right]}{(e^{\Lambda l}-e^{-\Lambda l})}+\frac{\lambda\left[2Q-Q'(e^{\lambda l}+e^{-\lambda l})\right]}{(e^{\lambda l}-e^{-\lambda l})}\right\}
\end{aligned}
$$

Similar expressions hold if $u_x < 0$ for $x \in (0, l)$ (by changing Q and Q' to $-Q$ and $-Q'$).

The above results are now applied to the solution u. Let $R_i^{\pm} = u(c_i^{\pm})$, $P_i^{\pm} = \epsilon u_{xx}(c_i^{\pm})$, $Q_i^{\pm} = \epsilon^2 u_{xxx}(c_i^{\pm})$ and $l_i = c_i - c_{i-1}$. Since $R_i^+ = R_i^-$ and $P_i^+ = P_i^-$, we have, for $i = 1, 2, \ldots N-1$:

$$
\begin{aligned}
&-\frac{2Q_{i-1}^+ - Q_i^-(e^{\Lambda l_i}+e^{-\Lambda l_i})}{\Lambda(e^{\Lambda l_i}-e^{-\Lambda l_i})}+\frac{2Q_{i-1}^+ - Q_i^-(e^{\lambda l_i}+e^{-\lambda l_i})}{\lambda(e^{\lambda l_i}-e^{-\lambda l_i})}\\
&\qquad = \frac{Q_i^+(e^{\Lambda l_{i+1}}+e^{-\Lambda l_{i+1}})-2Q_{i+1}^-}{\Lambda(e^{\Lambda l_{i+1}}-e^{-\Lambda l_{i+1}})}-\frac{Q_i^+(e^{\lambda l_{i+1}}+e^{-\lambda l_{i+1}})-2Q_{i+1}^-}{\lambda(e^{\lambda l_{i+1}}-e^{-\lambda l_{i+1}})}
\end{aligned}
\tag{119}
$$

and

$$
\begin{aligned}
&-\frac{\Lambda\left[2Q_{i-1}^+ - Q_i^-(e^{\Lambda l_i}+e^{-\Lambda l_i})\right]}{(e^{\Lambda l_i}-e^{-\Lambda l_i})}+\frac{\lambda\left[2Q_{i-1}^+ - Q_i^-(e^{\lambda l_i}+e^{-\lambda l_i})\right]}{(e^{\lambda l_i}-e^{-\lambda l_i})}\\
&\qquad = \frac{\Lambda\left[Q_i^+(e^{\Lambda l_{i+1}}+e^{-\Lambda l_{i+1}})-2Q_{i+1}^-\right]}{(e^{\Lambda l_{i+1}}-e^{-\Lambda l_{i+1}})}-\frac{\lambda\left[Q_i^+(e^{\lambda l_{i+1}}+e^{-\lambda l_{i+1}})-2Q_{i+1}^-\right]}{(e^{\lambda l_{i+1}}-e^{-\lambda l_{i+1}})}
\end{aligned}
\tag{120}
$$

For convenience, we introduce the following notations:

$$
\theta(l) = (1-e^{-2\lambda l}),\quad \eta(l) = \frac{e^{-\Lambda l}}{1-e^{-2\Lambda l}},\quad E(l) = \frac{1+e^{-2\Lambda l}}{1-e^{-2\Lambda l}}\quad \text{and } \sigma = \frac{\lambda}{\Lambda} = \epsilon + O(\epsilon^2).
$$

