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Nonuniqueness in a Free Boundary Problem from Combustion

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Abstract We study a parabolic free boundary problem with a fixed gradient condition which serves as a simplified model for the propagation of premixed equidiffusional flames. We give a rigorous justification of an example due to J.L. Vázquez that the initial data in the form of two circular humps leads to the nonuniqueness of limit solutions if the supports of the humps touch at the time of their maximal expansion.

Keywords Nonuniqueness · Free boundary problem · Singular perturbations · Flame propagation · Combustion

Mathematics Subject Classification (2000) 35R35 · 35K57 · 80A22 · 80A25

1 Introduction and Main Result

In this paper we consider a one-phase parabolic free boundary problem of finding a nonnegative function $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ such that

$$\begin{aligned} \Delta u - \partial_t u &= 0 \quad \text{in } \{u > 0\}, \\ |\nabla u| &= 1 \quad \text{on } \partial\{u > 0\}, \qquad (P) \\ u(\cdot, 0) &= u_0 \quad \text{on } \mathbb{R}^n, \end{aligned}$$

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where $u_0 \geq 0$ is a continuous function, typically assumed to have a compact support. The problem appears as the limit of the singular perturbation problem

$$\begin{aligned} \Delta u - \partial_t u &= \beta_\varepsilon(u) \quad \text{in } \mathbb{R}^n \times (0,\infty), \\ u(\cdot,0) &= u_0^\varepsilon \quad \text{on } \mathbb{R}^n \end{aligned} \tag{P_ε}$$

as $\varepsilon \to 0+$, where β_ε is a nonnegative Lipschitz function satisfying

$$\operatorname{supp} \beta_\varepsilon = [0,\varepsilon], \qquad \int_0^\varepsilon \beta_\varepsilon(s)ds = \frac{1}{2}$$

and u_0^ε approximate the initial data u_0 in a properly defined way (see Sect. 2). This singular perturbation problem is a simplified model for premixed equidiffusional flames, where u has the meaning of the normalized temperature of the reactant mixture with negative sign and 0 is the combustion temperature. More specifically, the sets $\{u = 0\}$, $\{0 < u \leq \varepsilon\}$, and $\{u > \varepsilon\}$ represent the *burnt*, *reaction*, and *unburnt* zones, respectively. For the details in combustion theory we refer to the book of Buckmaster and Ludford [5], or the lecture notes of Vázquez [19]. The limit $\varepsilon \to 0+$ corresponds to the regime of high activation energy in the thermo-diffusive model of Zeldovich and Frank-Kamenetski [23], developed in the late 1930s. The mathematical analysis of the free boundary problem (P) and the convergence of (P_ε) to (P) has been initiated by Caffarelli and Vázquez [9], and studied later by various authors.

Typically, *classical* (smooth) *solutions* of (P) exist and are unique only for a short time depending on the initial data, cf. Baconneau and Lunardi [3], so weaker notions of solutions are needed to study global in time solutions. One such notion, *limit solutions*, defined as limits of solutions of (P_ε), is ultimately related to the origin of this problem. We refer to the works of Caffarelli and Vázquez [9], Caffarelli, Lederman, and Wolanski [7, 8], as well as Weiss [21, 22] for fine properties of limit solutions. Other explored notions of solutions include *viscosity solutions*, which are essentially functions satisfying the comparison principle with classical sub- and supersolutions, see [8].

It has been shown that under certain geometric conditions on the initial data, such as the radial symmetry or starshapedness of level sets, the limit and viscosity solutions are unique, see Galaktionov, Hulshof, and Vázquez [13], Petrosyan [16], Kim [14], and Lederman, Vázquez, and Wolanski [15]. On the other hand, no uniqueness should be expected for general initial data. Indeed, in Sect. 13 of his lecture notes [19], Vázquez describes an explicit example where the nonuniqueness may occur. Here we give a slightly modified version of this example.

Example 1.1 (Nonuniqueness of limit solutions) Let $u_0 \geq 0$ be a continuous compactly supported and radially symmetric function in $\mathbb{R}^n$ such that

$$\{u_0 > 0\} = B_{r_0}, \quad r_0 > 0.$$

Assume also

$$u_0 \in C^\infty\left(\overline{B_{r_0}}\right), \qquad |\nabla u_0| = 1 \quad \text{on } \partial B_{r_0}.$$

For such hump-like initial data, problem (P) admits a unique classical radially symmetric solution u for some time interval $[0, T)$, $T > 0$, see [13], Theorem 3.1. For such u, the positivity set $\{u(\cdot, t) > 0\}$ is a ball of a certain radius $r(t) > 0$, for all $t \in [0, T)$, and $r(t)$ depends real-analytically on t. Furthermore, if T_{ext} denotes the maximum of all times T as above, then

$$\lim_{t \to T_{\mathrm{ext}}-} r(t) = 0.$$

Thus, in a sense, the classical solution u extincts at $t = T_{\mathrm{ext}}$. Now, if we extend u as identically 0 for $t \geq T_{\mathrm{ext}}$, the resulting function will be the unique limit solution of problem (P) with the initial data u_0.

Next, we note that for this solution u there exists a $t^* \in [0, T_{\mathrm{ext}})$ such that

$$r^* := r(t^*) = \max_{t \in [0, T_{\mathrm{ext}})} r(t). \tag{1.1}$$

Both cases $t^* = 0$ and $t^* > 0$ are possible, depending on the shape of u_0.

Consider now an initial data w_0 in the shape of the two humps as above with their centers separated by a distance of $2r^*$:

$$w_0(x) := u_0(x - r^* e_n) + u_0(x + r^* e_n). \tag{1.2}$$

Here $e_n = (0, \dots, 0, 1)$. Then there are two different ways to construct limit solutions of (P) with this initial data, possibly leading to different outcomes.

(1) *Minimal solution*. Approximate u_0 from *inside* by a sequence

$$\underline{u}_{0,j} \nearrow u_0, \qquad \operatorname{supp} \underline{u}_{0,j} \Subset B_{r_0},$$

and consider limit solutions $\underline{u}_j$ with the initial data $\underline{u}_{0,j}$. Then the supports of the space translates $\underline{u}_j(\cdot - r^* e_n, t)$ and $\underline{u}_j(\cdot + r^* e_n, t)$ will never touch. Thus, superimposing these solutions and passing to the limit, we will obtain that

$$\underline{w}(x, t) := u(x - r^* e_n, t) + u(x + r^* e_n, t)$$

is a limit solution of (P) with the initial data w_0. The positivity set of $\underline{w}(\cdot, t)$ will consist of two balls that touch at $t = t^*$ but then separate and collapse at $t = T_{\mathrm{ext}}$.

(2) *Maximal solution*. Next, if we approximate u_0 from *outside* by a sequence

$$\overline{u}_{0,j} \searrow u_0, \qquad \{\overline{u}_{0,j} > 0\} \Supset B_{r_0},$$

and consider limit solutions $\overline{u}_j$ with the initial data $\overline{u}_{0,j}$, then the supports of $\overline{u}_j(\cdot - r^* e_n, t)$ and $\overline{u}_j(\cdot + r^* e_n, t)$ will overlap at $t = t^*$. There is a possibility that once these supports overlap, they will merge and continue their evolution together. If this behavior persists under the limit $j \to \infty$, we will obtain a limit solution $\overline{w}$ of (P), which has the same initial data w_0 as $\underline{w}$, but is different from $\underline{w}$.

In a sense, the nonuniqueness phenomenon just described is a reflection of the fact that the solution depends *discontinuously* on the initial data. Qualitatively, the

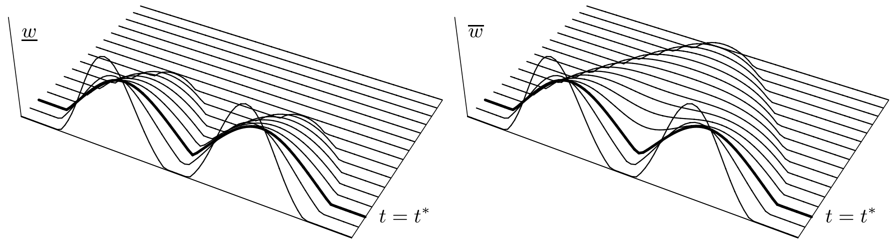


Fig. 1 Minimal and maximal solutions with the same "two-hump" initial data in dimension 1

approximation of the initial data from inside and outside may lead to different solutions.

In dimension $n = 1$, the above example can be validated by a simple use of the maximum principle (see Fig. 1 for an illustration). The main objective of this paper is to give a rigorous justification of Example 1.1 in dimensions $n \geq 2$, thus establishing the nonuniqueness of limit solutions. The following theorem is our main result.

Theorem 1.2 (Nonuniqueness of limit solutions) *Let w_0 be as in* (1.2) *in Example* 1.1 *in dimension $n \geq 2$. Then problem* (P) *has at least two different limit solutions with the initial data w_0. In particular, the maximal and minimal limit solutions are different from each other.*

Readers familiar with the geometric *motion by mean curvature* will notice immediately that the nonuniqueness result above is similar in nature to the phenomenon of the *fattening of level sets*, see e.g. Evans and Spruck [12] and Barles, Soner, and Souganidis [4]. For a better analogy, one should actually look at the evolution of the *graph* of $u(\cdot, t)$, rather than that of the free boundary $\partial\{u(\cdot, t) > 0\}$. In fact, the strategy and constructions behind the proof can be related to the properties of motion by mean curvature and minimal surfaces.

The paper is outlined as follows. Section 2 gives some properties of limit solutions. In particular, we show that the supremum and infimum of all limit solutions are still limit solutions (Theorem 2.1). Section 3 studies the minimizers of the Alt-Caffarelli functional which will be used in Sect. 4 to construct a subsolution v for (P). This subsolution has the form of two shrinking circles connected by a neck which *expands* in time in a selfsimilar fashion. Such a neck is obtained from an analogous stationary free boundary value problem to the Alt-Caffarelli case but solves $\Delta v = c > 0$ in the positivity set of v. Section 5 finishes the proof of Theorem 1.2, our main result. Section 6 discusses some connections between the flame propagation problem and motion by mean curvature and also mentions some future directions. In Appendix we prove the continuity of limit solutions up to $t = 0$ and the boundary continuity of Alt-Caffarelli minimizers. We conclude with an example of a solution of problem (P), which pinches in finite time.

2 Limit Solutions

We have chosen to work with the notion of limit solutions of (P) as we believe that this notion reflects best the origin of the problem in combustion theory. However, it should be noted that most of our results can be adapted also for viscosity solutions.

