DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **16**, Number **3**, October **2011**

doi:10.3934/dcdsb.2011.16.867

pp. **867–881**

LONG TIME BEHAVIOR OF SOME EPIDEMIC MODELS

Fang Li

Department of Mathematics, Purdue University,
150 N. University St., West Lafayette, IN 47907, USA

Nung Kwan Yip

Department of Mathematics, Purdue University,
150 N. University St., West Lafayette, IN 47907, USA

Abstract. In this paper, we prove two results concerning the long time behavior of two systems of reaction diffusion equations motivated by the S-I-R model in epidemic modeling. The results generalize and simplify previous approaches. In particular, we consider the presence of directed diffusions between the two species. The new system contains an ill-posed region for arbitrary parameters. Our result is established under the assumption of small initial data.

1. **Introduction.** The purpose of this paper is to study the long time behavior of two reaction-diffusion systems motivated by the S-I-R model. To model the transmission of an epidemic, in 1927, Kermack and McKendrick [7] first provided a kinetic system in an unstructured population. In this model, the population is assumed to be subdivided into three classes: the susceptible (S), infective (I), and recovered (R) populations, the total population is constant and the sizes of the susceptible and infective populations evolve according to the following system

$$\begin{cases} S_t = -\alpha SI & \text{for } t > 0, \\ I_t = \alpha SI - \gamma I & \text{for } t > 0, \end{cases}$$

where α and γ are positive constants. Many extensions incorporating, for example, age structure, time delays, spatial diffusion and variable infectivity have been considered (see, e.g., [1], [2], [4], [6]) during the past 80 years. In this paper, we focus on the models dealing with spatial issues.

In 1981, Webb [12] proposed a spatially inhomogeneous model in a bounded environment as follows

$$\begin{cases} S_t = k_1 \Delta S - \alpha SI & \text{for } x \in \Omega,\ t > 0, \\ I_t = k_2 \Delta I + \alpha SI - \gamma I & \text{for } x \in \Omega,\ t > 0, \\ \frac{\partial S}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0 & \text{for } x \in \partial\Omega,\ t > 0, \\ S(x,0) = S_0(x),\ I(x,0) = I_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where $\Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$ is the usual Laplace operator, Ω is a bounded smooth domain in $\mathbf{R}^N$ with unit outward normal vector ν on its boundary $\partial\Omega$, k_1, k_2 are two

2000 *Mathematics Subject Classification.* Primary: 35B40, 35K51; Secondary: 92B05, 92D30.
Key words and phrases. Long time behavior, epidemic models.
The second author is partially supported by an NSF grant from the Division of Mathematical Sciences.

positive constants representing the diffusion rates of the susceptible (S) and infected (I) population densities respectively. The constant α represents the infection rate of the susceptible while γ represents the recovery or death rate of the infected. Moreover, the author analyzed the long time behavior of solutions to (1) with *equal* diffusion coefficients in the one-dimensional case. We generalize Webb's result to higher dimensions without any restriction on the diffusion coefficients.

Theorem 1.1. *If $1 \le N \le 3$ and the initial values $S_0(x)$ and $I_0(x)$ of (1) are both nonnegative and not identically zero, then there exists $M \in (0, \gamma/\alpha)$ such that,*

$$\lim_{t\to+\infty} \| S(\cdot,t) - M \|_{C(\overline{\Omega})} = \lim_{t\to+\infty} \| I(\cdot,t) \|_{C(\overline{\Omega})} = 0. \tag{2}$$

Moreover, $\| I(\cdot,t) \|_{C(\overline{\Omega})}$ decays to zero exponentially.

Our second theorem concerns a modification of the above system by incorporating directed diffusion between the two species.

$$\begin{cases} S_t = k_1 S_{xx} + l_1(SS_x)_x + l_3(SI_x)_x - \alpha SI & \text{for } x \in \Omega,\ t>0, \\ I_t = k_2 I_{xx} + l_2(II_x)_x + l_4(IS_x)_x + \alpha SI - \gamma I & \text{for } x \in \Omega,\ t>0, \\ \frac{\partial S}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0 & \text{for } x \in \partial\Omega,\ t>0, \\ S(x,0) = S_0(x),\ I(x,0) = I_0(x) & \text{in } \Omega, \end{cases} \tag{3}$$

where l_1, l_2, l_3, l_4 are positive constants.

Theorem 1.2. *Suppose that $N = 1$. There exists a constant $c > 0$ such that if initially*

$$\overline{S}_0 \le c, \quad \int_\Omega \left[(S_0'(x))^2 + I_0^2(x) + (I_0'(x))^2\right] dx < c,$$

then

$$\lim_{t\to+\infty} \| S(\cdot,t) - M_d \|_{C(\overline{\Omega})} = \lim_{t\to+\infty} \| I(\cdot,t) \|_{C(\overline{\Omega})} = 0,$$

where $M_d \ge 0$ is a constant.

The key feature of the above system is the presence of diffusion fluxes depending on the population densities. While the terms $l_1(SS_x)_x$ and $l_2(II_x)_x$ enhance the smoothing effect of regular diffusions, the terms $l_3(SI_x)_x$ and $l_4(IS_x)_x$ on the other hand facilitate the *repulsion* between the two species. The main function behind these terms is that each type of species attempts to avoid high density of either its own or the other population. Thus this has the effect to *avoid overcrowding*. See [4, 5] for a discussion in the single species case. For systems with directed diffusion and general initial data, they might contain an ill-posed region. Thus care is needed in order to investigate long time behavior and even prove global in time existence of solutions. In the following, we explain intuitively the derivation of (3) and the ill-posedness phenomena.

1.1. **Derivation of (3).** Let $\{X_i\}_i$ and $\{Y_j\}_j$ be the position of individuals of the susceptible and infected species and S and I be some functions representing their population densities obtained by appropriate spatial averaging. Suppose that each of X_i's and Y_j's perform independent Brownian diffusion with drift given by the population densities:

$$dX_i = \sigma_1 dW_i^1(t) - \eta_1 \nabla S(X_i,t)\,dt - \eta_2 \nabla I(X_i,t)\,dt \tag{4}$$

$$dY_j = \sigma_1 dW_j^2(t) - \eta_3 \nabla S(Y_j,t)\,dt - \eta_4 \nabla I(Y_j,t)\,dt \tag{5}$$

