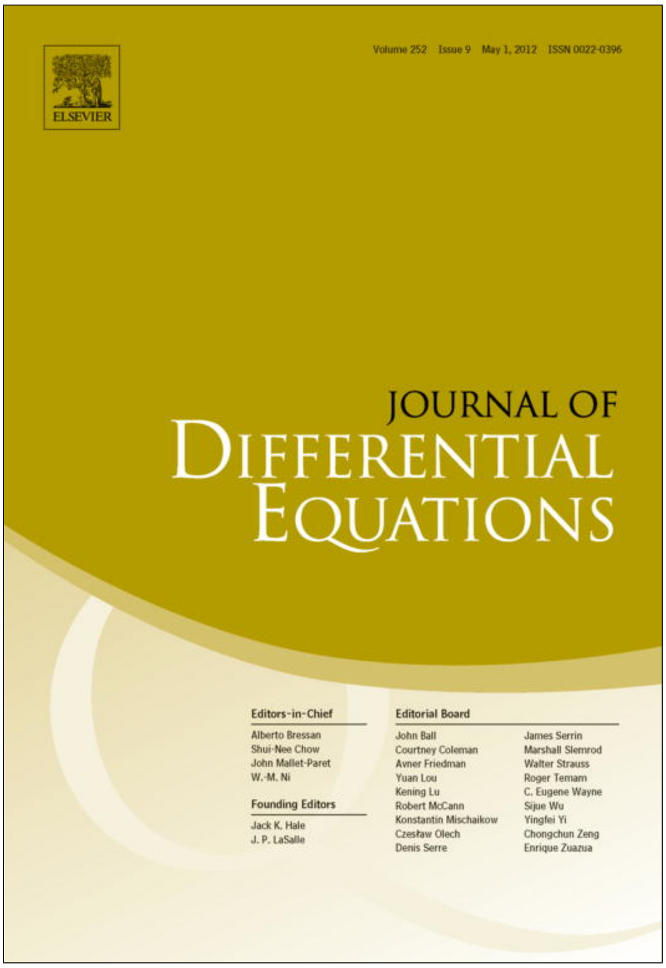
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On the parabolic Stefan problem for Ostwald ripening with kinetic undercooling and inhomogeneous driving force

D.C. Antonopoulou [a,b], G.D. Karali [a,b,*], N.K. Yip [c]

[a] Department of Applied Mathematics, University of Crete, GR-714 09 Heraklion, Crete, Greece
[b] Institute of Applied and Computational Mathematics, IACM, FORTH, Greece
[c] Department of Mathematics, Purdue University, West Lafayette, IN 47907-2067 USA

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ABSTRACT

Ostwald ripening is the coarsening phenomenon caused by the diffusion and solidification process which occurs in the last stage of a first-order phase transformation. The force that drives the system towards equilibrium is the gradient of the chemical potential that, according to the Gibbs–Thomson condition, on the interface, is proportional to its mean curvature. A quantitative description of Ostwald ripening has been developed by the Lifschitz–Slyozov–Wagner (LSW) theory. We extend the work of Niethammer (2000) [15] which deals with kinetic undercooling in the quasi-static case to the parabolic setting with temporally inhomogeneous driving forces on the solid–liquid interfaces. By means of *a priori* estimates, local and global existence results for the parabolic Stefan problem, we derive a first order approximation for the dynamical equations for the heat distribution and particle radii and then prove the convergence to a limiting description using a mean-field equation.

1. Introduction

1.1. The physical model

Ostwald ripening or coarsening [16] is a diffusion and solidification process occurring in the last stage of a first-order phase transformation. Usually, any first-order phase transformation process results in a two phase mixture with a dispersed (solid) second phase in a background (liquid) phase [17,18]. Initially the average size of the dispersed particles is very small. Hence, the interfacial energy

* Corresponding author at: Department of Applied Mathematics, University of Crete, GR-714 09 Heraklion, Crete, Greece.
E-mail address: gkarali@tem.uoc.gr (G.D. Karali).

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of the system is very large and the mixture is thus not in thermodynamical equilibrium. The force that drives the system towards equilibrium is the gradient of the chemical potential. According to the Gibbs–Thomson condition, on the interface between the two phases, the value of this driving force is proportional to the mean curvature of the interface. As a result, matter diffuses from regions of high curvature to regions of low curvature. This leads to the growth of large particles at the expense of small ones which eventually shrink to vanish. The outcome of this process, known as the Ostwald ripening, is the increase of the average particle size and the reduction of their number so that the mixture becomes coarser over time. A quantitative description of this process was first developed by Lifschitz and Slyozov [12] and independently by Wagner [19] under the assumption that the relative volume fraction of the dispersed phase is very small. The idea of the LSW theory is to make use of the growth velocity of an isolated particle. The interaction between the particles is captured through the average value of the background temperature field. This approach is thus called the *mean field approximation*.