Throughout the paper we fix a nonnegative Lipschitz continuous function $\beta : \mathbb{R} \to [0, \infty)$, which satisfies

$$\beta > 0 \quad \text{in } (0,1), \qquad \beta = 0 \quad \text{on } \mathbb{R} \setminus (0,1), \qquad \int_0^1 \beta(s)ds = \frac{1}{2}. \tag{2.1}$$

From this single function we construct a one-parameter family $\{\beta_\varepsilon\}_{\varepsilon>0}$ by scaling

$$\beta_\varepsilon(s) := \frac{1}{\varepsilon}\beta\left(\frac{s}{\varepsilon}\right), \tag{2.2}$$

so that we have

$$\operatorname{supp} \beta_\varepsilon = [0, \varepsilon], \qquad \int_0^\varepsilon \beta_\varepsilon(s)ds = \frac{1}{2}. \tag{2.3}$$

For a nonnegative continuous function u_0 with compact support in $\mathbb{R}^n$, we define the *limit solutions* of the free boundary problem (P) in the following way. Let nonnegative $u_0^\varepsilon \in C_0^\infty(\mathbb{R}^n)$ be approximations of u_0 in the sense that

$$\|u_0^\varepsilon - u_0\|_{L^\infty(\mathbb{R}^n)} \to 0, \qquad \operatorname{supp} u_0^\varepsilon \to \operatorname{supp} u_0 \tag{2.4}$$

as $\varepsilon \to 0+$, where the convergence of supports is understood in the sense of Hausdorff distance. Next, let u^ε be the solution of the approximating problem

$$\Delta u^\varepsilon - \partial_t u^\varepsilon = \beta_\varepsilon(u^\varepsilon) \quad \text{in } \mathbb{R}^n \times (0,\infty), \qquad u^\varepsilon(\cdot, 0) = u_0^\varepsilon. \tag{2.5}$$

Then the family $\{u^\varepsilon\}$ is uniformly bounded in $C_x^{0,1} \cap C_t^{0,1/2}$ norm on every compact subset of $\mathbb{R}^n \times (0, \infty)$, see [9] or [7]. Hence, for a subsequence $\varepsilon = \varepsilon_j \to 0+$, u^ε converge uniformly on compact subsets of $\mathbb{R}^n \times (0, \infty)$ to a certain function u that we call a *limit solution* of (P) with the initial data u_0.

We want to point out that if we put $u(\cdot, 0) = u_0$, the resulting function will become continuous on $\mathbb{R}^n \times [0, \infty)$. For the reader's convenience, we give a proof of this fact in Appendix A.1. Furthermore, if $u_0 \in C_0^{0,1}(\mathbb{R}^n)$, one can construct suitable approximations u_0^ε of the initial data for which the family $\{u^\varepsilon\}$ will be bounded in $C_x^{0,1} \cap C_t^{0,1/2}$ norm up to the time $t = 0$ (see [9]).

For the in-depth analysis of limit solutions and the sense in which they satisfy the free boundary conditions in (P), we refer to [8] and [21]. In this paper, we are mainly concerned with the question of uniqueness, or more precisely, nonuniqueness of limit solutions.

We start with a general result, which calibrates the possible nonuniqueness of limit solutions.

Theorem 2.1 (Maximal and minimal limit solutions) *For any nonnegative* $u_0 \in C_0(\mathbb{R}^n)$, *there exist* minimal *and* maximal *limit solutions with that initial data*, *i.e.*, *limit solutions* $\underline{u}$ *and* $\overline{u}$ *such that*

$$\underline{u}(x,t) \le u(x,t) \le \overline{u}(x,t), \quad (x,t) \in \mathbb{R}^n \times [0,\infty)$$

for any limit solution u.

Furthermore, $\underline{u}$ *and* $\overline{u}$ *can be obtained as the limits of sequences* $\{\underline{u}_j\}$ *and* $\{\overline{u}_j\}$ *of limit solutions that correspond to initial data* $\{\underline{u}_{0,j}\}$ *and* $\{\overline{u}_{0,j}\}$ *such that*

$$\underline{u}_{0,j} \nearrow u_0, \qquad \operatorname{supp} \underline{u}_{0,j} \Subset \operatorname{supp} u_0,$$

$$\overline{u}_{0,j} \searrow u_0, \qquad \operatorname{supp} \overline{u}_{0,j} \Supset \operatorname{supp} u_0.$$

We explicitly remark here that Theorems 1.2 and 2.1 are completely independent and rather complimentary to each other.

The rest of this section is dedicated to the proof of Theorem 2.1.

Lemma 2.2 (Limit solution as supersolution) *Let* u *be a limit solution of* (*P*). *Then* u *is a supersolution of problem* (*P*) *in the following sense*

- $u \in C(\mathbb{R}^n \times [0,\infty))$;
- u *satisfies* $\Delta u - \partial_t u = 0$ *in* $\{u > 0\}$;
- u *satisfies* $|\nabla u| \le 1$ *on* $\partial\{u > 0\} \cap \{t > 0\}$ *in the sense that*

$$\limsup_{\substack{(x,t)\to(x_0,t_0)\\ u(x,t)>0}} |\nabla u(x,t)| \le 1$$

for any $(x_0, t_0) \in \partial\{u > 0\}$ *with* $t_0 > 0$.

We refer to [7], Theorem 6.1 for the proof of the above result.

Lemma 2.3 (Strict comparison principle) *Let nonnegative* $u_{0,1}, u_{0,2} \in C_0(\mathbb{R}^n)$ *be such that*

$$u_{0,1} > u_{0,2} \quad \text{on } \operatorname{supp} u_{0,2}.$$

If u_1 *and* u_2 *are any two limit solutions with the initial data* $u_{0,1}$ *and* $u_{0,2}$, *respectively, then* $u_1 \ge u_2$ *in* $\mathbb{R}^n \times [0,\infty)$.

This statement is essentially proved by the first author in [16], see Proposition 2.5 and Remark 2.2 there. For the reader's convenience, we reproduce the proof here, since the conditions in [16] are slightly different. However, the reader may skip this proof in the first reading as it is not used outside this section.

Proof First observe that the statement would be elementary if we knew that u_1 and u_2 are the limits of the solutions u_1^ε and u_2^ε of (P_ε) over the *same* sequence $\varepsilon = \varepsilon_j \to 0+$: the result would follow from the standard comparison principle, since the nonlinearity

β_ε is assumed to be Lipschitz. So the difficulty lies in the comparison of solutions that correspond to different sequences $\varepsilon \to 0+$.

The idea of the proof is to construct a family of supersolutions $\tilde{u}_1^\varepsilon$ of (P_ε) that converges to a small modification $\tilde{u}_1$ of u_1 over *any* sequence $\varepsilon \to 0+$. Then we apply the comparison principle over the sequence that generates u_2 and thereby conclude the proof of the lemma.

To this end, consider the ordinary differential equation

$$(\phi^\varepsilon)'' = \tilde{\beta}_\varepsilon(\phi^\varepsilon), \tag{2.6}$$

where $\tilde{\beta}_\varepsilon(s) := (1/\varepsilon)\tilde{\beta}(s/\varepsilon)$ and $\tilde{\beta}$ is given by

$$\tilde{\beta}(s) := \begin{cases} 0, & s \in (a, 1), \\ \frac{1}{(1-\kappa)^2}\,\beta(s), & s \notin (a, 1). \end{cases} \tag{2.7}$$

Here $\kappa > 0$ is a small parameter and $a = a_\kappa \in (0, 1)$ is chosen so that $\int_a^1 \beta(s)ds = \frac{1}{2}(1-\kappa)^2$. Note that the latter implies that $\int_a^1 \tilde{\beta}(s)ds = \frac{1}{2}$. Next, there exists a unique $C^{1,1}_{\mathrm{loc}}$ solution of (2.6), which satisfies

$$\phi^\varepsilon(s) = a\varepsilon \quad \text{for } s \le 0, \qquad \phi^\varepsilon(s) > a\varepsilon \quad \text{for } s > 0.$$

In fact, the family $\{\phi^\varepsilon\}$ can be recovered through the scaling $\phi^\varepsilon(s) = \varepsilon\phi(s/\varepsilon)$ from a single function ϕ which solves the equation

$$\phi'' = \tilde{\beta}(\phi)$$

with an appropriate normalization. It is easy to see that the function ϕ^ε is monotone nondecreasing and convex and becomes linear with slope 1 as soon as it reaches the level ε at $s = C\varepsilon$ for some constant $C > 0$. The latter follows from the identity

$$[(\phi^\varepsilon)']^2 = 2\tilde{\mathcal{B}}_\varepsilon(\phi^\varepsilon), \quad \tilde{\mathcal{B}}_\varepsilon(s) = \int_0^s \tilde{\beta}_\varepsilon(\sigma)d\sigma$$

which is obtained from (2.6) by multiplying its both sides by $(\phi^\varepsilon)'$ and integrating from 0 to s. In particular, we see that

$$\phi^\varepsilon(s) \to s^+ \quad \text{uniformly on } \mathbb{R} \text{ as } \varepsilon \to 0+. \tag{2.8}$$

Next, with $\kappa > 0$ as in the definition of $\tilde{\beta}$ above and two additional small parameters $\tau, \eta > 0$, define a small modification of the function u_1:

$$\tilde{u}_1(x,t) := \frac{1}{1+2\kappa}(u_1(x, t+\tau) - \eta)^+.$$

The parameters can be chosen so that

$$\tilde{u}_{0,1} := \frac{1}{1+2\kappa}(u_1(\cdot, \tau) - \eta)^+ > u_{0,2} \quad \text{on } \operatorname{supp} u_{0,2}.$$

Moreover, using Lemma 2.2, we may adjust the parameters so that

$$|\nabla \tilde{u}_1| \le 1-\kappa \quad \text{in } 0<\tilde{u}_1<\eta. \tag{2.9}$$