As $Q_i^+ + Q_i^- = 2$, we can write $Q_i^{\pm} = 1 \pm \delta_i$ for some number δ_i. Then (119) and (120) become: $(i = 1, 2, \ldots N-1)$

$$
\begin{aligned}
&2\theta(l_{i+1})e^{-\lambda l_i}\delta_{i-1} + 2\theta(l_i)e^{-\lambda l_{i+1}}\delta_{i+1} + \left[\theta(l_{i+1})(1+e^{-2\lambda l_i}) + \theta(l_i)(1+e^{-2\lambda l_{i+1}})\right]\delta_i\\
&-2\sigma\theta(l_i)\theta(l_{i+1})\left[\eta(l_i)\delta_{i-1} + \eta(l_{i+1})\delta_{i+1} + \left(\frac{E(l_i)+E(l_{i+1})}{2}\right)\delta_i\right]\\
= \;&\theta(l_{i+1})(1-e^{-\lambda l_i})^2 - \theta(l_i)(1-e^{-\lambda l_{i+1}})^2\\
&-\sigma\theta(l_i)\theta(l_{i+1})\left[E(l_i) - 2\eta(l_i) - (E(l_{i+1}) - 2\eta(l_{i+1}))\right];
\end{aligned}
\tag{121}
$$

and

$$\begin{aligned}
& 2\theta(l_{i+1})e^{-\lambda l_i}\delta_{i-1} + 2\theta(l_i)e^{-\lambda l_{i+1}}\delta_{i+1} + \left[\theta(l_{i+1})(1+e^{-2\lambda l_i}) + \theta(l_i)(1+e^{-2\lambda l_{i+1}})\right]\delta_i \\
& -2\frac{\theta(l_i)\theta(l_{i+1})}{\sigma}\left[\eta(l_i)\delta_{i-1} + \eta(l_{i+1})\delta_{i+1} + \left(\frac{E(l_i)+E(l_{i+1})}{2}\right)\delta_i\right] \\
= \; & \theta(l_{i+1})(1-e^{-\lambda l_i})^2 - \theta(l_i)(1-e^{-\lambda l_{i+1}})^2 \\
& -\frac{\theta(l_i)\theta(l_{i+1})}{\sigma}\left[E(l_i) - 2\eta(l_i) - (E(l_{i+1}) - 2\eta(l_{i+1}))\right] \qquad (122)
\end{aligned}$$

Let $\mathbf{l} = (l_1, l_2, \ldots l_N)^T$. We now introduce the following vectors $\mathbf{a}, \mathbf{b}(\mathbf{l}), \mathbf{e}(\mathbf{l}) \in R^{N-1}$ and matrices $\mathbf{A}(\mathbf{l}), \mathbf{D}(\mathbf{l}), \Theta(\mathbf{l}) \in R^{(N-1)\times(N-1)}$: (In the following, $(\mathbf{M})_i$ refers to the i-th component of $\mathbf{M}$ and the subscript i ranges from 1 to $N-1$.)

$$\begin{aligned}
(\mathbf{a})_i &= \delta_i \\
(\mathbf{b}(\mathbf{l}))_i &= \theta(l_{i+1})(1-e^{-\lambda l_i})^2 - \theta(l_i)(1-e^{-\lambda l_{i+1}})^2 = \theta(l_i)\theta(l_{i+1})\left(\frac{1-e^{-\lambda l_i}}{1+e^{-\lambda l_i}} - \frac{1-e^{-\lambda l_{i+1}}}{1+e^{-\lambda l_{i+1}}}\right) \\
(\mathbf{e}(\mathbf{l}))_i &= E(l_i) - 2\eta(l_i) - (E(l_{i+1}) - 2\eta(l_{i+1})) = \left(\frac{1-e^{-\Lambda l_i}}{1+e^{-\Lambda l_i}} - \frac{1-e^{-\Lambda l_{i+1}}}{1+e^{-\Lambda l_{i+1}}}\right) \\
(\mathbf{A}(\mathbf{l})\mathbf{a})_i &= 2\theta(l_{i+1})e^{-\lambda l_i}\delta_{i-1} + 2\theta(l_i)e^{-\lambda l_{i+1}}\delta_{i+1} \\
&\quad + \left[\theta(l_{i+1})(1+e^{-2\lambda l_i}) + \theta(l_i)(1+e^{-2\lambda l_{i+1}})\right]\delta_i \\
(\mathbf{D}(\mathbf{l})\mathbf{a})_i &= \eta(l_i)\delta_{i-1} + \eta(l_{i+1})\delta_{i+1} + \left(\frac{E(l_i)+E(l_{i+1})}{2} - 1\right)\delta_i \\
\Theta(\mathbf{l}) &= \text{the } (N-1)\times(N-1) \text{ diagonal matrix with } \Theta(\mathbf{l})_{ii} = \theta(l_i)\theta(l_{i+1}) \\
\mathbf{I} &= \text{the } (N-1)\times(N-1) \text{ identity matrix}
\end{aligned}$$