The above means that each individual can sense the spatial variation of population densities and attempt to go downhill of the density functions. With the above, the key is to derive (continuum) equations for S and I as the number of particles goes to infinity. This is obtained by testing with smooth functions. Let φ be any smooth spatial function. By Ito's formula [11], we have:

$$\begin{aligned} & d\varphi(X_i(t)) \\ = \; & \nabla\varphi(X_i)\cdot dX_i + \frac{1}{2}\triangle\varphi(X_i)\, d\langle X_i\rangle \\ & \text{(where } \langle\cdot\rangle \text{ refers to the quadratic variation of the process)} \\ = \; & \nabla\varphi(X_i)\cdot\big(\sigma_1 dW_i^1(t) - \eta_1\nabla S(X_i,t)\,dt - \eta_2\nabla I(X_i,t)\,dt\big) + \frac{\sigma_1^2}{2}\triangle\varphi(X_i)\,dt. \end{aligned}$$

Integrating in time, we have:

$$\begin{aligned} \varphi(X_i(t)) &= \varphi(X_i(0)) + \sigma_1\int_0^t \nabla\varphi(X_i)\cdot dW_i^1(s) \\ &\quad - \int_0^t \nabla\varphi(X_i(s))\cdot\Big(\eta_1\nabla S(X_i,s) + \eta_2\nabla I(X_i,s)\Big)\,ds + \int_0^t \frac{\sigma_1^2}{2}\triangle\varphi(X_i(s))\,ds. \end{aligned}$$

Upon summing over all the X_i's and dividing by its total number N gives:

$$\begin{aligned} \frac{1}{N}\sum_i \varphi(X_i(t)) \;&=\; \frac{1}{N}\sum_i \varphi(X_i(0)) + \sigma_1\frac{1}{N}\sum_i\int_0^t \nabla\varphi(X_i)\cdot dW_i^1(s) \\ &\quad -\frac{1}{N}\sum_i\int_0^t \nabla\varphi(X_i(s))\cdot\Big(\eta_1\nabla S(X_i,s) + \eta_2\nabla I(X_i,s)\Big)\,ds \\ &\quad +\frac{1}{N}\sum_i\int_0^t \frac{\sigma_1^2}{2}\triangle\varphi(X_i(s))\,ds. \end{aligned}$$

By Law of Large Numbers, as $N \longrightarrow \infty$, the empirical density $\frac{1}{N}\sum_i \delta_{X_i(t)}$ formally converges to the deterministic spatial density function S. Hence the above gives, for each smooth function φ, the following identity for S:

$$\begin{aligned} \Big\langle S(t),\varphi\Big\rangle = \Big\langle S(0),\varphi\Big\rangle - \eta_1\int_0^t \Big\langle S(s),\nabla\varphi\cdot\nabla S\Big\rangle\,ds - \eta_2\int_0^t \Big\langle S(s),\nabla\varphi\cdot\nabla I\Big\rangle\,ds \\ + \frac{\sigma_1^2}{2}\int_0^t \Big\langle S(s),\triangle\varphi\Big\rangle\,ds. \end{aligned}$$

which is equivalent to the weak form of the first equation of (3). (In the above, $\Big\langle\;,\;\Big\rangle$ refers to the inner product between functions.) The equation for I can be obtained in a similar fashion. The above derivation can be made rigorous by using the general theory of particle representations of PDEs (see for example [8]).

1.2. **Ill-posedness of (3).** Here we indicate the ill-posedness phenomena of the system with general coefficients. Consider the linearization around a constant state $\bar S$ and $\bar I$ keeping only the highest order derivative terms:

$$\begin{aligned} S_t &= k_1 S_{xx} + l_1\bar S S_{xx} + l_3\bar S I_{xx} \\ I_t &= k_2 I_{xx} + l_2\bar I I_{xx} + l_4\bar I S_{xx} \end{aligned}$$

which is written in the following matrix form:

$$\frac{d}{dt}\begin{pmatrix} S \\ I \end{pmatrix} = \begin{pmatrix} k_1 + l_1\bar S & l_3\bar S \\ l_4\bar I & k_2 + l_2\bar I \end{pmatrix}\begin{pmatrix} S_{xx} \\ I_{xx} \end{pmatrix}. \qquad (6)$$

It is easy to see that the above system is well-posed if and only if the coefficient matrix has only eigenvalues with positive real parts. This is similar to the requirement that the diffusion coefficient for heat equation must be positive for well-posedness. This can also be seen more concretely by taking the Fourier transform of equation (6):

$$\frac{d}{dt}\begin{pmatrix}\hat{S}(\xi,t)\\ \hat{I}(\xi,t)\end{pmatrix} = -\xi^2\begin{pmatrix}k_1+l_1\bar{S} & l_3\bar{S}\\ l_4\bar{I} & k_2+l_2\bar{I}\end{pmatrix}\begin{pmatrix}\hat{S}(\xi,t)\\ \hat{I}(\xi,t)\end{pmatrix}.$$

The imposed condition implies that the growth rates of $\hat{S}$ and $\hat{I}$ do not increase as $|\xi| \longrightarrow \infty$. Now we compute the eigenvalues of the matrix:

$$\det\begin{pmatrix}k_1+l_1\bar{S}-\lambda & l_3\bar{S}\\ l_4\bar{I} & k_2+l_2\bar{I}-\lambda\end{pmatrix} = 0$$

which gives:

$$\lambda^2 - (k_1+l_1\bar{S}+k_2+l_2\bar{I})\lambda + (k_1+l_1\bar{S})(k_2+l_2\bar{I}) - l_3l_4\bar{S}\bar{I} = 0.$$

Setting $a = k_1+l_1\bar{S}$ and $b = k_2+l_2\bar{I}$ gives

$$\lambda_{1,2} = \frac{1}{2}\left(a+b \pm \sqrt{(a-b)^2+4l_3l_4\bar{S}\bar{I}}\right).$$

Hence we need

$$a+b \geq \sqrt{(a-b)^2+4l_3l_4\bar{S}\bar{I}}$$

i.e.

$$ab \geq l_3l_4\bar{S}\bar{I}$$

or equivalently,

$$k_1k_2 + l_1k_2\bar{S} + k_1l_2\bar{I} \geq (l_3l_4 - l_1l_2)\bar{S}\bar{I}. \tag{7}$$

From the above, we notice that formally, the system is in general *ill-posed*, in particular when l_3 and l_4 are large. Nevertheless, it can still be well-posed (at the linearized level) if $\bar{S}$ and $\bar{I}$ are *small*. This is the case on which we will concentrate in this paper.

This ill-posedness phenomenon also appears in the avoidance and overcrowding model considered in [9]. They study the following model:

$$\begin{cases} S_t &=& k_1(SN_x)_x + k_2(SI_x)_x - \alpha SI\\ I_t &=& k_1(IN_x)_x + \alpha SI - \gamma I\\ R_t &=& k_1(RN_x)_x + \gamma I\end{cases}$$

where $N = S+I+R$ denotes the total population and R denotes the recovered population. Using the condition (7), we see that it is ill-posed at any constant state.