Now consider the composition

$$\tilde{u}_1^\varepsilon(x,t)=\phi^\varepsilon(\tilde{u}_1(x,t)). \tag{2.10}$$

First, we claim that $\tilde{u}_1^\varepsilon \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n\times(0,\infty))$. The only nontrivial place where we need to check this regularity is near $\partial\{\tilde{u}_1>0\}$. To this end, note that the identity $\phi^\varepsilon(s)\equiv\phi^\varepsilon(0)=a\varepsilon$ for $s\le 0$ implies that $\tilde{u}_1^\varepsilon=\phi^\varepsilon(\frac{1}{1+2\kappa}(u_1(x,t+\tau)-\eta))$ in $\{u_1>0\}\supset\partial\{\tilde{u}_1>0\}$. Thus, $\tilde{u}_1^\varepsilon$ is indeed locally $C^{1,1}$. Next, we have

$$\Delta\tilde{u}_1^\varepsilon-\partial_t\tilde{u}_1^\varepsilon=(\phi^\varepsilon)'(\tilde{u}_1)(\Delta\tilde{u}_1-\partial_t\tilde{u}_1)+(\phi^\varepsilon)''(\tilde{u}_1)|\nabla\tilde{u}_1|^2$$

a.e. in $\mathbb{R}^n\times(0,\infty)$. Note that the first term on the right hand side vanishes almost everywhere. To estimate the second term, assume that $\varepsilon>0$ is so small that the condition $\phi^\varepsilon(s)\in\operatorname{supp}\tilde{\beta}_\varepsilon=[a\varepsilon,\varepsilon]$ implies $s<\eta$. Then, using (2.6), (2.9), and (2.10), we obtain

$$\Delta\tilde{u}_1^\varepsilon-\partial_t\tilde{u}_1^\varepsilon=\tilde{\beta}_\varepsilon(\phi^\varepsilon(\tilde{u}_1))|\nabla\tilde{u}_1|^2\le\tilde{\beta}_\varepsilon(\phi^\varepsilon(\tilde{u}_1))(1-\kappa)^2\le\beta_\varepsilon(\tilde{u}_1^\varepsilon)$$

a.e. in $\mathbb{R}^n\times(0,\infty)$. Thus, $\tilde{u}_1^\varepsilon$ is a supersolution of (P_ε). Now, if $\varepsilon=\varepsilon_j>0$ is sufficiently small, from (2.8) we obtain that

$$\tilde{u}_{0,1}^\varepsilon:=\tilde{u}_1^\varepsilon(\cdot,0)>u_{0,2}^\varepsilon \quad \text{on } \operatorname{supp} u_{0,2}^\varepsilon,$$

where $u_{0,2}^\varepsilon$ for $\varepsilon=\varepsilon_j\to 0+$ is the sequence of initial data that generates the limit solution u_2. Applying the comparison principle and passing to the limit, we obtain that $\tilde{u}_1\ge u_2$ in $\mathbb{R}^n\times(0,\infty)$. Letting the parameters $\tau,\eta,\kappa\to 0$, we conclude the proof of the lemma. □

Remark 2.4 Note that the only property of u_1 used in the previous proof is that u_1 is a supersolution in the sense of Lemma 2.2.

Lemma 2.5 *The limit of limit solutions is a limit solution. More precisely, if $\{u_j\}$ is a sequence of limit solutions with initial data $\{u_{0,j}\}$ such that $u_{0,j}\to u_0$ in $L^\infty(\mathbb{R}^n)$,* supp $u_{0,j}\to$ supp u_0 *and $u_j\to u$ uniformly on compact subsets of* $\mathbb{R}^n\times(0,\infty)$, *then u is a limit solution with the initial data u_0.*

Proof The proof is simply achieved by the Cantor diagonalization process. □

We are now ready to prove the main result of this section.

Proof of Theorem 2.1 Let $\{\underline{u}_{0,j}\}$ and $\{\overline{u}_{0,j}\}$ be two sequences of functions in $C_0(\mathbb{R}^n)$ which are strictly increasing and decreasing, respectively, in the sense that

$$\underline{u}_{0,j+1} > \underline{u}_{0,j} \quad \text{on } \operatorname{supp} \underline{u}_{0,j},$$
$$\overline{u}_{0,j} > \overline{u}_{0,j+1} \quad \text{on } \operatorname{supp} \overline{u}_{0,j+1},$$

and which converge uniformly to u_0 on $\mathbb{R}^n$. Let $\{\underline{u}_j\}$ and $\{\overline{u}_j\}$ be the sequences of limit solutions with these initial data. By Lemma 2.3 these sequences are monotone and thus convergent on compact subsets of $\mathbb{R}^n \times (0, \infty)$. Their respective limits $\underline{u}$ and $\overline{u}$ are limit solutions, in virtue of Lemma 2.5.

Now, let u be any limit solution with the initial data u_0. Applying Lemma 2.3 with $u_{0,1} = u_0$ and $u_{0,2} = \underline{u}_{0,j}$, we obtain that $u \geq \underline{u}_j$ for any j. Hence, $u \geq \underline{u}$ and consequently $\underline{u}$ is the minimal limit solution. Applying Lemma 2.3 again but with $u_{0,1} = \overline{u}_{0,j}$ and $u_{0,2} = u_0$, we obtain that $\overline{u}$ is the maximal limit solution.

The theorem is thus proved. □

3 Catenoidal Alt-Caffarelli Minimizers

In this section we establish some properties of nonnegative minimizers of the energy functional studied in the seminal paper of Alt and Caffarelli [1]. The results will be used to construct a subsolution of (P) in the proof of our main result.

Given an open set $D \subset \mathbb{R}^n$ with a smooth boundary and a boundary data in the form of $v_0 \in W^{1,2}(D) \cap L^\infty(D)$, $v_0 \geq 0$, consider the problem of minimizing the functional

$$J(v) := \int_D \Big(|\nabla v|^2 + \chi_{\{v>0\}} \Big)\, dx \tag{AC}$$

over the class of functions $v \in W^{1,2}(D)$ taking boundary values v_0 on ∂D in the Sobolev trace sense, i.e. $v - v_0 \in W_0^{1,2}(D)$. The minimizers of (AC) are known to solve the elliptic free boundary problem with a fixed gradient condition

$$\begin{aligned} \Delta v &= 0 \quad \text{in } \{v > 0\}, \\ |\nabla v| &= 1 \quad \text{on } \Gamma := \partial\{v > 0\} \cap D, \end{aligned} \tag{3.1}$$

in a certain weak sense. The above elliptic problem can be regarded as the stationary version of problem (P). The same free boundary problem appears also in many applications ranging from the modeling of jets and cavities to electro-chemical machining.

Alt and Caffarelli [1] have shown that the minimizers of (AC) are locally Lipschitz continuous in D. Moreover, they have established that the reduced part of the free boundary $\Gamma_{\text{red}} := \partial_{\text{red}}\{v > 0\} \cap D$ is locally analytic and that the singular set $\Sigma := \Gamma \setminus \Gamma_{\text{red}}$ has Hausdorff measure $H^{n-1}(\Sigma) = 0$. Later, Weiss [20] has established the existence of a critical dimension k^*, with the property that Σ is empty for $n < k^*$, consists of isolated points if $n = k^*$, and has Hausdorff dimension at most $n - k^*$ for $n \geq k^*$. It is currently known that $4 \leq k^* \leq 7$. The lower bound follows from the work of Caffarelli, Jerison, and Kenig [6] and the upper bound has been proved by De Silva and Jerison [10].

We will also use the following fact concerning the boundary regularity of v: if the boundary data v_0 is continuous on ∂D, the minimizers v will also be continuous up to ∂D. This fact is somewhat similar to the continuity up to the time $t = 0$ of the limit solutions of problem (P). For the convenience of the reader, we provide a proof of this result in Appendix A.2. On the other hand, note that the Lipschitz continuity of v on $\overline{D}$ does not necessarily follow from the Lipschitz continuity of v_0, not even for positive harmonic functions, which are special solution of (3.1).

Next, we note that the functional J may admit more than one minimizer with the same boundary values, since it is not convex. Nevertheless, the minimizers of J enjoy their own version of the strict comparison principle, similar to Lemma 2.3.

Lemma 3.1 (Strict comparison principle for minimizers of (AC)) *Let v_1 and v_2 be nonnegative minimizers of the functional* (AC) (*with respect to their own boundary values*) *which are continuous up to ∂D and such that $v_1 > v_2$ on $\partial D \cap \overline{\{v_2 > 0\}}$. Then $v_1 \geq v_2$ in D.*

Proof Let $V = \{x \in D : v_2 > v_1\}$. We claim that $v_1 = v_2 = 0$ on ∂V. To this end, pick an arbitrary $x_0 \in \partial V$ and consider the following two possibilities:

(1) $x_0 \in \partial D$. From the assumption on the boundary values of v_1 and v_2, we must have $v_2(x_0) = 0$. Moreover, since $0 \leq v_1 \leq v_2$ on $\overline{V}$, we must also have $v_1(x_0) = 0$.

(2) $x_0 \in D$. In this case the continuity implies that $v_1(x_0) = v_2(x_0) =: a$. Consider then the functions $\underline{v} = \min\{v_1, v_2\}$ and $\overline{v} = \max\{v_1, v_2\}$. Then $\underline{v}$ and $\overline{v}$ are also minimizers of J in D with the boundary values v_2 and v_1 on ∂D, respectively. This follows from the easily verifiable identity (known as the *lattice property*)

$$J(v_1) + J(v_2) = J(\underline{v}) + J(\overline{v}).$$

We now have $\underline{v} \leq \overline{v}$ in D and $\underline{v}(x_0) = \overline{v}(x_0) = a$. Thus, if $a > 0$ both $\underline{v}$ and $\overline{v}$ must be harmonic near x_0. Then from the strong comparison principle for harmonic functions we will have that $\underline{v} = \overline{v}$ in a neighborhood of x_0. This implies that $v_1 = v_2$ in the same neighborhood, which contradicts to the choice $x_0 \in \partial\{v_2 > v_1\}$. Hence $a = 0$.

Now, to finish the proof, we note that if $V \neq \emptyset$, the minimizing property of v_1 and v_2 will imply that both functions must vanish identically on V, contradicting the definition of the set V. □

The main result of this section is the construction of a minimizer of J in a cylindrical-shaped domain D with a high radius-to-width ratio so that its free boundary is catenoidal-like. Such a qualitative behavior is similar to that of minimal surfaces (soap films) attached to two parallel planar circular wires. If the wires are very far apart from each other, the minimal surface consists of two separate planes, while if they are close together, the resultant surface is a *connected catenoid*.