More specifically, the LSW theory produces an equation for $n = n(R,t)$ the *number density* of the particles at time t as a function of radius R. This function is shown to satisfy the following equation:

$$\frac{\partial n(R,t)}{\partial t} + \frac{\partial}{\partial R}\big(V(R,t)n(R,t)\big) = 0, \tag{1}$$

where V is the *growth rate* of a particle of radius R:

$$V(R,t) = \frac{1}{R(t)}\left(\frac{1}{\overline{R}(t)} - \frac{1}{R(t)}\right), \tag{2}$$

and $\overline{R}(t)$ is the *average particle radius*:

$$\overline{R}(t) = \frac{\int R n(R,t)\,dR}{\int n(R,t)\,dR}. \tag{3}$$

Note that by definition, $n(R,t)\,dR$ gives the number of particles at time t with radius in the range $[R, R+dR]$. Hence $\int n(R,t)\,dR$ is the total number of particles present at time t. The system (1)–(3) is analyzed in [12,19]. It is argued that there exist infinitely many *self-similar solutions*, but only one is believed to describe the typical behavior of the system for large times. This is given by

$$n_s(R,t) \cong \frac{1}{t^{\frac{4}{3}}} G\left(\frac{R(t)}{\overline{R}(t)}\right) \quad \text{where } G(\cdot) \text{ is some scaling function.} \tag{4}$$

Based on this, the following temporal laws are derived for the average radius and the total number of particles:

$$\overline{R}(t) \cong \left(\overline{R}^3(0) + \frac{4}{9}t\right)^{\frac{1}{3}} \quad \text{and} \quad N(t) \cong \left(\overline{R}^3(0) + \frac{4}{9}t\right)^{-1}. \tag{5}$$

There have been many mathematical works concerning the above description. It is a nontrivial step to connect statements (1) and (5) rigorously to the underlying diffusion and solidification process. Note that the above is a *mean field description* – the velocity function V involves the *average of all the radii*. Hence the first mathematical task is to understand under what realistic assumption this mean field model is justified. It turns out that this is true only when the overall *capacity* of the solid particles *vanishes*. This is a much stronger condition than the requirement that the *volume fraction* of the solid phase vanishes. The necessity of this will become clear from the estimates we derived in the later sections.

Another important ingredient is the *boundary conditions* at the solid–liquid interphase. The most common ones are the *equilibrium Gibbs–Thomson condition* and the more general *kinetic undercooling*. The purpose of this work is to understand the effects coming from the presence of spatially inhomogeneous driving forces, in particular, *at the interfaces*. The ultimate goal is to incorporate realistic stochastic driving forces. Even though we restrict our attention to deterministic driving forces in this paper, we believe that our result can shed light on the plausible approaches and the desired estimates to handle the stochastic case.

We now describe in detail the mathematical formulation of the above solidification phenomena.

1.2. Mathematical formulation – free boundary value problem

In this section, we describe the mathematical set-up for the diffusion and solidification process. In the following, we consider the growth of the solid phase of a substance in an undercooled liquid phase of the same substance. Assuming isotropic growth, one possible model is the following Stefan problem for the temperature field θ and the solid–liquid-interface Γ [7,10]:

$$\begin{aligned} C\partial_t\theta &= K\Delta\theta \quad \text{in } \Omega_L,\\ HV &= -K\nabla\theta\cdot n \quad \text{on } \Gamma,\\ V &= -M\big(\theta_M\sigma k + H(\theta-\theta_M)\big) \quad \text{on } \Gamma, \end{aligned} \tag{6}$$

where the liquid and solid phases are denoted by Ω_L and $\Omega_S = \mathbb{R}^3\backslash\Omega_L$ respectively, while $\Gamma = \partial\Omega_S$ is the solid–liquid interface. Note that these sets are all time dependent. In the above, K is the thermal diffusivity, C is the heat capacity, θ_M is the melting temperature at a flat interface, H is the latent heat, σ is the surface tension, M is a mobility coefficient, k denotes the mean curvature of Γ (which is positive for a ball), n is the outward normal to the solid phase, and V is the normal velocity of the interface. The first interfacial condition on Γ, also known as the *Stefan condition*, ensures local conservation of heat. The second interfacial condition, known as the *kinetic undercooling*, couples the geometry of the interface with the evolution of the temperature in the liquid phase Ω_L. The curvature term forces the system to reduce the surface area of the interface Γ. But in the case of undercooled liquid, the second term gives a growing tendency for the solid phase. In other words, these two terms compete against each other. The following equilibrium condition