(In the above definitions of $\mathbf{A}(\mathbf{l})$ and $\mathbf{D}(\mathbf{l})$, we set $\delta_0 = \delta_N = 0$.) The expressions (121) and (122) can now be concisely written as

$$\begin{aligned}
\mathbf{A}(\mathbf{l})\mathbf{a} - 2\sigma\Theta(\mathbf{l})(\mathbf{I}+\mathbf{D}(\mathbf{l}))\mathbf{a} &= \mathbf{b}(\mathbf{l}) - \sigma\Theta(\mathbf{l})\mathbf{e}(\mathbf{l}) \\
\text{and } \mathbf{A}(\mathbf{l})\mathbf{a} - 2\frac{1}{\sigma}\Theta(\mathbf{l})(\mathbf{I}+\mathbf{D}(\mathbf{l}))\mathbf{a} &= \mathbf{b}(\mathbf{l}) - \frac{1}{\sigma}\Theta(\mathbf{l})\mathbf{e}(\mathbf{l})
\end{aligned}$$

The above leads to $2(\mathbf{I}+\mathbf{D}(\mathbf{l}))\mathbf{a} = \mathbf{e}(\mathbf{l})$ and $\mathbf{A}(\mathbf{l})\mathbf{a} = \mathbf{b}(\mathbf{l})$. Hence

$$\mathbf{b}(\mathbf{l}) = \frac{\mathbf{A}(\mathbf{l})}{2}(\mathbf{I}+\mathbf{D}(\mathbf{l}))^{-1}\mathbf{e}(\mathbf{l}) \text{ or equivalently } \Theta(\mathbf{l})^{-1}\mathbf{b}(\mathbf{l}) = \frac{\Theta(\mathbf{l})^{-1}\mathbf{A}(\mathbf{l})}{2}(\mathbf{I}+\mathbf{D}(\mathbf{l}))^{-1}\mathbf{e}(\mathbf{l}).$$

Component-wise, the above is equal to, for $i = 1, 2, \dots N-1$:

$$\frac{1-e^{-\lambda l_i}}{1+e^{-\lambda l_i}} - \frac{1-e^{-\lambda l_{i+1}}}{1+e^{-\lambda l_{i+1}}}$$
$$= \frac{1}{\theta(l_i)\theta(l_{i+1})} \left\{ \sum_{k=1}^{N-1} \frac{1}{2} \left(\mathbf{A}(\mathbf{l})(\mathbf{I}+\mathbf{D}(\mathbf{l}))^{-1}\right)_{ik} \left(\frac{1-e^{-\Lambda l_k}}{1+e^{-\Lambda l_k}} - \frac{1-e^{-\Lambda l_{k+1}}}{1+e^{-\Lambda l_{k+1}}} \right) \right\}$$

Consider the function

$$f_a(l) = \frac{1-e^{-al}}{1+e^{-al}} \text{ and its derivative } f_a'(l) = \frac{2ae^{-al}}{(1+e^{-al})^2}.$$