To proceed, let

$$Q_{a,b} := \{x = (x', x_n) : |x'| < a, |x_n| < b\} = B'_a \times (-b, b), \quad a, b > 0,$$

where B'_r denotes the ball of radius r in $\mathbb{R}^{n-1}$ centered at the origin. For a fixed small constant $\eta > 0$ (say $\eta = 1/10$) and a given large $a > 0$ let $D = D_a$ be a C^∞ domain such that

$$Q_{a-\eta,1} \cup Q_{a,1-\eta} \subset D_a \subset Q_{a,1}.$$

Proposition 3.2 (Catenoidal minimizer) *Let D_a be as above and v be a minimizer of the Alt-Caffarelli functional* (AC) *with the boundary values $g = g_\kappa \in C^\infty(\partial D_a)$ satisfying*

$$\begin{aligned} g &= 0 && \text{on } \partial D_a \cap \{|x'| \geq a-2\},\\ g &= 1-\kappa && \text{on } \partial D_a \cap \{|x'| \leq a-3\},\\ 0 \leq g &\leq 1-\kappa && \text{on } \partial D_a. \end{aligned}$$

Then there exist constants $\kappa_0 > 0$ and $1 < a_0 < \infty$, depending only on dimension n, such that if $0 < \kappa < \kappa_0$ and $a > a_0$, then

$$Q_{1,1} \subset \{v > 0\} \subset Q_{a-1,1}.$$

In particular, the free boundary

$$\Gamma := \partial\{v > 0\} \cap D_a$$

is contained in $\overline{Q_{a-1,1}} \setminus Q_{1,1}$, see Fig. 2.

Proof Consider the solution ψ of the Dirichlet problem

$$\Delta\psi = 0 \quad \text{in } B_1 \setminus \overline{B_{1/2}}, \tag{3.2}$$

$$\psi = 0 \quad \text{on } \partial B_1, \qquad \psi = c \quad \text{on } \partial B_{1/2}, \tag{3.3}$$

where the constant $c > 0$ is chosen so that

$$|\nabla\psi| = 1 \quad \text{on } \partial B_1. \tag{3.4}$$

More explicitly,

$$\psi(x) = \begin{cases} \frac{1}{n-2}\left(\frac{1}{|x|^{n-2}} - 1\right), & n > 2,\\ \log\frac{1}{|x|}, & n = 2. \end{cases}$$

Note that if we extend ψ by 0 outside B_1, it will become a minimizer of the Alt-Caffarelli functional (AC) in $B_\rho \setminus \overline{B_{1/2}}$ for any $\rho \geq 1$ (see [1], Sect. 2.6). Next, fix a large $\lambda > 0$ and consider the scaled function

$$\psi_\lambda(x) = \lambda\,\psi\left(\frac{x}{\lambda}\right).$$

Define also the translates

$$\psi_\lambda^\pm(x) = \psi_\lambda\left(x \mp \tilde\lambda e_n\right), \quad \tilde\lambda = \lambda + \sqrt{\kappa_0},$$

where $\kappa_0 > 0$ is a small constant to be specified later. We now claim that

$$u(x) \geq \psi_\lambda^\pm(x) \quad \text{in } D_a, \tag{3.5}$$

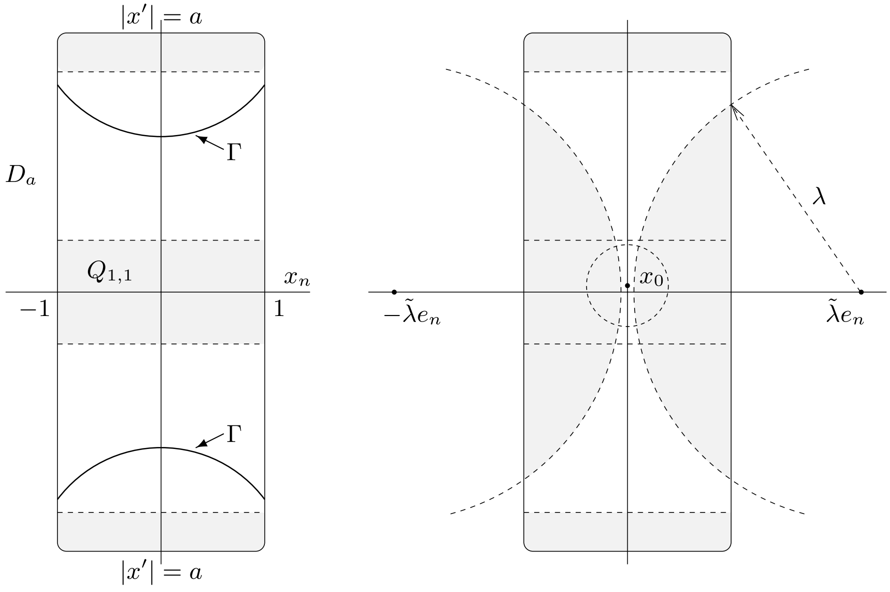


Fig. 2 Cross-section of the catenoidal minimizer

provided $0 < \kappa < \kappa_0$ and $a > a_0 = a_0(\lambda, \kappa_0)$. This will follow from the strict comparison principle (Lemma 3.1), once we verify that

$$\psi_\lambda^+ < 1 - \kappa = v \quad \text{on } \operatorname{supp} \psi_\lambda^+ \cap \partial D_a \tag{3.6}$$

and similarly for ψ_λ^-.

Indeed, we have $\operatorname{supp} \psi_\lambda^+ = \overline{B_\lambda(\tilde{\lambda} e_n)}$ and therefore

$$\operatorname{supp} \psi_\lambda^+ \cap \{x_n = 1\} \subset B'_{\sqrt{\lambda^2 - (\lambda-1)^2}} \times \{1\}.$$

Thus, if $a > 3 + \sqrt{\lambda^2 - (\lambda - 1)^2}$, we will have

$$\operatorname{supp} \psi_\lambda^+ \cap \partial D_a \subset B'_{a-3} \times \{1\}.$$

Suppose now $x \in \partial D_a \cap \{x_n = 1\}$. Then

$$\begin{aligned} \psi_\lambda^+(x) \le \psi_\lambda^+(e_n) &= \lambda \psi \left(\frac{1 - \lambda - \sqrt{\kappa_0}}{\lambda} e_n \right) \\ &\le \lambda \frac{1 - \sqrt{\kappa_0}}{\lambda} \max_{B_1 \setminus B_{1-1/\lambda}} |\nabla \psi| \le (1 - \sqrt{\kappa_0})(1 + \sqrt{\kappa_0}) = 1 - \kappa_0. \end{aligned}$$

In the last step we have used the fact that $|\nabla \psi| \le 1 + \sqrt{\kappa_0}$ in the annulus $B_1 \setminus B_{1-1/\lambda}$, provided λ is large. This proves (3.6) and consequently (3.5) for large a.

The latter inequality implies that $v > 0$ in $D_a \cap B_\lambda(\pm(\lambda + \sqrt{\kappa_0})e_n)$, see Fig. 2. This leads to the inclusion

$$\{v = 0\} \cap Q_{2,1} \subset Q_{2,\delta},$$

where $\delta > 0$ can be made as small as we like if λ is taken sufficiently large and κ_0 small. We next show that this implies that in fact

$$\{v = 0\} \cap Q_{1,1} = \emptyset.$$

Indeed, if there is an $x_0 \in \partial\{v > 0\} \cap Q_{1,1}$, then by the density property satisfied by the minimizers of Alt-Caffarelli functional (see [1], Lemma 3.1) one should have

$$0 < c \le \frac{|\{v = 0\} \cap B_{1/2}(x_0)|}{|B_{1/2}(x_0)|} \le 1 - c$$

for a dimensional constant c. However, the inequality from below will fail in our case if δ is small. Hence

$$v > 0 \quad \text{in } Q_{1,1}.$$

To complete the proof of the proposition, it remains to show that

$$v = 0 \quad \text{in } \{|x'| \ge a - 1\} \cap D_a.$$

This follows easily from the strict comparison in D_a with the family of functions

$$(x' \cdot e' - (a - 1))^-$$

for unit vectors $e' \in \mathbb{R}^{n-1}$, which are minimizers of the Alt-Caffarelli functional (AC) on any compact subset of $\mathbb{R}^n$ (e.g. see [1]) and are strictly larger than the boundary values g of v on ∂D_a. □

Assume now that the domain D_a is rotationally symmetric with respect to x_n-axis and so is the boundary data in Proposition 3.2, i.e. $g(x) = g(|x'|, x_n)$. Then the Alt-Caffarelli functional admits a minimizer with the same symmetry

$$v(x) = v(|x'|, x_n).$$

Indeed, the minimal (or maximal) minimizer necessarily has this property—by taking the inf and sup of all rotated versions of the minimizer. For such rotationally symmetric minimizers we can say a little more about the free boundary.

Proposition 3.3 *Let v be as in Proposition 3.2 with $a > a_0$ and $0 < \kappa < \kappa_0$. Under the additional rotational symmetry condition with respect to x_n-axis, as described above, the free boundary Γ is C^∞ (actually, real-analytic).*

Proof This is a simple corollary from the general free boundary regularity theory known for Alt-Caffarelli minimizers, as described at the beginning of this section.

Namely, it is know that the singular set $\Sigma = \Gamma \setminus \Gamma_{\text{red}}$ is empty if $n < 4$ and has Hausdorff dimension at most $n-4$ for $n \geq 4$. Now, if Σ is nonempty and $x_0 \in \Sigma$, x_0 cannot be on x_n-axis by Proposition 3.2, since $\Gamma \subset Q_{a-1,1} \setminus Q_{1,1}$. Therefore, from the rotational symmetry of Σ around the x_n-axis, Σ will contain an $(n-2)$-dimensional sphere and thereby will have Hausdorff dimension at least $n-2$, a contradiction. Thus, $\Sigma = \emptyset$, implying that Γ is real analytic. □

4 Construction of the Subsolution

In this section, we construct a subsolution of the free boundary problem (P) in the form of two disjoint shrinking circular solutions joined by a "catenoidal neck". This is carried out in several steps. First, the neck v is constructed by finding a stationary function which solves a free boundary problem similar to (P) but satisfies $\Delta v = c > 0$ in the positivity set of v. In principle, the desired v can be obtained by minimizing the modified Alt-Caffarelli functional

$$J_c(v) = \int_D \Big(|\nabla v|^2 + \chi_{\{v>0\}}(1+cv)\Big)\,dx.$$

However, instead of re-doing the existence and regularity properties of the minimizers as in [1], we find it more elementary to make use of the original functional (AC) and perform approximations from there. After that, a time dependent subsolution of (P) is obtained by forming a selfsimilar scaling of v which is then pasted together with two shrinking circular solutions.

Consider a minimizer v as in Proposition 3.2 for some $a > a_0$ and $0 < \kappa < \kappa_0$. Additionally, assume the rotational symmetry as in Proposition 3.3 and let

$$V := \{v > 0\}, \qquad \Gamma = \partial V \cap D.$$

Then we know that Γ is real analytic, $v \in C^\infty(\overline{V} \cap D)$, and

$$v = 0, \qquad |\nabla v| = 1 \quad \text{on } \Gamma,$$

where the latter condition is understood as the limit from inside V.