Let us state that the main contribution of the current paper is the simplicity of our approaches, compared with the abstract functional analysis setting using semi-group theory. (See [3] for a general survey about reaction diffusion equations and their long time behaviors.) We mainly employ energy estimates combined with maximum principle. The key element of our approach is the observation of the fact that $\int I(x,t)\,dx \longrightarrow 0$ as $t \longrightarrow \infty$. Several uses of embedding theorems imply convergence in higher order norms. This seems to be a novelty and might be useful in other systems arised in mathematical biology. In addition, we also generalize some earlier results. For example, the work [12] does not consider directed cross diffusion and requires the diffusion coefficients of the two species to be the same; the results in [10] about long time behaviors require a priori the uniform bounds of the solutions. The disadvantage of our approach appears to be that our results depend on spatial dimensions even though we believe this can be overcome by using higher order regularity and embedding theorems. But in order to make the key idea transparent, in this paper, we prove **Theorem 1.1** and **Theorem 1.2** for dimensions $1 \leq N \leq 3$ and $N = 1$ respectively.

2. **System without directed difussion.** In this section, we study the large time behavior of the system (1). In the following, we use $\overline{F}(t)$ to denote the spatial average of a function $F(x,t)$:

$$\overline{F}(t) = \frac{1}{|\Omega|}\int_\Omega F(x,t)dx.$$

2.1. **Preliminaries.** By the maximum principle, it is easy to see that

$$S(x,t) > 0, \ \ I(x,t) > 0 \ \text{ for } \ t > 0.$$

Hence w.l.o.g., throughout this section, we always assume that $S_0(x) > 0$ and $I_0(x) > 0$ on $\overline{\Omega}$. For later convenience, denote

$$S^* = \max_{\overline{\Omega}} S_0(x) > 0, \ S_* = \min_{\overline{\Omega}} S_0(x) > 0,$$

$$I^* = \max_{\overline{\Omega}} I_0(x) > 0, \ I_* = \min_{\overline{\Omega}} I_0(x) > 0.$$

Since $S_t < k_1 \Delta S$ for $x \in \Omega, \ t > 0$ satisfying $\frac{\partial S}{\partial \nu} = 0$ for $x \in \partial\Omega, \ t > 0$, it follows immediately from the maximum principle that

$$S(x,t) \leq S^* \text{ for } t > 0.$$

By integrating the first equation in the system (1) over the domain Ω, we have

$$\frac{d}{dt}\overline{S}(t) = -\alpha\overline{SI}(t) < 0, \ \ \text{with} \ \ \overline{S}(0) > 0, \tag{8}$$

This implies that $\overline{S}(t)$ decreases monotonically to a nonnegative constant M as $t \to +\infty$. Similarly, integrating the sum of the first two equations in the system (1) over the domain Ω gives

$$\frac{d}{dt}\overline{S+I}(t) = -\gamma\overline{I}(t) < 0, \ \ \text{with} \ \ \overline{S+I}(0) > 0, \tag{9}$$

which yields the monotone convergence of $\overline{S+I}(t)$ as $t \to +\infty$. Therefore $\lim_{t\to+\infty}\overline{I}(t)$ also exists. Moreover, by integrating (9) from 0 to t, we get

$$\overline{S+I}(t) - \overline{S+I}(0) = -\gamma\int_0^t \overline{I}(\tau)d\tau.$$

Letting $t \to +\infty$ in the above equation, the convergence of $\overline{S+I}(t)$ and $\overline{I}(t)$ guarantees that $\lim_{t\to+\infty}\overline{I}(t) = 0$.

Let us first summarize the basic properties of the solutions $S(x,t)$ and $I(x,t)$ obtained from the above preliminary analysis as follows.

Lemma 2.1. *For $N \geq 1$ and $S_0(x) > 0$, $I_0(x) > 0$ on $\overline{\Omega}$, we have*

(a) $S(x,t) \leq S^*$ *for* $t > 0$.
(b) $\lim_{t\to+\infty}\overline{S}(t) = M \geq 0$.
(c) $\lim_{t\to+\infty}\overline{I}(t) = 0$.

Next we provide a simple lemma which will be used frequently.

Lemma 2.2. *If* $h(t) \geq 0$ *satisfies*

$$h'(t) \leq -\lambda h(t) + g(t), \quad h(0) = h_0 > 0, \tag{10}$$

where the constant $\lambda > 0$, $g(t) > 0$ *and* $\lim_{t\to+\infty} g(t) = 0$, *then* $\lim_{t\to+\infty} h(t) = 0$.

This lemma is rather standard. We include a proof here for the convenience of the readers.

Proof. For any fixed $\epsilon > 0$, there exists $T_1 > 0$ such that $g(t) < \epsilon$ for $t \geq T_1$. Suppose that $\frac{\lambda}{2}h(t) \geq \epsilon$ for any $t \geq T_1$, then we have

$$h'(t) \leq -\lambda h(t) + g(t) \leq -\frac{\lambda}{2}h(t), \text{ for } t \geq T_1,$$

which implies that $\lim_{t\to+\infty} h(t) = 0$. This is a contradiction. Thus there exists $T_2 \geq T_1$ such that

$$\frac{\lambda}{2}h(t) < \epsilon \text{ at } t = T_2. \tag{11}$$

Notice that for $t \geq T_2 \geq T_1$, $h'(t) \leq -\lambda h(t) + \epsilon$. Hence by direct computation and (11), it is easy to obtain that for $t \geq T_2$

$$h(t) \leq \exp(\lambda(T_2 - t))h(T_2) + \frac{\epsilon}{\lambda} < \frac{3\epsilon}{\lambda}.$$

□

2.2. **Proof of Theorem 1.1.** First we state the main estimates for $S(x,t) - M$ and $I(x,t)$ derived in this paper.

Proposition 1. *For* $1 \leq N \leq 4$,

$$\lim_{t\to+\infty} \| S(\cdot,t) - M \|_{W^{2,2}(\Omega)} = \lim_{t\to+\infty} \| I(\cdot,t) \|_{W^{2,2}(\Omega)} = 0.$$

Remark 1. It follows immediately from the Sobolev Embedding Theorem that for $1 \leq N \leq 3$

$$\| S(\cdot,t) - M \|_{C^{1,\mu}(\overline{\Omega})} < c \| S(\cdot,t) - M \|_{W^{2,2}(\Omega)},$$

and

$$\| I(\cdot,t) \|_{C^{1,\mu}(\overline{\Omega})} < c \| I(\cdot,t) \|_{W^{2,2}(\Omega)},$$

where $0 < \mu < 1/2$ and the constant c is independent of t. Therefore,

$$\lim_{t\to+\infty} \| S(\cdot,t) - M \|_{C(\overline{\Omega})} = \lim_{t\to+\infty} \| I(\cdot,t) \|_{C(\overline{\Omega})} = 0,$$

for $1 \leq N \leq 3$. Estimate (2) in **Theorem 1.1** is thus verified.