Hence there are numbers $\left\{ \tilde{l}_i, \hat{l}_i \right\}_{i=1}^{N-1}$ such that

$$\frac{2\lambda e^{-\lambda \tilde{l}_i}}{(1+e^{-\lambda \tilde{l}_i})^2}(l_i - l_{i+1}) = \frac{1}{\theta(l_i)\theta(l_{i+1})} \left\{ \sum_{k=1}^{N-1} \frac{1}{2} (\mathbf{A}(\mathbf{l})(\mathbf{I}+\mathbf{D}(\mathbf{l}))^{-1})_{ik} \left(\frac{2\Lambda e^{-\Lambda \hat{l}_k}}{(1+e^{-\Lambda \hat{l}_k})^2}(l_k - l_{k+1}) \right) \right\}. \tag{123}$$

Note that $\frac{E(l_i)+E(l_{i+1})}{2} - 1 = \frac{2e^{-2\Lambda l_i}}{1-e^{-2\Lambda l_i}} + \frac{2e^{-2\Lambda l_{i+1}}}{1-e^{-2\Lambda l_{i+1}}}$. Since $10\epsilon|\ln\epsilon| \le l_i \le o(1)$, all the entries of $\mathbf{D}(\mathbf{l})$ are bounded by $O(\epsilon^{10})$ which implies that the entries of $(\mathbf{I}+\mathbf{D}(\mathbf{l}))^{-1}$ can be bounded by some $O(1)$ constants. Furthermore, we have

$$\frac{2\lambda e^{-\lambda \tilde{l}_i}}{(1+e^{-\lambda \tilde{l}_i})^2} = O(1), \quad \frac{2\Lambda e^{-\Lambda \hat{l}_k}}{(1+e^{-\Lambda \hat{l}_k})^2} \le O(\epsilon^9) \quad \text{and} \quad \frac{1}{2} l_i \le \theta(l_i) \le 2 l_i.$$

so that the entries of $\Theta(\mathbf{l})^{-\mathbf{1}}\mathbf{A}(\mathbf{l})(\mathbf{I}+\mathbf{D}(\mathbf{l}))^{-\mathbf{1}}$ can be bounded by $O(\max_i(l_i^{-1}))$. Applying these estimates to (123), we have $\sum_i |l_i - l_{i+1}|^2 \le o(1) \sum_i |l_i - l_{i+1}|^2$ which leads to $l_i = l_{i+1}$ for all i. Hence $\mathbf{e}(\mathbf{l}) = \mathbf{0}$, $\mathbf{a} = \mathbf{0}$ and $Q_i^{\pm} = 1$. By **Proposition 3.3**, u equals $Q^N(x)$ given by (4). **Claim III** is thus proved.

6 Proof of Theorem 1.3

The **Theorem** follows easily by the formulas and approach we have been using. Let $l = K\epsilon|\ln\epsilon|$ for some K, $N = \frac{1}{l}$ be an integer and $u = Q^N(x)$. Then,

$$\nu^2 = e^{-\frac{l}{\epsilon}} = \epsilon^K \quad \text{and} \quad u(B) = \frac{l}{2} + o(l) = O(\epsilon|\ln\epsilon|).$$

Now the graphs of u and u_x are shown in the following figure.

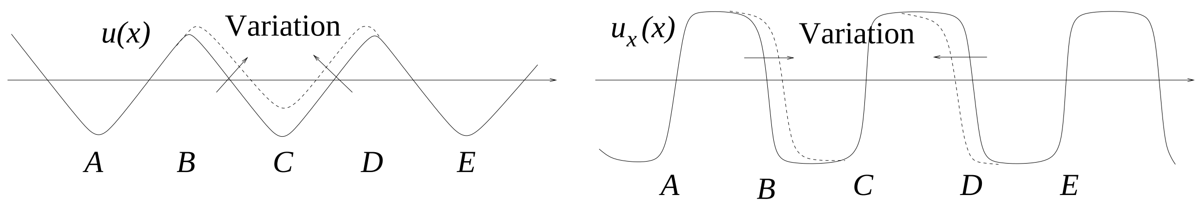

Figure 14.