Let now V_0 be the connected component of V containing the cylinder $Q_{1,1}$. Even though $\partial V_0 \cap D$ is regular, it may have a complicated behavior near ∂D. To avoid any problems caused by this, we consider a C^∞ domain $\hat{V}_0$ such that

$$\{|x_n| < (1-\eta)^2\} \cap V_0 \subset (1-\eta)\hat{V}_0 \subset \{|x_n| < 1-\eta\} \cap V_0$$

for a small $\eta > 0$ to be specified later. The boundary of $\hat{V}_0$ is naturally subdivided into three parts:

$$S^- = \partial \hat{V}_0 \cap \{x_n \leq -(1-\eta)\}, \tag{4.1}$$

$$S = \partial \hat{V}_0 \cap \{|x_n| \leq 1-\eta\}, \tag{4.2}$$

$$S^+ = \partial \hat{V}_0 \cap \{x_n \geq 1-\eta\}. \tag{4.3}$$

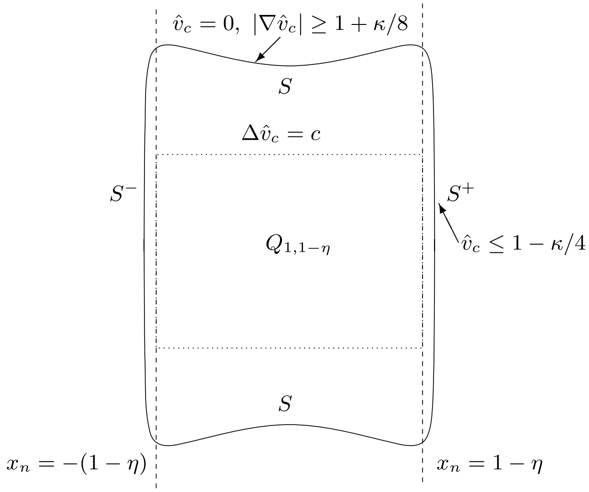


Fig. 3 The domain $\hat{V}_0$ and the function $\hat{v}_c$

Now let

$$\hat{v}_0(x) := (1+\kappa/4)\frac{v((1-\eta)x)}{1-\eta}, \quad x \in \hat{V}_0.$$

The function $\hat{v}_0$ and the domain $\hat{V}_0$ have the following properties:

$$Q_{1,1-\eta} \subset \hat{V}_0 \subset Q_{a,1}, \tag{4.4}$$

$$\hat{v}_0 > 0, \qquad \Delta \hat{v}_0 = 0 \quad \text{in } \hat{V}_0, \tag{4.5}$$

$$\hat{v}_0 = 0, \qquad |\nabla \hat{v}_0| = 1 + \kappa/4 \quad \text{on } S, \tag{4.6}$$

$$\hat{v}_0 \le 1 - \kappa/2 \quad \text{on } S^- \cup S^+, \tag{4.7}$$

if $\eta > 0$ is sufficiently small. These properties follow easily from the corresponding ones of v in Proposition 3.2.

Next, for small $c > 0$, let $\hat{v}_c$ be the solution of the Dirichlet problem

$$\Delta \hat{v}_c = c \quad \text{in } \hat{V}_0, \qquad \hat{v}_c = \hat{v}_0 \quad \text{on } \partial \hat{V}_0.$$

Since $\partial \hat{V}_0$ is regular, the solutions $\hat{v}_c$ satisfy uniform $C^{1,\alpha}$ estimates on $\overline{\hat{V}_0}$ and therefore,

$$\hat{v}_c \to \hat{v}_0, \quad \text{as } c \to 0+.$$

Hence, we can find a small $c > 0$ such that $\hat{v}_c$ will satisfy the following properties (see Fig. 3):

$$\Delta \hat{v}_c = c \quad \text{in } \hat{V}_0, \tag{4.8}$$

$$\hat{v}_c > 0 \quad \text{in } \{|x_n| \le 1-\eta\} \cap \hat{V}_0 \supset Q_{1,1-\eta}, \tag{4.9}$$

$$\hat{v}_c = 0, \qquad |\nabla \hat{v}_c| \ge 1 + \kappa/8 \quad \text{on } S, \tag{4.10}$$

$$\hat{v}_c \le 1 - \kappa/4 \quad \text{on } S^- \cup S^+. \tag{4.11}$$

Next, we use the above $\hat{v}_c$ to construct a time dependent subsolution of (P). To this end, take a small parameter $\alpha > 0$ and consider the function

$$\upsilon(x,t) := \sqrt{\alpha t}\, \hat{v}_c\left(\frac{x}{\sqrt{\alpha t}}\right),$$

which is defined in the space-time domain

$$\Upsilon := \left\{(x,t) : t > 0, \frac{x}{\sqrt{\alpha t}} \in \hat{V}_0\right\}.$$

We claim that if α is sufficiently small, then υ is a subcaloric function in Υ. Indeed, we have

$$\begin{aligned}\Delta\upsilon - \partial_t \upsilon &= \frac{\Delta\hat{v}_c(\frac{x}{\sqrt{\alpha t}})}{\sqrt{\alpha t}} - \frac{\alpha\, \hat{v}_c(\frac{x}{\sqrt{\alpha t}})}{2\sqrt{\alpha t}} + \frac{x\cdot\nabla\hat{v}_c(\frac{x}{\sqrt{\alpha t}})}{2t}\\ &= \frac{1}{\sqrt{\alpha t}}\left(\Delta\hat{v}_c(\xi) + \frac{\alpha}{2}(\xi\cdot\nabla\hat{v}_c(\xi) - \hat{v}_c(\xi))\right),\end{aligned}$$

where $\xi = \frac{x}{\sqrt{\alpha t}}$. Since $\Delta\hat{v}_c = c > 0$ and both $\hat{v}_c$ and $|\nabla\hat{v}_c|$ are bounded, taking α sufficiently small, we can guarantee that

$$\Delta\upsilon - \partial_t\upsilon > 0 \quad \text{in } \Upsilon. \tag{4.12}$$

Next, consider the t-slices of Υ:

$$\Upsilon(t) := \{x : (x,t) \in \Upsilon\} = \sqrt{\alpha t}\, \hat{V}_0.$$

Then the boundary $\partial\Upsilon(t)$ is naturally subdivided into three parts, corresponding to $S^{\pm}$ and S in (4.1)–(4.3):

$$\Sigma^{\pm}(t) := \sqrt{\alpha t}\, S^{\pm}, \qquad \Sigma(t) := \sqrt{\alpha t}\, S.$$

Now properties (4.9)–(4.11) of $\hat{v}_c$ translate into the following statements for υ:

$$\upsilon > 0 \quad \text{in } \Upsilon(t) \cap \left\{|x_n| < (1-\eta)\sqrt{\alpha t}\right\}, \tag{4.13}$$

$$\upsilon = 0, \qquad |\nabla\upsilon| \geq 1 + \kappa/8 \quad \text{on } \Sigma(t), \tag{4.14}$$

$$\upsilon \leq (1-\kappa/4)\sqrt{\alpha t} \quad \text{on } \Sigma^-(t) \cup \Sigma^+(t). \tag{4.15}$$

The function υ will serve as a "neck" joining two disjoint "shrinking circles" that we construct in the next step.

Let $\theta_0 \geq 0$ be a continuous, compactly supported, and radially symmetric function in $\mathbb{R}^n$, which satisfies

$$\{\theta_0 > 0\} = B_{\rho_0}, \qquad \theta_0 \in C^\infty(\overline{B_{\rho_0}}), \qquad |\nabla\theta_0| = 1 \quad \text{on } \partial B_{\rho_0} \tag{4.16}$$

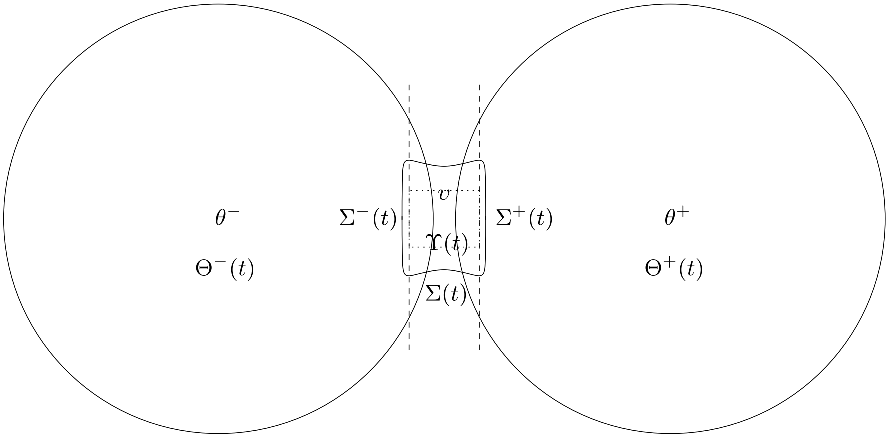


Fig. 4 Subsolution with a "connecting neck"

and let θ be the radially symmetric classical solution of the free boundary problem (P)

$$\begin{aligned} \Delta\theta - \partial_t\theta = 0 &\quad \text{in } \{\theta > 0\}, \\ |\nabla\theta| = 1 &\quad \text{on } \partial\{\theta > 0\}, \\ \theta(\cdot, 0) = \theta_0. & \end{aligned}$$

By [13], Theorem 3.1, the solution θ exists in some time interval $t \in [0, \tau)$, $\tau > 0$. We additionally assume that θ_0 is strictly concave in B_{ρ_0} which implies that $\partial_t\theta \le 0$ on $\mathbb{R}^n \times [0, \tau)$. In particular, the positivity sets

$$\Theta(t) := \{\theta(\cdot, t) > 0\} = B_{\rho(t)}$$

shrink as t increases, which means that the radius $\rho(t)$ is nonincreasing. On the other hand, for t before the extinction time, the speed of propagation of the free boundary, given by $\partial_t\theta/|\nabla\theta|$, is bounded and therefore we have

$$\rho_0 - C_0 t \le \rho(t) \le \rho_0, \quad 0 \le t \le \tau_0, \tag{4.17}$$

for some $0 \le C_0 < \infty$ and $\tau_0 > 0$.

Now, consider the translates of θ:

$$\theta^{\pm}(x, t) := \theta(x \mp \rho_0 e_n, t).$$

The positivity sets

$$\Theta^{\pm}(t) := \{\theta^{\pm}(\cdot, t) > 0\}$$

are two disjoint balls that touch at $t = 0$ and separate for $t > 0$.