Since the proof of Proposition 1 is quite lengthy, we will demonstrate it later and continue to prove **Theorem 1.1** here.

Proof of **Theorem 1.1**. Suppose that $M > \gamma/\alpha$, then because of the estimate (2), there exists $T > 0$ such that

$$\alpha S(x,t) - \gamma > 0 \text{ for } x \in \Omega, \ t \geq T.$$

Thus from the second equation in the system (1), it is easy to see that

$$\frac{d}{dt}\overline{I}(t) > 0 \text{ for } t \geq T,$$

which implies that $\overline{I}(t) > \overline{I}(T) > 0$ contradicting Lemma 2.1(c). Therefore, $0 \leq M \leq \gamma/\alpha$ for $1 \leq N \leq 3$.

Let us first show that $M < \gamma/\alpha$. Suppose that $M = \gamma/\alpha$. Set $w(x,t) = S(x,t) - M$ and rewrite the system (1) as follows

$$\begin{cases} w_t = k_1\Delta w - \alpha w I - \gamma I & \text{for } x \in \Omega,\ t > 0, \\ I_t = k_2 \Delta I + \alpha w I & \text{for } x \in \Omega,\ t > 0, \\ \frac{\partial w}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0 & \text{for } x \in \partial\Omega,\ t > 0, \\ w(x,0) = w_0(x) = S_0(x) - M,\ I(x,0) = I_0(x) & \text{in } \Omega. \end{cases} \tag{12}$$

Integrating the first two equations in the above system over the domain Ω, we have

$$\begin{cases} \overline{w}'(t) = -\alpha\overline{wI}(t) - \gamma\overline{I}(t) & \text{for } t > 0, \\ \overline{I}'(t) = \alpha\overline{wI}(t) & \text{for } t > 0. \end{cases}$$

Then it follows from direct calculations that

$$\overline{w}(T) + \overline{I}(T) - \overline{w}(t) - \overline{I}(t) = -\gamma \int_t^T \overline{I}(\tau)d\tau,$$

where $T > t > 0$. Letting $T \to +\infty$, because of Lemma 2.1(b), (c), we have

$$\int_t^{+\infty} \overline{I}(\tau)d\tau = \frac{1}{\gamma}(\overline{w}(t) + \overline{I}(t)).$$

Hence

$$\lim_{t\to+\infty} \int_t^{+\infty} \overline{I}(\tau)d\tau = 0. \tag{13}$$

Due to (13), we can apply L'Hôpital's rule to obtain

$$\lim_{t\to+\infty} \frac{\overline{w}(t)}{\int_t^{+\infty}\overline{I}(\tau)d\tau} = \lim_{t\to+\infty} \frac{-\alpha\overline{wI}(t) - \gamma\overline{I}(t)}{-\overline{I}(t)} = \gamma,$$

where the second equation holds because of (2). This positive limit guarantees that there exists $T_1 > 0$ such that $\overline{w}(t) > 0$ for $t > T_1$.

Using the second equation in the system (12) again, we deduce that

$$\frac{d}{dt}\frac{1}{|\Omega|}\int_\Omega \ln I(x,t)dx = k_2\frac{1}{|\Omega|}\int_\Omega \frac{|\nabla I|^2}{I^2}dx + \alpha\overline{w}(t) > 0, \quad \text{for } t > T_1.$$

(Here recall the assumption that $M = \frac{\gamma}{\alpha}$.) Hence

$$\frac{1}{|\Omega|}\int_\Omega \ln I(x,t)dx > \frac{1}{|\Omega|}\int_\Omega \ln I(x,T_1)dx, \quad \text{for } t > T_1.$$

Thanks to (2), we derive a contradiction by letting $t \to +\infty$ in the above inequality. Therefore $M < \gamma/\alpha$ is proved.

Secondly, we will verify that $\| I(\cdot,t) \|_{C(\overline{\Omega})}$ decays to zero exponentially. Let $\theta = (\gamma - \alpha M)/2$. Since $M < \gamma/\alpha$, $\theta > 0$. Then due to (2), there exists $T > 0$ such that

$$\gamma - \alpha S(x,t) > \theta, \quad \text{for } t > T,$$

and the equation satisfied by $I(x,t)$ in the system (1) yields that

$$I_t < k_2\Delta I - \theta I, \quad \text{for } t > T.$$

Now compare $I(x,t)$ with the $l(t)$ given by:

$$l'(t) = -\theta l(t), \quad l(T) = \max_{\overline{\Omega}} I(x,T).$$

It follows from the maximum principle that

$$I(x,t) \le l(t) = l(T)\exp(-\theta(t-T)), \quad \text{for } t > T. \tag{14}$$

Now it only remains to show that $M > 0$. Using the estimate (14), the equation satisfied by $S(x,t)$ in the system (1) becomes

$$S_t \geq k_1 \Delta S - c\exp(-\theta t) S, \quad \text{for} \quad t > T,$$

where $c = \alpha l(T)\exp(\theta T)$. Similarly, compare $S(x,t)$ with the function $h(t)$ which satisfies

$$h'(t) = -c\exp(-\theta t)h(t), \quad h(T) = \min_{\overline{\Omega}} S(x,T) > 0, \quad t > T. \tag{15}$$

By the maximum principle, we have

$$S(x,t) \geq h(t), \quad \text{for} \quad t > T.$$

While, at the same time, let us solve the initial value problem (15) for $t > T$.

$$h(t) = h(T)\exp\big(\frac{c}{\theta}\exp(-\theta t)\big)\exp\big(-\frac{c}{\theta}\exp(-\theta T)\big).$$

Consequently,

$$M \geq h(T)\exp\big(-\frac{c}{\theta}\exp(-\theta T)\big) > 0.$$

Theorem 1.1 is thus proved. □

The rest of this section is dedicated to the proof of Proposition 1. To better explain our idea, we divide our proof into several lemmas. The estimates of $S(x,t) - M$ and $I(x,t)$ are improved gradually during the presentation of these lemmas. For convenience, we always use c_i and C_i, $i \geq 1$, to denote constants which are independent of x and t.