Consider the variation of u_x given by:

$$\Psi_x = \begin{cases} F_x\left(x - \frac{A+B}{2}, \frac{B-A}{2}\right), & x \in (A,B) \\ F_x\left(-x + \frac{B+C}{2}, \frac{C-B}{2}\right), & x \in (B,C) \\ -F_x\left(x - \frac{C+D}{2}, \frac{D-C}{2}\right), & x \in (C,D) \\ -F_x\left(-x + \frac{D+E}{2}, \frac{E-D}{2}\right), & x \in (D,E) \end{cases}$$

With the above, we have

$$\begin{aligned} \mathcal{D}(u,\Psi) &= \int_A^E \left(\epsilon^2 \Psi_{xx}^2 + \Psi_x^2\right) dx - 2\frac{\Psi_x^2(B)}{|u_{xx}(B)|} - 2\frac{\Psi_x^2(D)}{|u_{xx}(D)|} \\ &= \frac{4}{\epsilon}(1 - 4\nu^2 + o(\nu^4)) - \frac{4}{\epsilon|u_{xx}(B)|}\frac{1}{\epsilon}\left(\frac{1-\nu^2}{1+\nu^2}\right)^2 \\ &= \frac{4(1-4\nu^2+o(\nu^2))}{\epsilon}\left(1 - \frac{1}{1 - 2\nu^2 - \epsilon u(B) + O(\epsilon^2) + o(\nu^2)}\right) \\ &\le -\frac{7\nu^2 + O(\epsilon^2)}{\epsilon}. \end{aligned}$$

where in the above, we have used (67) with $Q = \epsilon^2 u_{xxx}(B) = 1$ to express $\epsilon u_{xx}(B)$. Hence the overall second variation for the test function Ψ is given by

$$\partial^2 \mathcal{E}(u,\Psi) \le -\frac{7\nu^2 + O(\epsilon^2)}{\epsilon} + 2l + o(l) = \frac{-7\epsilon^K + O(\epsilon^2|\ln\epsilon|)}{\epsilon} < 0$$

as long as $0 < K < 2$ and ϵ is small enough. Thus **Theorem 1.3** is proved.

A Explicit Examples

In this section, we provide some simple explicit examples of unstable solutions of (10) to illustrate that the various cases considered in the proof of **Theorem 1.2** can actually occur.

We will find $4L$-periodic functions which solves the following unit cell problem:

$$u_x > 0 \text{ for } x \in (-l, l);$$
$$u_x < 0 \text{ for } x \in (-L, -l) \bigcup (l, L);$$
$$\text{and } u(x) = -u(-x) \text{ for } x \in (-L, L).$$

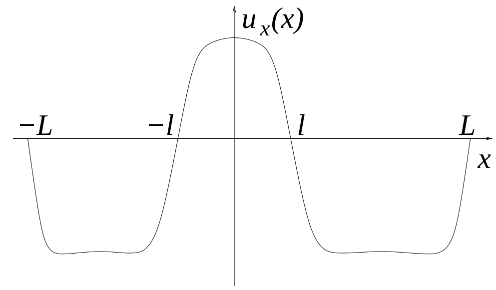

Figure 15.

By symmetry and (47), it suffices to consider u given by the form:

$$u = \begin{cases} A(e^{\Lambda x} - e^{-\Lambda x}) + B(e^{\lambda x} - e^{-\lambda x}), & x \in (-l, l) \\ Ce^{\Lambda x} + De^{-\Lambda x} + Ee^{\lambda x} + Fe^{-\lambda x}, & x \in (l, L) \end{cases}$$

Given $L > 0$, there are seven unknowns $A, B \dots F$ and l. They are found by the following boundary and jump conditions for u:

$$u_x(l^-) = 0; \;\; u_x(l^+) = 0;$$
$$u(l^-) = u(l^+);$$
$$u_{xx}(l^-) = u_{xx}(l^+);$$
$$\epsilon^2 \left(u_{xxx}(l^+) - u_{xxx}(l^-)\right) = 2;$$
$$u_x(L) = 0; \;\; \epsilon^2 u_{xxx}(L) = 1.$$