The final step in the construction is given by the following proposition.

Proposition 4.1 (Subsolution) *Let*

$$\phi(x,t) := \begin{cases} \max\{\theta^-, \upsilon, \theta^+\}, & \text{in } \Upsilon, \\ \theta^-, & \text{in } \operatorname{supp}\theta^- \setminus \Upsilon, \\ \theta^+, & \text{in } \operatorname{supp}\theta^+ \setminus \Upsilon, \\ 0, & \text{elsewhere} \end{cases}$$

and denote (*see Fig.* 4)

$$\Phi := \{\phi > 0\}, \qquad \Phi(t) := \{\phi(\cdot,t) > 0\} = \Theta^-(t) \cup \Upsilon(t) \cup \Theta^+(t).$$

Then there exists $\tau_0 > 0$ *such that*

- ϕ *is continuous in* $\mathbb{R}^n \times (0,\tau_0)$;
- ϕ *satisfies*
$$\Delta\phi - \partial_t\phi \geq 0 \quad \text{in } \Phi \cap (\mathbb{R}^n \times (0,\tau_0))$$
in the sense of distributions; *and*
- ϕ *satisfies the free boundary condition* $|\nabla\phi| \geq 1$ *on* $\partial\Phi \cap (\mathbb{R}^n \times (0,\tau_0))$ *in the sense*
$$\liminf_{\substack{x\to x_0 \\ x\in\Phi(t)}} \frac{\phi(x,t)}{\operatorname{dist}(x,\partial\Phi(t))} \geq 1$$
for any $x_0 \in \partial\Phi(t)$, $0 < t < \tau_0$.

Proof The positivity sets $\Theta^\pm(t) = \{\theta^\pm(\cdot,t) > 0\}$ are the balls $B_{\rho(t)}(\pm\rho_0 e_n)$ and because of the finite speed of propagation (4.17), they will be separated by a distance of at most $2C_0 t$ for $0 < t < \tau_0$.

On the other hand, from the definition of the function υ, the set $\Upsilon(t)$ grows with a factor of $\sqrt{t}$. Namely, we have

$$\Upsilon(t) = \sqrt{\alpha t}\,\hat{V}_0.$$

Therefore, if we take $0 < t \leq \tau_0$ sufficiently small, we will have $\Sigma^\pm(t) \Subset \Theta^\pm(t)$, or more quantitatively,

$$\operatorname{dist}(\Sigma^\pm(t), \partial\Theta^\pm(t)) \geq (1-2\eta)\sqrt{\alpha t}.$$

Next, using the fact that $|\nabla\theta^\pm| = 1$ on the free boundaries $\partial\Theta^\pm(t)$, by C^2 continuity of $\theta^\pm$ there exists $\delta > 0$ such that

$$\theta^\pm(x,t) \geq (1-\kappa/16)\operatorname{dist}(x,\partial\Theta^\pm(t)),$$

whenever the latter distance is less than δ and $t < \tau_0$. Therefore, for small $t < \tau_0$ and $x \in \Sigma^\pm(t)$, we have

$$\theta^\pm(x,t) \geq (1-2\eta)(1-\kappa/16)\sqrt{\alpha t} \geq (1-\kappa/8)\sqrt{\alpha t} \quad \text{on } \Sigma^\pm(t),$$

provided one takes $\eta < \kappa/32$ in the construction of υ. In particular, comparing this inequality to (4.15), we obtain that

$$\theta^{\pm}(x,t) > \upsilon(x,t) \quad \text{on } \Sigma^{-}(t) \cup \Sigma^{+}(t).$$

Since $\upsilon = 0$ on the remaining part of $\partial\Upsilon(t)$, it holds that

$$\theta^{\pm} > \upsilon \quad \text{on } \partial\Upsilon(t) \cap \Theta^{\pm}(t) \tag{4.18}$$

for small $0 < t < \tau_0$.

It is now straightforward to show that the function ϕ defined in the statement of the proposition is continuous in $\{0 < t < t_0\}$ and subcaloric (i.e. a subsolution of the heat equation) in $\Phi \cap \{0 < t < t_0\}$.

We will concentrate on subcaloricity, the proof of continuity being analogous. Recall that by definition

$$\Phi(t) = \Theta^{-}(t) \cup \Upsilon(t) \cup \Theta^{+}(t).$$

Then for $x \in \Phi(t)$, we have the following three possibilities:

(1) $x \in \Upsilon(t)$. Since $\phi = \max\{\theta^{-}, \upsilon, \theta^{+}\}$ in Υ, ϕ is subcaloric in a neighborhood of (x,t), as the maximum of three subcaloric functions.
(2) $x \in \partial\Upsilon(t) \cap \Theta^{\pm}(t)$. In this case the inequality (4.18) shows that $\phi = \theta^{\pm}$ in a neighborhood of (x,t) and therefore ϕ is again (sub)caloric there.
(3) $x \in \Theta^{\pm}(t) \setminus \overline{\Upsilon(t)}$. Here again $\phi = \theta^{\pm}$ in a neighborhood of (x,t) and therefore is (sub)caloric.

The above argument thus establishes that ϕ is subcaloric in its positivity set Φ for $0 < t < \tau_0$.

Finally, we show that ϕ satisfies the condition $|\nabla\phi| \geq 1$ on $\partial\Phi$ in the sense

$$\liminf_{\substack{x \to x_0 \\ x \in \Phi(t)}} \frac{\phi(x,t)}{\operatorname{dist}(x, \partial\Phi(t))} \geq 1 \quad \text{for any } x_0 \in \partial\Phi(t),\ 0 \leq t < \tau_0.$$

This follows from the fact that

$$\partial\Phi(t) \subset \partial\Theta^{-}(t) \cup \Sigma(t) \cup \partial\Theta^{+}(t)$$

and from the inequalities (which are satisfied by our construction)

$$\liminf_{\substack{x \to x_0 \\ x \in \Theta^{\pm}(t)}} \frac{\theta^{\pm}(x,t)}{\operatorname{dist}(x, \partial\Theta^{\pm}(t))} \geq 1, \quad \text{for } x_0 \in \partial\Theta^{\pm}(t)$$

and

$$\liminf_{\substack{x \to x_0 \\ x \in \Upsilon(t)}} \frac{\upsilon(x,t)}{\operatorname{dist}(x, \partial\Upsilon(t))} \geq 1, \quad \text{for } x_0 \in \Sigma(t),$$

since $\phi \geq \theta^{\pm}$ in $\Theta^{\pm}$ and $\phi \geq \upsilon$ in Υ. □

5 Proof of the Main Theorem

In this section we give the proof of Theorem 1.2.

To proceed, we fix u_0 as in Example 1.1 and let r^* be as in (1.1). Also, throughout this section, we use the following notations for the spatial translates

$$f^{\pm}(x) := f(x \mp r^* e_n), \qquad g^{\pm}(x,t) := g(x \mp r^* e_n, t).$$

Theorem 1.2 is a direct consequence of the next two propositions.

Proposition 5.1 (Minimal solution) *Let the initial data w_0 be as in* (1.2). *Then the minimal limit solution of problem* (P) *with the initial data w_0 is given by the formula*

$$\underline{w} = u^- + u^+,$$

where u is the unique limit solution with the initial data u_0.

Proof Let $w_* := u^- + u^+$. We want to show that $w_* = \underline{w}$.

First, note that w_* is a supersolution of problem (P) in the sense of Lemma 2.2 and therefore we can use it as the function u_1 in the strict comparison principle (Lemma 2.3), see Remark 2.4. Let now $\underline{u}_{0,j} \geq 0$ be such that

$$\underline{u}_{0,j} < u_0 \quad \text{on } \operatorname{supp} \underline{u}_{0,j}$$

and let $\underline{u}_j$ be a limit solution of problem with the initial data $\underline{u}_{0,j}$. Let also

$$\underline{w}_{0,j} := \underline{u}_{0,j}^- + \underline{u}_{0,j}^+$$

and $\underline{w}_j$ be the corresponding limit solution.

Applying Lemma 2.3 to the pair w_* and $\underline{w}_j$, we obtain that

$$w_* \geq \underline{w}_j \quad \text{in } \mathbb{R}^n \times [0,\infty).$$

Now, letting $\underline{u}_{0,j} \nearrow u_0$ and making use of the second conclusion of Theorem 2.1, we have $w_j \nearrow \underline{w}$ and hence $w_* \geq \underline{w}$.

On the other hand, applying again Lemma 2.3 to the pairs $\underline{w}$ and $\underline{u}_j^{\pm}$, we obtain that

$$\underline{w} \geq \max\{\underline{u}_j^-, \underline{u}_j^+\} = \underline{u}_j^- + \underline{u}_j^+ \quad \text{in } \mathbb{R}^n \times [0,\infty).$$

Since u is the unique classical and limit solution with the initial data u_0, it holds that

$$\underline{u}_j^{\pm} \to \underline{u}^{\pm} = u^{\pm},$$

leading to

$$\underline{w} \geq u^- + u^+ = w_* \quad \text{in } \mathbb{R}^n \times [0,\infty).$$

This completes the proof of the proposition. □

To state our next proposition, consider the time t^* of the maximal expansion of the solution u as in (1.1). We now specify the function θ_0 in the construction of the subsolution in Proposition 4.1. Take $\rho_0 = r^*$ and choose θ_0 to be a strictly concave smooth function in B_{ρ_0} satisfying (4.16) and such that

$$\theta_0(x) \le u(x, t^*) \quad \text{in } B_{\rho_0}. \tag{5.1}$$

This is possible since u is C^2 up to the free boundary.

Proposition 5.2 (Maximal solution) *Let w_0 be as in (1.2) and θ_0 as above. Then the maximal limit solution of problem (P) with the initial data w_0 satisfies*

$$\overline{w}(x, t^* + t) \ge \phi(x, t),$$

for $x \in \mathbb{R}^n$ and small $0 \le t < \tau_0$, where ϕ is the function constructed in Proposition 4.1.