Lemma 2.3. *For $N \geq 1$, $\lim_{t\to+\infty} \| S(\cdot,t) - M \|_{L^2(\Omega)} = 0$.*

Proof. Denote $\tilde{w}(x,t) = S(x,t) - \overline{S}(t)$. Then $\tilde{w}(x,t)$ satisfies

$$\tilde{w}_t = k_1\Delta\tilde{w} - \alpha\tilde{w}I - \alpha\overline{S}I + \alpha\overline{SI}, \quad \text{for } x \in \Omega, \ t > 0,$$

with $\partial\tilde{w}/\partial\nu = 0$ for $x \in \partial\Omega, \ t > 0$. (Note that $\overline{S}'(t) = -\alpha\overline{SI}$.) Multiplying this equation by $\tilde{w}$, integrating by parts and applying Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega \tilde{w}^2 dx &= -k_1\int_\Omega |\nabla\tilde{w}|^2 dx - \alpha\int_\Omega \tilde{w}^2 I dx - \alpha\overline{S}\int_\Omega \tilde{w} I dx \\
&\leq -k_1 c_1 \int_\Omega \tilde{w}^2 dx - \alpha\int_\Omega \tilde{w}^2 I dx + \alpha\int_\Omega \tilde{w}^2 I dx + \frac{1}{4}\alpha\overline{S}^2\int_\Omega I dx \\
&\leq -k_1 c_1 \int_\Omega \tilde{w}^2 dx + C_1\overline{I}.
\end{aligned}$$

Then it follows easily from Lemma 2.1(c) and Lemma 2.2 that

$$\lim_{t\to+\infty} \| \tilde{w}(\cdot,t) \|_{L^2(\Omega)} = 0.$$

Combined with Lemma 2.1(b), the lemma is proved. □

The following lemma also holds for $N \geq 1$.

Lemma 2.4. *For $N \geq 1$,*

$$\lim_{t\to+\infty} \| I(\cdot,t) \|_{W^{1,2}(\Omega)} = \lim_{t\to+\infty} \| S(\cdot,t) - M \|_{W^{1,2}(\Omega)} = 0.$$

Proof. Recall that $I(x,t)$ satisfies

$$I_t = k_2\Delta I + \alpha SI - \gamma I, \quad \text{for } x \in \Omega,\ t > 0,$$

with $\partial I/\partial \nu = 0$ for $x \in \partial\Omega,\ t > 0$. Standard calculations give that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega I^2 dx = -k_2\int_\Omega |\nabla I|^2 dx + \int_\Omega (\alpha S - \gamma) I^2 dx,$$

and

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla I|^2 dx = -k_2\int_\Omega |\Delta I|^2 dx - \gamma\int_\Omega |\nabla I|^2 dx - \alpha\int_\Omega SI\Delta I dx.$$

Then, it follows from Lemma 2.1(a) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \big(I^2 + |\nabla I|^2\big) dx \\
= \ & -(k_2+\gamma)\int_\Omega |\nabla I|^2 dx + \int_\Omega (\alpha S-\gamma) I^2 dx - k_2\int_\Omega |\Delta I|^2 dx - \alpha\int_\Omega SI\Delta I dx \\
\le \ & -(k_2+\gamma)\int_\Omega |\nabla I|^2 dx + \int_\Omega (\alpha S-\gamma) I^2 dx - k_2\int_\Omega |\Delta I|^2 dx \\
& + k_2\int_\Omega |\Delta I|^2 dx + \frac{\alpha^2}{4k_2}\int_\Omega S^2 I^2 dx \\
\le \ & -k_2\int_\Omega \big(I^2 + |\nabla I|^2\big) dx + C_2\int_\Omega I^2 dx.
\end{aligned}$$

Setting $h(t) = \| I(\cdot,t) \|_{W^{1,2}(\Omega)}$, the above inequality can be rewritten as

$$h'(t) \le -k_2 h(t) + \frac{C_2}{h(t)}\int_\Omega I^2 dx. \tag{16}$$

Choose $\theta = \min\{\frac{1}{2}, \frac{4}{N+2}\} \in (0,1)$. By Hölder's inequality, the Sobolev Embedding Theorem and the fact that $I \ge 0$, we derive that

$$\int_\Omega I^2 dx \le \Big(\int_\Omega I dx\Big)^\theta \Big(\int_\Omega I^{\frac{2-\theta}{1-\theta}} dx\Big)^{1-\theta} \le c_2 \overline{I}^\theta(t) h^{2-\theta}(t).$$

This, together with (16), yields that

$$\frac{1}{\theta}\frac{d}{dt} h^\theta(t) \le -k_2 h^\theta(t) + c_2 C_2 \overline{I}^\theta(t).$$

Hence $\lim_{t\to+\infty} \| I(\cdot,t) \|_{W^{1,2}(\Omega)} = 0$ follows again from Lemma 2.2.

Next we prove the long time behavior regarding $S(x,t)$. Consider $w(x,t) = S(x,t) - M$ which satisfies

$$w_t = k_1\Delta w - \alpha wI - \alpha MI, \quad \text{for } x \in \Omega,\ t > 0, \tag{17}$$

with $\partial w/\partial \nu = 0$ for $x \in \partial\Omega,\ t > 0$. By direct computations again, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega \big(w^2 + |\nabla w|^2\big) dx \ = \ & -k_1\int_\Omega |\nabla w|^2 dx - \alpha\int_\Omega w^2 I dx - \alpha M\int_\Omega wI dx \\
& -k_1\int_\Omega |\Delta w|^2 dx + \alpha\int_\Omega wI\Delta w dx + \alpha M\int_\Omega I\Delta w dx.
\end{aligned}$$

Since Lemma 2.1(a) tells us that $w(x,t)$ is bounded, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \big(w^2 + |\nabla w|^2\big) dx \\
\le \ & -k_1\int_\Omega \big(w^2 + |\nabla w|^2\big) dx + k_1\int_\Omega w^2 dx + c_3\overline{I}(t) + C_3\int_\Omega I^2 dx.
\end{aligned}$$

Obviously, because of Lemma 2.3 and $\lim_{t\to+\infty} \| I(\cdot,t) \|_{W^{1,2}(\Omega)}= 0$, we are now ready to apply Lemma 2.2 to derive that

$$\lim_{t\to+\infty} \| S(\cdot,t) - M \|_{W^{1,2}(\Omega)}= 0.$$

□

Proof of Proposition 1. Using (17), it is routine to check that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta w|^2 dx \\
= \;& \int_\Omega (\Delta w)\big(k_1\Delta(\Delta w) - \alpha\Delta(wI) - \alpha M\Delta I\big)dx \\
= \;& -k_1\int_\Omega |\nabla(\Delta w)|^2 dx + \alpha\int_\Omega \nabla(\Delta w)\cdot\nabla(wI)dx + \alpha M\int_\Omega \nabla(\Delta w)\cdot\nabla I dx \\
\leq \;& C_4'\int_\Omega \big(|\nabla(wI)|^2 + |\nabla I|^2\big)dx \\
\leq \;& C_4\int_\Omega \big(I^2|\nabla w|^2 + |\nabla I|^2\big)dx.
\end{aligned}$$

We only have the boundedness of w due to Lemma 2.1(a), thus $I(x,t)$ has to be kept in the last inequality above. Now let us put together all three terms $\int_\Omega w^2dx$, $\int_\Omega |\nabla w|^2dx$, $\int_\Omega |\Delta w|^2dx$, and continue our calculations further.