The above correspond to the following system of equations:

$$\Lambda A(e^{\Lambda l} + e^{-\Lambda l}) + \lambda B(e^{\lambda l} + e^{-\lambda l}) = 0 \tag{124}$$
$$\Lambda(Ce^{\Lambda l} - De^{-\Lambda l}) + \lambda(Ee^{\lambda l} - Fe^{-\lambda l}) = 0 \tag{125}$$
$$A(e^{\Lambda l} - e^{-\Lambda l}) + B(e^{\lambda l} - e^{-\lambda l}) = Ce^{\Lambda l} + De^{-\Lambda l} + Ee^{\lambda l} + Fe^{-\lambda l} \tag{126}$$
$$\Lambda^2 A(e^{\Lambda l} - e^{-\Lambda l}) + \lambda^2 B(e^{\lambda l} - e^{-\lambda l}) = \Lambda^2(Ce^{\Lambda l} + De^{-\Lambda l}) + \lambda^2(Ee^{\lambda l} + Fe^{-\lambda l}) \tag{127}$$
$$-\Lambda^3 A(e^{\Lambda l} + e^{-\Lambda l}) - \lambda^3 B(e^{\lambda l} + e^{-\lambda l}) + \Lambda^3(Ce^{\Lambda l} - De^{-\Lambda l}) + \lambda^3(Ee^{\lambda l} - Fe^{-\lambda l}) = \tfrac{2}{\epsilon^2} \tag{128}$$
$$\Lambda(Ce^{\Lambda L} - De^{-\Lambda L}) + \lambda(Ee^{\lambda L} - Fe^{-\lambda L}) = 0 \tag{129}$$
$$\Lambda^3(Ce^{\Lambda L} - De^{-\Lambda L}) + \lambda^3(Ee^{\lambda L} - Fe^{-\lambda L}) = \tfrac{1}{\epsilon^2} \tag{130}$$

Let $\gamma = \frac{\lambda}{\Lambda}$. Using (124) – (128), we obtain:

$$\begin{aligned}
A &= -\frac{\lambda}{\Lambda}\left(\frac{e^{\lambda l}+e^{-\lambda l}}{e^{\Lambda l}+e^{-\Lambda l}}\right)B \\
C &= -\frac{\lambda}{\Lambda}\left(\frac{e^{\lambda l}+e^{-\lambda l}}{e^{\Lambda l}+e^{-\Lambda l}}\right)B + \frac{e^{-\Lambda l}}{\epsilon^2\Lambda^3(1-\gamma^2)} \\
D &= \frac{\lambda}{\Lambda}\left(\frac{e^{\lambda l}+e^{-\lambda l}}{e^{\Lambda l}+e^{-\Lambda l}}\right)B - \frac{e^{\Lambda l}}{\epsilon^2\Lambda^3(1-\gamma^2)} \\
E &= B - \frac{e^{-\lambda l}}{\epsilon^2\Lambda^3\gamma(1-\gamma^2)} \\
F &= -B + \frac{e^{\lambda l}}{\epsilon^2\Lambda^3\gamma(1-\gamma^2)}
\end{aligned}$$

Substituting the above into (129) and (130) leads to the following two expressions for B:

$$B = \frac{e^{\lambda(L-l)}+e^{-\lambda(L-l)}-1}{\epsilon^2\Lambda^3\gamma(1-\gamma^2)(e^{\lambda L}+e^{-\lambda L})} \quad \text{and} \quad B = \frac{\left(e^{\Lambda(L-l)}+e^{-\Lambda(L-l)}-1\right)\left(e^{\Lambda l}+e^{-\Lambda l}\right)}{\epsilon^2\Lambda^3\gamma(1-\gamma^2)(e^{\Lambda L}+e^{-\Lambda L})(e^{\lambda l}+e^{-\lambda l})}$$

and hence the following relationship between l and L:

$$\frac{\left(e^{\lambda(L-l)}+e^{-\lambda(L-l)}-1\right)\left(e^{\lambda l}+e^{-\lambda l}\right)}{e^{\lambda L}+e^{-\lambda L}} = \left(\frac{1+e^{-2\Lambda l}}{1+e^{-2\Lambda L}}\right)\left(1-e^{-\Lambda(L-l)}+e^{-2\Lambda(L-l)}\right) \tag{131}$$