Proof Let radially symmetric $\overline{u}_{0,j}$ with the same properties as u_0 be such that

$$\overline{u}_{0,j} > u_0 \quad \text{on } \operatorname{supp} u_0$$

and let $\overline{u}_j$ be the corresponding solution of problem (P). From the strict comparison principle stated in Lemma 2.3, we have that $\overline{u}_j \ge u$ in $\mathbb{R}^n \times [0, \infty)$. In particular,

$$\{u(\cdot, t) > 0\} \subset \{\overline{u}_j(\cdot, t) > 0\}.$$

Moreover, we claim that in fact

$$\{u(\cdot, t) > 0\} \Subset \{\overline{u}_j(\cdot, t) > 0\}, \tag{5.2}$$

as long as the former set is nonempty, or, equivalently that the free boundaries $\partial\{\overline{u}_j(\cdot, t) > 0\}$ and $\partial\{u(\cdot, t) > 0\}$ never touch. This is a very well known property of classical solutions. Indeed, assume the contrary. Since the compact inclusion is true for $t = 0$, let t_0 be the first positive time when the two free boundaries touch. Then we obtain a contradiction with the Hopf boundary principle, since we will have that the normal derivatives $\partial_\nu u = \partial_\nu \overline{u}_j = 1$ at the touching points.

Now, let $\overline{w}_j$ be a limit solution with the initial data

$$\overline{w}_{0,j}(x) := \overline{u}_{0,j}^- + \overline{u}_{0,j}^+.$$

Since $\overline{w}_{0,j} \ge \overline{u}_{0,j}^\pm$, we obtain that

$$\overline{w}_j \ge \max\{\overline{u}_j^-, \overline{u}_j^+\},$$

since $\overline{u}_j$ is the only and therefore minimal limit solution with the initial data $\overline{u}_{0,j}$. In particular, on $\overline{B_{\rho_0}(-\rho_0 e_n)} \cup \overline{B_{\rho_0}(\rho_0 e_n)}$,

$$\overline{w}_j(x,t^*) > \max\{u^-(x,t^*), u^+(x,t^*)\}$$
$$= u^-(x,t^*) + u^+(x,t^*) \geq \theta_0^-(x) + \theta_0^+(x).$$

Hence we can find a very small parameter $\tau > 0$ (that we will later take to 0) such that

$$\overline{w}_j(x,t^*) > (1+\tau)\phi(x,\tau) \quad \text{on } \overline{\Phi(\tau)}, \tag{5.3}$$

where ϕ and Φ are as constructed in Proposition 4.1.

We now claim that if τ_0 is as in the same proposition, then for $0 \leq t < \tau_0 - \tau$

$$\overline{w}_j(x,t^*+t) > (1+\tau)\phi(x,\tau+t) \quad \text{on } \overline{\Phi(\tau+t)}. \tag{5.4}$$

Assume the contrary. Since (5.4) is satisfied for $t = 0$, let t_0 be the first positive time when (5.4) is violated. For this t_0 we still have

$$\overline{w}_j(x,t^*+t_0) \geq (1+\tau)\phi(x,\tau+t_0) \quad \text{on } \overline{\Phi(\tau+t_0)} \tag{5.5}$$

with the equality attained at least at one point $x_0 \in \overline{\Phi(\tau+t_0)}$:

$$\overline{w}_j(x_0,t^*+t_0) = (1+\tau)\phi(x,\tau+t_0).$$

Consider then two cases.

(1) $x_0 \in \Phi(\tau+t_0)$. In this case we obtain a simple contradiction with the parabolic interior maximum principle, since ϕ is subcaloric and $\overline{w}_j$ caloric in their respective positivity sets.

(2) $x_0 \in \partial\Phi(\tau+t_0)$. In this case we also have $x_0 \in \partial\{\overline{w}_j(\cdot,t^*+t_0) > 0\}$. Further note that the domain $\Phi(t)$ satisfies the interior ball condition for any $t > 0$ as the union of smooth domains $\Theta^{\pm}(t)$ and $\Upsilon(t)$. So let ν be the interior normal at x_0 to a ball contained in $\Phi(\tau+t_0)$ and touching $\partial\Phi(\tau+t_0)$ at x_0. Consider then the points $x(s) = x_0 + s\nu$ for small $s > 0$. It is easy to see that

$$\operatorname{dist}(x(s), \partial\Phi(\tau+t_0)) = \operatorname{dist}(x(s), \partial\{\overline{w}_j(\cdot,t^*+t_0) > 0\}) = s.$$

Thus, on one hand we have by Lemma 2.2 that

$$\limsup_{s\to 0+} \frac{\overline{w}_j(x(s),t^*+t_0)}{s} \leq 1,$$

while on the other, by (5.5) and Proposition 4.1, we have

$$\liminf_{s\to 0+} \frac{\overline{w}_j(x(s),t^*+t_0)}{s} \geq (1+\tau)\liminf_{s\to 0+} \frac{\phi(x(s),\tau+t_0)}{s} \geq 1+\tau,$$

which is clearly a contradiction. Hence (5.4) follows.

To finish the proof, we first let $\tau \to 0+$ and then $\overline{u}_{0,j} \searrow u_0$ in L^∞-norm to obtain

$$\overline{w}(x,t^*+t) \geq \phi(x,t) \quad \text{on } \overline{\Phi(t)},\ t \in [0,\tau_0],$$

where $\tau_0 > 0$ is as in Proposition 4.1. (Note that we have again invoked the second part of Theorem 2.1.) The proof is thus complete. □

Proof of Theorem 1.2 The proof follows immediately from Propositions 5.1 and 5.2. □

6 Further Questions and Connection with Motion by Mean Curvature

This section describes some further directions which are currently under investigation.

It appears that the flame propagation problem is very much related to the widely studied *motion by mean curvature* (MMC)—the geometric motion of a hypersurface with the normal velocity at a point equaling its mean curvature. The linearized version of MMC is given by the linear heat equation which is exactly the equation solved by u on $\partial\{u > 0\}$. This partly explains why the two problems are invariant under the same scaling $(x,t) \mapsto (x/a, t/a^2)$. In addition, both problems exhibit solutions of the selfsimilar form

$$u(x,t) = \sqrt{T-t}\, U\left(\frac{x}{\sqrt{T-t}}\right),$$

where $T > 0$ corresponds to some singular or blowup time. Furthermore, the MMC of a 2-dimensional hypersurface in $\mathbb{R}^3$, which is reflection symmetric with respect to the x_1x_2-plane, can also be viewed as a flame propagation problem on the x_1x_2-plane with "infinite" normal gradient condition. Thus, many qualitative and quantitative results between the flame propagation and MMC can be anticipated. Here we mention two such results. Rigorous justification in more general situation is work in progress.

6.1 Shrinking Torus and Radially Symmetric Annulus Heat Distribution

The work [17] describes in detail the MMC of the torus initial data:

$$\mathcal{T}_{(R,r)}(0) : \left(\sqrt{x_1^2 + x_2^2} - R\right)^2 + x_3^2 = r^2, \quad 0 < r < R.$$

It is shown that there is an $r_* > 0$ with the following property. For $r_* < r < R$, before the whole surface extincts, the inner radial portion of the torus comes together at the origin and then opens up vertically—this is called *focusing*. For $0 < r < r_*$, the surface evolves in such a way that the curvature blows up in finite time along a circle on the x_1x_2-plane and then the whole circle disappears. At $r = r_*$, the focusing and the extinction times *coincide*. In all cases, there is a *unique* solution up to the time the surface disappears.

An analogous situation for the flame propagation is also studied in [13]. The initial data is taken to be radially symmetric and supported on an annulus:

$$u_0(x_1, x_2) = u_0\left(\sqrt{x_1^2 + x_2^2}\right) \quad \text{for } 0 < a_0 < \sqrt{x_1^2 + x_2^2} < b_0.$$

Then $u(\cdot, t)$ is supported on a time dependent annulus: $a(t) \le \sqrt{x_1^2 + x_2^2} \le b(t)$. It is shown that there is an $a_* > 0$ such that for appropriate initial data, there is a time $T_* = T_*(a_0, u_0) > 0$ such that (i) if $a_* < a_0$, then $0 = a(T_*) < b(T_*)$; (ii) if $0 < a_0 < a_*$, then $0 < a(T_*) = b(T_*)$ and the solution vanishes at T_*; (iii) if $a_0 = a_*$, then $0 = a(T_*) = b(T_*)$ (see [13], p. 595). The three cases are called *focusing*, *extinction on a circle*, and *extinction on annulus*, respectively. They are very similar to the situation for the MMC of a torus.

6.2 Radially Symmetric Surfaces and Heat Humps with Connecting Neck

The work [2] investigates the MMC of radially symmetric surfaces, in particular, the formation of neck pinching type singularities and the number of times such singularity can occur. An interesting example is provided by the following initial data ([2], p. 354):

$$\Gamma_\lambda(0) : x_1^2 + x_3^2 = (1 - x_2^2)(1 - \lambda + \lambda x_2^2)^2, \quad 0 \le \lambda \le 1.$$

Note that for λ close to 1, the initial surface has a neck at the origin while for λ close to 0, the surface is convex. The above work proves the existence of a $\lambda_* \in (0, 1)$ such that the neck structure of $\Gamma_{\lambda_*}(0)$ persists during the whole evolution up to the extinction time T of $\Gamma_{\lambda_*}(t)$. In addition, the blow-up rate is *faster* than $(T - t)^{-\frac{1}{2}}$. (The corresponding solution is called the Hamilton's incredibly shrinking dumbbell.)

The above example is in fact analogous to the situation investigated in this paper. Instead of the two circular heat humps touching initially, consider two disjoint humps connected by a narrow channel of neck. It is conceivable that if the neck is too narrow, it will pinch off and the two humps then becomes disjoint. (See Appendix A.3 for an actual example of this scenario.) On the other hand, if the neck is too thick, the overall support might eventually become convex and remains so up to the vanishing time. Analogous to the MMC case, at some critical thickness of the neck, the support of the heat distribution might stay non-convex up to the vanishing time. This situation is also anticipated in [19].

It would be interesting to perform rigorous analysis for the above mentioned phenomena, in particular, the characterization of blow-up rate(s), the number of singularities, their locations and stability properties. Furthermore, from a practical point of view (in terms of physical modeling and numerical simulation), it is important to understand if there is any *selection principle* of the non-unique solutions. See [11, 18] for results along these lines.

Appendix

A.1 Up to $t = 0$ Continuity of Limit Solution of (P)

Proposition A.1 *Let u be a limit solution of problem (P) with a nonnegative initial data $u_0 \in C_0(\mathbb{R}^n)$, as defined in the beginning of Sect. 2. Then u is continuous up to $t = 0$ and $u(\cdot, 0) = u_0$.*

Proof First, note that u is a subsolution of the heat equation in $\mathbb{R}^n \times (0, \infty)$ as the limit of subsolutions u^ε. Therefore, we immediately have the inequality from above

$$\limsup_{(x,t)\to(x_0,0)} u(x,t) \le u_0(x_0)$$

for any $x_0 \in \mathbb{R}^n$. Hence, if $u_0(x_0) = 0$, then the continuity at x_0 follows.