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \big(w^2 + |\nabla w|^2 + |\Delta w|^2\big)dx \\
= \;& -k_1\int_\Omega |\nabla w|^2dx - \alpha\int_\Omega w^2 I dx - \alpha M\int_\Omega wI dx \\
& -k_1\int_\Omega |\Delta w|^2 dx + \alpha\int_\Omega wI\Delta w dx + \alpha M\int_\Omega I\Delta w dx \\
& -k_1\int_\Omega |\nabla(\Delta w)|^2 dx + \alpha\int_\Omega \nabla(\Delta w)\cdot\nabla(wI)dx + \alpha M\int_\Omega \nabla(\Delta w)\cdot\nabla I dx \\
\leq \;& -k_1\int_\Omega |\nabla w|^2 dx + c_4\overline{I}(t) - \frac{k_1}{2}\int_\Omega |\Delta w|^2 dx + c_4\int_\Omega I^2 dx \\
& +C_4\int_\Omega \big(I^2|\nabla w|^2 + |\nabla I|^2\big)dx \\
\leq \;& -\frac{k_1}{2}\int_\Omega \big(w^2 + |\nabla w|^2 + |\Delta w|^2\big)dx + \frac{k_1}{2}\int_\Omega w^2 dx + c_4\overline{I}(t) + c_4\int_\Omega I^2 dx \\
& +C_4\int_\Omega |\nabla I|^2 dx + C_4\int_\Omega I^2|\nabla w|^2 dx.
\end{aligned}$$

In order to take care of the last term in the last inequality above, we restrict the dimensions of the domain Ω to $1 \leq N \leq 4$. Then, it follows that

$$\int_\Omega I^2|\nabla w|^2 dx \leq \Big(\int_\Omega I^4 dx\Big)^{1/2}\Big(\int_\Omega |\nabla w|^4 dx\Big)^{1/2} \leq c \,\| I(\cdot,t) \|^2_{W^{1,2}(\Omega)} \| w(\cdot,t) \|^2_{W^{2,2}(\Omega)} .$$

Together with Lemma 2.4, there exists $T > 0$ such that for $t > T$,

$$C_4\int_\Omega I^2|\nabla w|^2 dx < \frac{k_1}{4}\int_\Omega \big(w^2 + |\nabla w|^2 + |\Delta w|^2\big)dx.$$

Consequently, when $t > T$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \big(w^2+|\nabla w|^2+|\Delta w|^2\big)dx \\
\le\ & -\frac{k_1}{4}\int_\Omega \big(w^2+|\nabla w|^2+|\Delta w|^2\big)dx+\frac{k_1}{2}\int_\Omega w^2dx+c_4\overline{I}(t)+c_4\int_\Omega I^2dx \\
&+C_4\int_\Omega |\nabla I|^2dx,
\end{aligned}$$

which, combined with Lemmas 2.2 and 2.4, implies that

$$\lim_{t\to+\infty}\| S(\cdot,t)-M\|_{W^{2,2}(\Omega)}=0.$$

We can deal with $I(x,t)$ by the similar procedure. For the convenience of the readers, we write down some calculations here, especially the parts which are different. Recall that $I(x,t)$ satisfies

$$I_t=k_2\Delta I+\alpha SI-\gamma I,\quad \text{for } x\in\Omega,\ t>0.$$

Then we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta I|^2dx \\
=\ & -k_2\int_\Omega |\nabla(\Delta I)|^2dx-\alpha\int_\Omega \nabla(\Delta I)\cdot\nabla(SI)dx-\gamma\int_\Omega(\Delta I)^2dx \\
\le\ & C_5\int_\Omega \big(I^2|\nabla w|^2+|\nabla I|^2\big)dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \big(I^2+|\nabla I|^2+|\Delta I|^2\big)dx \\
\le\ & -(k_2+\gamma)\int_\Omega |\nabla I|^2dx+\int_\Omega(\alpha S-\gamma)I^2dx-k_2\int_\Omega|\Delta I|^2dx-\alpha\int_\Omega SI\Delta Idx \\
&+C_5\int_\Omega \big(I^2|\nabla w|^2+|\nabla I|^2\big)dx \\
\le\ & -\frac{k_2}{2}\int_\Omega \big(I^2+|\nabla I|^2+|\Delta I|^2\big)dx+c_5\int_\Omega I^2dx+C_5\int_\Omega|\nabla I|^2dx \\
&+C_5\int_\Omega I^2|\nabla w|^2dx.
\end{aligned}$$

When $1\le N\le 4$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \big(I^2+|\nabla I|^2+|\Delta I|^2\big)dx \\
\le\ & -\frac{k_2}{2}\int_\Omega \big(I^2+|\nabla I|^2+|\Delta I|^2\big)dx+c_5\int_\Omega I^2dx+C_5\int_\Omega|\nabla I|^2dx \\
&+C_5'\| I(\cdot,t)\|^2_{W^{1,2}(\Omega)}\| w(\cdot,t)\|^2_{W^{2,2}(\Omega)},
\end{aligned}$$

and then by applying Lemma 2.2, we deduce that

$$\lim_{t\to+\infty}\| I(\cdot,t)\|_{W^{2,2}(\Omega)}=0.$$

□

3. **System with directed difussion.** In this section, we study the large time behavior of the system (3) with directed diffusion in the one dimensional case.

As stated earlier, extra care is needed due to the possibility of ill-posedness of the system. In addition, in the current case we do not have the maximum principle at our disposal so that a priori we do not have $\| S(\cdot,t) \|_{L^\infty(\Omega)} < C < \infty$ for all time.

Now we proceed to the proof of **Theorem 1.2**. First by applying similar arguments at the beginning of Section 2.1, we obtain the estimates for $\overline{S}$ and $\overline{I}$.

Lemma 3.1. *For $N \geq 1$ and $S_0(x) > 0,\ I_0(x) > 0$ on $\overline{\Omega}$, we have*

(a) $\overline{S}'(t) < 0$, $\lim_{t\to+\infty} \overline{S}(t) = M_d \geq 0$.

(b) $\lim_{t\to+\infty} \overline{I}(t) = 0$.

Proof of **Theorem 1.2**. Denote

$$u(x,t) = S(x,t) - \overline{S}(t)$$

and rewrite the system (3) as follows.

$$\begin{cases} u_t = k_1 u_{xx} + l_1 (S u_x)_x + l_3 (S I_x)_x - \alpha S I + \alpha \overline{SI} & \text{for } x \in \Omega,\ t > 0, \\ I_t = k_2 I_{xx} + l_2 (I I_x)_x + l_4 (I u_x)_x + \alpha S I - \gamma I & \text{for } x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0 & \text{for } x \in \partial\Omega,\ t > 0, \\ u(x,0) = S_0(x) - \overline{S}_0,\ I(x,0) = I_0(x) & \text{in } \Omega. \end{cases} \tag{18}$$

(Note again that $\overline{S}'(t) = -\alpha \overline{SI}$.)