We will use the above to determine the value(s) of l.

To simplify the following computations, we will assume that $L = K\epsilon|\ln\epsilon|$ for some sufficiently large K. Note that (131) has the trivial solution $l = \frac{L}{3}$ which corresponds to the $\frac{4L}{3}$-periodic solution. We look for two additional solutions. (By **Claim III** of the proof of **Theorem 1.2**, these are the only three solutions.)

1. $l = h\epsilon|\ln\epsilon|$ for some $h < \frac{K}{3}$.

   In this case, $e^{-2\Lambda L}$, $e^{-\Lambda(L-l)}$ and $e^{-2\Lambda(L-l)}$ are all transcendentally small and thus can be ignored. Then we have:

$$\begin{aligned}
e^{-2\Lambda l} &= \frac{\left(e^{\lambda(L-l)}+e^{-\lambda(L-l)}-1\right)\left(e^{\lambda l}+e^{-\lambda l}\right)}{e^{\lambda L}+e^{-\lambda L}} - 1 = \frac{e^{\lambda(L-2l)}+e^{-\lambda(L-2l)}-(e^{\lambda l}+e^{-\lambda l})}{e^{\lambda L}+e^{-\lambda L}} \\
&= \frac{(\lambda(L-2l))^2-(\lambda l)^2+O(L^3)}{e^{\lambda L}+e^{-\lambda L}}.
\end{aligned}$$

   The above has a solution for l satisfying:

$$e^{-2\Lambda l} = O(L^2)$$

   which belong to the case of $\mathbf{LLS_1}$ — see (95). The graph of u is shown in Figure 16.

2. $y = L - l = h\epsilon|\ln\epsilon|$ for some $h < \frac{K}{3}$.

Let $y = L - l$. By ignoring all the transcendentally small terms again, we obtain:

$$\begin{aligned} -e^{-\Lambda y} + e^{-2\Lambda y} &= \frac{e^{\lambda(L-2y)} + e^{-\lambda(L-2y)} - \left(e^{\lambda(L-y)} + e^{-\lambda(L-y)}\right)}{e^{\lambda L} + e^{-\lambda L}} \\ &= -\frac{2\lambda^2 Ly + o(Ly)}{e^{\lambda L} + e^{-\lambda L}} \end{aligned}$$

which has a solution for y such that

$$e^{-\Lambda y} = O(Ly).$$

This belongs to the case of **SLSS** — see (114). The graph of u is shown in Figure 17.

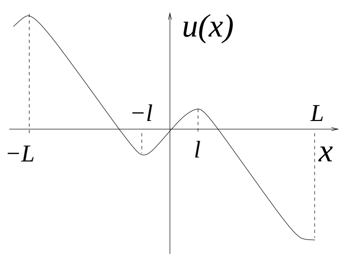

Figure 16.

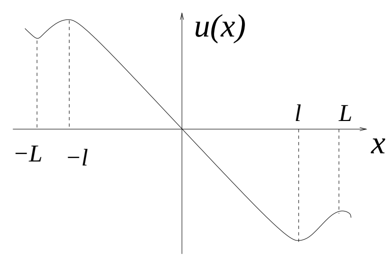

Figure 17.

By using the above approach together with some perturbation argument, it seems possible to construct other non-periodic unstable solutions. It is certainly interesting to characterize the structure of all critical points.

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