Now, if $u_0(x_0) > 0$, we claim that $u > 0$ in a cylinder $B_\delta(x_0) \times (0, \delta)$ for a small $\delta > 0$. This follows from the fact that one can place a translate of a small Caffarelli-Vazquez selfsimilar solution $U(x,t)$, see Sect. 1 in [9], under u_0 with $\{U(\cdot, 0) > 0\} \ni x_0$. Since every limit solution satisfies a comparison principle with the selfsimilar solution, see Corollary 1.4 in [9], we obtain that u must stay above that solution and therefore $u > 0$ in $B_\delta(x_0) \times (0, \delta)$ for a small $\delta > 0$. In fact, using the above argument, we obtain also that the approximating functions u^ε will stay above the approximating functions U^ε of U and therefore we will also have that $u^\varepsilon > 0$ in $B_\delta(x_0) \times (0, \delta)$. But this means that u^ε satisfies the heat equation in $B_\delta(x_0) \times (0, \delta)$ with initial data $u^\varepsilon(\cdot, 0) = u_0^\varepsilon$. Passing to the limit $\varepsilon = \varepsilon_j \to 0+$, we obtain that u solves the heat equation in $B_\delta(x_0) \times (0, \delta)$ with initial data $u(\cdot, 0) = u_0$ and is therefore continuous at $(x_0, 0)$. □

A.2 Up to Boundary Continuity of Alt-Caffarelli Minimizers

Proposition A.2 *Let $v \ge 0$ be a minimizer of the Alt-Caffarelli functional in a smooth domain D with the boundary data v_0, as described in the beginning of Sect. 3. If v_0 is continuous on ∂D then v is continuous up to ∂D and $v|_{\partial D} = v_0$.*

Proof The proof is quite similar to that of Proposition A.1.

First, note that since v is subharmonic in D, we readily have

$$\limsup_{\substack{x\to x_0\\ x\in D}} v(x) \le v(x_0)$$

for $x_0 \in \partial D$. Hence, if x_0 is such $v_0(x_0) = 0$, then the continuity at that point follows.

Now suppose $v_0(x_0) > 0$. Let ψ be as in (3.2)–(3.4). Note that this ψ is the unique minimizer of Alt-Caffarelli in the annulus $R_1 = B_1 \setminus \overline{B_{1/2}}$ with boundary data (3.3). Now, let $\delta > 0$ be so small that there is a ball of radius $\delta/2$ touching ∂D at x_0 from *outside* D. Let ξ_0 be the center of this ball. Then consider the rescaled translate

$$\psi_{\delta,\xi_0}(x) := \delta\, \psi\left(\frac{x - \xi_0}{\delta}\right),$$

which will be the unique minimizer in the annulus $R_\delta(\xi_0) = B_\delta(\xi_0) \setminus \overline{B_{\delta/2}(\xi_0)}$. Assume additionally that

$$v_0 \ge 2c\,\delta \quad \text{on } B_\delta(\xi_0) \cap \partial D,$$

where c is as in (3.3). Such $\delta > 0$ exists, since $v_0(x_0) > 0$. Now we claim that

$$v \ge \psi_{\delta,\xi_0} \quad \text{in } \Omega_\delta(\xi_0) := B_\delta(\xi_0) \cap D.$$

Assume the contrary and consider the functions

$$\underline{\psi} := \min\{v, \psi_{\delta,\xi_0}\}, \quad \overline{v} := \max\{v, \psi_{\delta,\xi_0}\} \quad \text{in } \Omega_\delta(\xi_0).$$

It is easy to see that $\underline{\psi}$ and $\overline{v}$ have the same traces on $\partial\Omega_\delta(\xi_0)$ as ψ_{δ,ξ_0} and v, respectively. But then, from the minimizing property of both ψ_{δ,ξ_0} and v, we must have

$$J(\psi_{\delta,\xi_0}) \le J(\underline{\psi}),$$
$$J(v) \le J(\overline{v}),$$

where the functional J above is taken over the set $\Omega_\delta(\xi_0)$.

On the other hand, we have the lattice property

$$J(\psi_{\delta,\xi_0}) + J(v) = J(\underline{\psi}) + J(\overline{v}),$$

which implies that both $\underline{\psi}$ and $\overline{v}$ are be minimizers of J in $\Omega_\delta(\xi_0)$. Now, note that ψ_{δ,ξ_0} is the unique minimizer of J in the annulus $R_\delta(\xi_0)$ with its own boundary values. Hence it is also the unique minimizer of J in $\Omega_\delta(\xi_0)$. This implies that actually $\underline{\psi} = \psi_{\delta,\xi_0}$ which is equivalent to the inequality $v \ge \psi_{\delta,\xi_0}$ in $\Omega_\delta(\xi_0)$. In particular, we obtain that v is positive and harmonic in $\Omega_\delta(\xi_0)$. Hence, v is continuous at x_0. □

A.3 A Pinching Example

This section constructs a simple example of a (super)solution of problem (P) which pinches, i.e. the positivity set of the function becomes *disconnected*. The idea comes from the pinching example of radially symmetric cylindrical solutions for motion by mean curvature.

Define

$$u(x_1, x_2, t) = \left[\frac{h^2(x_1,t) - x_2^2}{h(x_1,t)}\right]^+ = \left[h(x_1,t) - \frac{x_2^2}{h(x_1,t)}\right]^+$$

where

$$h(x_1,t) = \min\{g(x_1), 2\} - at \quad \text{with } g(x_1) = \sqrt{1 + (bx_1)^2},$$

and a, b are positive constants (to be determined). The function is constructed so that

$$\operatorname{supp} u(\cdot,t) = \{|x_2| \le h(x_1,t)\}, \qquad |\partial_{x_2} u| = 2,$$

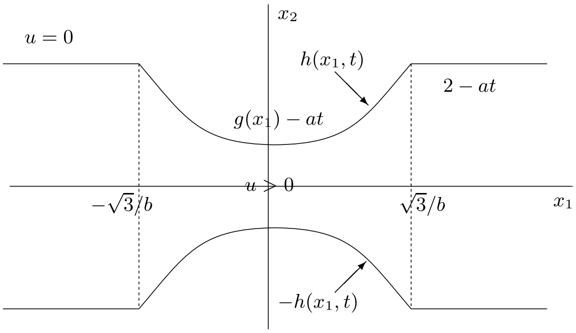
see Fig. 5 for visualization. Note that the pinching occurs at $t_1 = 1/a$ and the whole function vanishes identically at $t_2 = 2/a$.

The function u defined above satisfies the following properties:

(1) Boundedness of $|\nabla u|$ on $\partial\{u > 0\}$. Note that on $\partial\{u > 0\}$,

$$|\nabla u| = 2\sqrt{1 + (h')^2}$$

Fig. 5 A pinching example



for $|bx_1| \le \sqrt{3}$ and $x_2 = \pm h(x_1, t)$ (here $(\cdot)' = \partial_{x_1}$) and

$$|\nabla u| = 2$$

for $|bx_1| \ge \sqrt{3}$ and $x_2 = \pm(2 - at)$. Hence $|\nabla u|$ is uniformly bounded from above and below.

(2) Negativity of $\Delta u - \partial_t u$ in $\{u > 0\}$. For $|bx_1| \le \sqrt{3}$ and $|x_2| \le h(x_1, t)$, observe:

$$\partial_t u = \partial_t h + \frac{\partial_t h}{h^2} x_2^2 = -a\left(1 + \frac{x_2^2}{h^2}\right) \ge -2a$$

and

$$\Delta u = \partial_{x_1 x_1} u + \partial_{x_2 x_2} u = \left(h' + \frac{h'}{h^2} x_2^2\right)' - \frac{2}{h}$$
$$= h''\left(1 + \frac{x_2^2}{h^2}\right) - \frac{2}{h} - 2\frac{h'^2}{h}\frac{x_2^2}{h^2} \le 2h'' - \frac{2}{h}.$$

Hence $\Delta u - \partial_t u \le 2a + 2h'' - 2/h$ so that we just need to check that $a \le -h'' + 1/h$, i.e.

$$a \le -\frac{b^2}{(1 + (bx_1)^2)^{3/2}} + \frac{1}{\sqrt{1 + (bx_1)^2} - at}$$

which is true provided $a + b^2 \le 1/2$.

For $|bx_1| \ge \sqrt{3}$ and $|x_2| \le (2 - at)$, we have:

$$\Delta u - \partial_t u = -\frac{2}{2 - at} + a\left(1 + \frac{x_2^2}{(2 - at)^2}\right) \le 2a - 1$$

which is nonpositive provided $0 \le a \le 1/2$.

Hence, replacing u by its small and positive multiple—this is to ensure that $|\nabla u| \le 1$ for *all* $x \in \partial\{u > 0\}$—we obtain a supersolution of the flame propagation problem (P), which pinches in finite time.

Furthermore, from this function u one can construct a family of pinching supersolutions u^ε of (P_ε) by taking compositions with one-dimensional profiles. Namely, let ϕ be a positive increasing function on $\mathbb{R}$, which solves $\phi'' = \beta(\phi)$ and satisfies $\phi(s) \to 0$ as $s \to -\infty$ and $\phi'(s) = 1$ as $s \to +\infty$. We may normalize ϕ by asking $\phi(0) = 1$. Then the rescalings $\phi^\varepsilon(s) = \varepsilon\phi(s/\varepsilon)$ satisfy $(\phi^\varepsilon)''(s) = \beta_\varepsilon(\phi^\varepsilon(s))$ and moreover $\phi^\varepsilon(s) \to s^+$ uniformly on $\mathbb{R}$ as $\varepsilon \to 0+$. Now, define $u^\varepsilon = \phi^\varepsilon(u)$. It is easy to verify that u^ε are supersolution of (P_ε):

$$\Delta\phi^\varepsilon(u) - \partial_t\phi^\varepsilon(u) = (\phi^\varepsilon)'(u)(\Delta u - \partial_t u) + (\phi^\varepsilon)''(u)|\nabla u|^2$$
$$\leq \beta_\varepsilon(\phi^\varepsilon(u))|\nabla u|^2 \leq \beta_\varepsilon(\phi^\varepsilon(u)),$$

where in the last step we have used that in fact $|\nabla u| \leq 1$ everywhere. Finally, note that by our construction the unburnt zones $\{u^\varepsilon > \varepsilon\}$ coincide with $\{u > 0\}$ and therefore pinch in finite time, independent of ε.

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