Using the system (18), it is routine to check that

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx \\ = & -k_1 \int_\Omega u_x^2 dx - l_1 \int_\Omega S u_x^2 dx - l_3 \int_\Omega S u_x I_x dx - \alpha \int_\Omega S I u dx \\ < & -k_1 \int_\Omega u_x^2 dx - l_3 \overline{S} \int_\Omega u_x I_x dx - l_3 \int_\Omega u u_x I_x dx - \alpha \overline{S} \int_\Omega I u dx, \end{aligned}$$

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\int_\Omega I^2 dx \\ = & -k_2 \int_\Omega I_x^2 dx - l_2 \int_\Omega I I_x^2 dx - l_4 \int_\Omega I u_x I_x dx + \alpha \int_\Omega S I^2 dx - \gamma \int_\Omega I^2 dx \\ < & -\gamma \int_\Omega I^2 dx - k_2 \int_\Omega I_x^2 dx - l_4 \int_\Omega I u_x I_x dx + \alpha \overline{S} \int_\Omega I^2 dx + \alpha \int_\Omega u I^2 dx, \end{aligned}$$

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\int_\Omega u_x^2 dx \\ = & -k_1 \int_\Omega u_{xx}^2 dx - l_1 \int_\Omega (S u_x)_x u_{xx} dx - l_3 \int_\Omega (S I_x)_x u_{xx} dx + \alpha \int_\Omega S I u_{xx} dx \\ < & -k_1 \int_\Omega u_{xx}^2 dx - l_3 \overline{S} \int_\Omega I_{xx} u_{xx} dx - l_3 \int_\Omega u I_{xx} u_{xx} dx - l_3 \int_\Omega I_x u_x u_{xx} dx \\ & -\alpha \overline{S} \int_\Omega I_x u_x dx - \alpha \int_\Omega u I_x u_x dx. \end{aligned}$$

Note that in the above, we have used the following computation:

$$
\begin{aligned}
& -\int_\Omega (Su_x)_x u_{xx}\,dx = -\int_\Omega S_x u_x u_{xx}\,dx - \int_\Omega S u_{xx}^2\,dx \\
< \quad & -\int_\Omega S_x u_x u_{xx}\,dx = -\int_\Omega u_x^2 u_{xx}\,dx = -\int_\Omega \left(\frac{u_x^3}{3}\right)_x dx = 0.
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\int_\Omega I_x^2 dx \\
= \quad & -k_2\int_\Omega I_{xx}^2 dx - l_2\int_\Omega (II_x)_x I_{xx} dx - l_4\int_\Omega (Iu_x)_x I_{xx} dx - \alpha\int_\Omega SII_{xx} dx \\
& +\gamma\int_\Omega II_{xx} dx \\
< \quad & -k_2\int_\Omega I_{xx}^2 dx - \gamma\int_\Omega I_x^2 dx - l_4\int_\Omega Iu_{xx}I_{xx} dx - l_4\int_\Omega I_x u_x I_{xx} dx \\
& +\alpha\overline{S}\int_\Omega I_x^2 dx + \alpha\int_\Omega uI_x^2 dx + \alpha\int_\Omega Iu_x I_x dx.
\end{aligned}
$$

Similarly, we have used:

$$
-\int_\Omega (II_x)_x I_{xx}\,dx = -\int_\Omega I_x^2 I_{xx}\,dx - \int_\Omega II_{xx}^2\,dx < -\int_\Omega I_x^2 I_{xx}\,dx = 0.
$$

Putting together the above four inequalities and setting

$$
\ell^2(t) = \int_\Omega \big(u^2 + u_x^2 + I^2 + I_x^2\big)dx,
$$

$$
\rho^2(t) = \int_\Omega \big(u_x^2 + u_{xx}^2 + I^2 + I_x^2 + I_{xx}^2\big)dx,
$$

we obtain that

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\ell^2(t) \\
< \quad & -k\rho^2(t) - l_3\overline{S}\int_\Omega u_x I_x dx - l_3\overline{S}\int_\Omega I_{xx}u_{xx} dx \\
& -l_3\int_\Omega uu_x I_x dx - l_4\int_\Omega Iu_x I_x dx - l_3\int_\Omega uI_{xx}u_{xx} dx - l_4\int_\Omega Iu_{xx}I_{xx} dx \\
& -l_3\int_\Omega I_x u_x u_{xx} dx - l_4\int_\Omega I_x u_x I_{xx} dx + H(t),
\end{aligned} \tag{19}
$$

where $k = \min\{k_1,\ k_2,\ \gamma\}$ and

$$
H(t) = \alpha\overline{S}\int_\Omega \big(-Iu + I^2 - I_x u_x + I_x^2\big)dx + \alpha\int_\Omega \big(uI^2 - uI_x u_x + uI_x^2 + Iu_x I_x\big)dx.
$$

Recall that $u(x,t) = S(x,t) - \overline{S}(t)$. Hence by Lemma 3.1, the Poincaré inequality and the Embedding Theorem, we have

$$\begin{aligned} H(t) &\leq \alpha\overline{S}_0 \int_\Omega \big(\frac{1}{2}I^2 + \frac{1}{2}u^2 + I^2 + \frac{1}{2}I_x^2 + \frac{1}{2}u_x^2 + I_x^2\big)dx \\ &\quad + \alpha \parallel u(\cdot,t) \parallel_{L^\infty(\Omega)} \int_\Omega \big(I^2 + \frac{1}{2}I_x^2 + \frac{1}{2}u_x^2 + I_x^2\big)dx \\ &\quad + \alpha \parallel I(\cdot,t) \parallel_{L^\infty(\Omega)} \int_\Omega \big(\frac{1}{2}I_x^2 + \frac{1}{2}u_x^2\big)dx \\ &\leq c_1\alpha\overline{S}_0\rho^2(t) + c_2\alpha\ell^3(t). \end{aligned}$$

Then applying Lemma 3.1, the Poincaré inequality and the Embedding Theorem again, it follows from (19) that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\ell^2(t) &\leq -k\rho^2(t) + l_3\overline{S}_0 \int_\Omega \big(\frac{1}{2}I_x^2 + \frac{1}{2}u_x^2\big)dx + l_3\overline{S}_0 \int_\Omega \big(\frac{1}{2}I_{xx}^2 + \frac{1}{2}u_{xx}^2\big)dx \\ &\quad + l_3 \parallel u(\cdot,t) \parallel_{L^\infty(\Omega)} \int_\Omega \big(\frac{1}{2}I_x^2 + \frac{1}{2}u_x^2 + \frac{1}{2}I_{xx}^2 + \frac{1}{2}u_{xx}^2\big)dx \\ &\quad + l_4 \parallel I(\cdot,t) \parallel_{L^\infty(\Omega)} \int_\Omega \big(\frac{1}{2}I_x^2 + \frac{1}{2}u_x^2 + \frac{1}{2}I_{xx}^2 + \frac{1}{2}u_{xx}^2\big)dx \\ &\quad + l_3 \parallel I_x(\cdot,t) \parallel_{L^\infty(\Omega)} \big(\int_\Omega u_x^2 dx\big)^{\frac{1}{2}} \big(\int_\Omega u_{xx}^2 dx\big)^{\frac{1}{2}} \\ &\quad + l_4 \parallel u_x(\cdot,t) \parallel_{L^\infty(\Omega)} \big(\int_\Omega I_x^2 dx\big)^{\frac{1}{2}} \big(\int_\Omega I_{xx}^2 dx\big)^{\frac{1}{2}} + H(t) \\ &\leq -k\rho^2(t) + \frac{1}{2}l_3\overline{S}_0\rho^2(t) + \frac{1}{2}cl_3\ell(t)\rho^2(t) + \frac{1}{2}cl_4\ell(t)\rho^2(t) \\ &\quad + cl_3\ell(t)\rho^2(t) + cl_4\ell(t)\rho^2(t) + H(t) \\ &< -k\rho^2(t) + \frac{1}{2}l_3\overline{S}_0\rho^2(t) + c_3l_3\ell(t)\rho^2(t) + c_3l_4\ell(t)\rho^2(t) \\ &\quad + c_1\alpha\overline{S}_0\rho^2(t) + c_2\alpha\ell^3(t). \end{aligned} \tag{20}$$

Moreover, notice that $\ell^2(t) \leq c_4\rho^2(t)$ since $\overline{u}(t) = 0$. It follows immediately from (20) that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\ell^2(t) \\ < \ &-k\rho^2(t) + \frac{1}{2}l_3\overline{S}_0\rho^2(t) + c_3l_3\ell(t)\rho^2(t) + c_3l_4\ell(t)\rho^2(t) \\ &+ c_1\alpha\overline{S}_0\rho^2(t) + c_2c_4\alpha\ell(t)\rho^2(t) \\ = \ &-k\rho^2(t) + \big[\big(\frac{1}{2}l_3 + c_1\alpha\big)\overline{S}_0 + \big(c_3l_3 + c_3l_4 + c_2c_4\alpha\big)\ell(t)\big]\rho^2(t) \end{aligned} \tag{21}$$

Therefore, if initially

$$\big(\frac{1}{2}l_3 + c_1\alpha\big)\overline{S}_0 + \big(c_3l_3 + c_3l_4 + c_2c_4\alpha\big)\ell(0) \leq \frac{k}{2},$$

then $\left.\frac{d\ell^2(t)}{dt}\right|_{t=0} < 0$ and hence

$$\ell(t) \leq \ell(0) \quad \text{for} \quad t > 0,$$

which immediately yields that

$$\big(\frac{1}{2}l_3 + c_1\alpha\big)\overline{S}_0 + \big(c_3l_3 + c_3l_4 + c_2c_4\alpha\big)\ell(t) \leq \frac{k}{2} \quad \text{for} \quad t > 0. \tag{22}$$

At the end, (21), (22) and $\ell^2(t) \le c_4\rho^2(t)$ together imply that

$$\frac{1}{2}\frac{d}{dt}\ell^2(t) < -\frac{k}{2c_4}\ell^2(t).$$

Consequently,

$$\lim_{t\to+\infty}\ell^2(t) = \lim_{t\to+\infty}\int_\Omega \left(u^2 + u_x^2 + I^2 + I_x^2\right)dx = 0.$$

This implies that

$$\lim_{t\to+\infty} \| u(\cdot,t) \|_{C(\overline{\Omega})} = \lim_{t\to+\infty} \| I(\cdot,t) \|_{C(\overline{\Omega})} = 0,$$

because of the Embedding Theorem. And Theorem 1.2 follows from Lemma 3.1. □

REFERENCES

[1] S. Busenberg and C. Castillo-Chavez, *Interaction, pair formation and force of infection terms in sexually transmitted diseases*, Mathematical and Statistical Approaches to AIDS Epidemiology, 289–300, Lecture Notes in Biomath., **83**, Springer, Berlin, 1989.

[2] C. Castillo-Chavez and H. R. Thieme, *On the role of variable infectivity in the dynamics of the human immunodeficiency virus epidemic*, Mathematical and Statistical Approaches to AIDS Epidemiology, 157–176, Lecture Notes in Biomath., **83**, Springer, Berlin, 1989.

[3] P. C. Fife, *Asymptotic States for Equations of Reaction and Diffusion*, Bull. AMS, **84** (1978), 693–726.

[4] M. E. Gurtin and R. C. MacCamy, *On the diffusion of biological populations*, Math. Biosci., **33** (1977), 35–49.

[5] W. S. C. Gurney and R. M. Nisbet, *The Regulation of Inhomogeneous Populations*, J. Theor. Biol., **52** (1975), 441–457.

[6] M. Iannelli, R. Loro, F. A. Milner, A. Pugliese and G. Rabbiolo, *An AIDS model with distributed incubation and variable infectiousness: applications to i.v. drug users in Latium, Italy*, Eur. J. Epidemiol., **8** (1992), 585–593.

[7] W. O. Kermack and A. G. McKendrick, *A Contribution to the Mathematical Theory of Epidemics*, Proc. Roy. Soc. Lond. A, **115** (1927), 700–721

[8] T. G. Kurtz and J. Xiong, *Particle representation for a class of nonlinear SPDEs*, Stoch. Proc. and Their Appl., **83** (1999), 103–126.

[9] F. A. Milner and R. Zhao, *S-I-R Model with Directed Spatial Diffusion*, Mathematical Population Studies, **15** (2008), 160–181.

[10] M. Mimura and M. Yamaguti, *Pattern formation in interacting and diffusing systems in population biology*, Adv. Biophys., **15** (1982), 19–65.

[11] B. K. Oksendal, "Stochastic Differential Equations: An Introduction with Applications," 6th edition, Springer-Verlag, Berlin, 2003.

[12] G. F. Webb, *A reaction-diffusion model for a deterministic diffusive epidemic*, J. Math. Anal. Appl., **84** (1981), 150–161.

Received August 2010; revised March 2011.

E-mail address: fli@math.purdue.edu
E-mail address: yip@math.purdue.edu