$$\theta_M\sigma k + H(\theta-\theta_M) = 0, \tag{7}$$

formally derived by setting $V = 0$ or $M = \infty$ is called the *Gibbs–Thomson law* on the interface. It predicts that the melting temperature is reduced for small particles. It is this effect which provides the barrier for nucleation of solid and thus allows for the existence of undercooled liquid phase. Since during Ostwald ripening, the interfacial velocities are relatively small, the Gibbs–Thomson condition is often used as an approximation of the general growth law. Nevertheless, even for small interfacial velocities, the kinetic term in the boundary condition has a strong regularizing effect on small particles.

System (6) is one type of *free boundary value problems*. There are many mathematical works that tackle these problems. See for example [11,3] for the existence of weak solution with the Gibbs–Thomson condition. A local existence result of classical solution with kinetic undercooling is given by [4]. The key feature of the problem currently undertaken is to describe the system under a *large* number of particles. This problem appears to be in the realm of homogenization procedure. However, standard techniques of homogenization such as asymptotic expansion, two-scale- or G-convergence do not suffice due to the highly nonlinear interaction between the heat distribution and the solid–liquid interface. The intricacy is already seen in the more simplified, stationary, elliptic problems in perforated domains. In this case, in order to derive the average equations that capture the behavior of the solutions in large spatial scales, it is found out that the *capacity of the holes* is a crucial quantity.

Most closely related is the work [5] that considers Dirichlet problems in domains with holes in a similar setting. It proves that if the capacity does not vanish, the type of the limit equation changes. In [6], the simpler Stefan problem with zero boundary condition for the heat distribution at the solid–liquid interface was studied in which the solid phase is not allowed to melt completely. This last mentioned work handles the case of finite capacity and hence it does not get a mean-field model in the limit.

The connection between (1)–(2) and (6) has been studied in [14] and [15]. The author is able to rigorously justify the mean field description under the vanishing capacity assumption. The former work considers the Gibbs–Thomson condition in both the *quasi-static* ($K\triangle\theta = 0$) and *parabolic* ($C\partial_t\theta = K\triangle\theta$) case. The latter work considers the kinetic undercooling condition in the quasi-static case. In both works, the vanishing capacity plays a crucial role.

A comment about the geometric set-up in the above two works. They both consider an isotropic approximation in which the solid particles are disjoint spherical balls which are stationary in space, i.e. the center of the particles do not move during the evolution. The works [1,2,8] remove this restrictive hypothesis by obtaining precise expressions for the equations of the centers and also radii by taking into account the geometry of the solid particles. However, the overall mean field description remains unchanged.

1.3. Motivation for the current work

The motivations of the current work are two folds. First we want to extend the work of [15] to the parabolic setting. The cited work deals with kinetic undercooling in the quasi-static case. Even though the strategy of attack follows closely to [14,15], due to the combined presence of the parabolicity and the kinetic undercooling, some additional terms appear in the derivation of energy estimates and the construction of sub- and super-solutions. These terms require extra care in the analysis. Thus we feel that it is worthwhile to investigate more rigorously this case.

Second, we want to consider the effect of inhomogeneous driving forces both in the spatial and temporal setting. Ideally, we would like to incorporate stochastic perturbations. Possible modification of (6) is the following

$$\begin{aligned} C\partial_t\theta &= K\Delta\theta + \xi(x,t) \quad \text{in } \Omega_L, \\ HV &= -K\nabla\theta\cdot n \quad \text{on } \Gamma, \\ V &= -M\big(\theta_M\sigma k + H(\theta-\theta_M)\big) + \zeta(x,t) \quad \text{on } \Gamma, \end{aligned} \tag{8}$$

where ξ and ζ are stochastic driving forces. A choice often used is some *white noise* in time and/or space (even though this is far from clear from a modeling point of view). However, a general theory of stochastic perturbation in moving boundary value problems, in particular the incorporation of white noise into the free boundaries, is not currently available.

In order to understand the estimates involved, in the current paper, we restrict our attention to deterministic driving forces which perturb in time the dynamics of the solid–liquid interface Γ. Specifically, we set $\xi \equiv 0$ and ζ to be some time dependent function which can take on different values on separate parts of Γ. We believe the results obtained here can lead to useful understanding to the ultimate, more general stochastic case.

An outline of this paper and the underlying method is in place. As mentioned before, we follow the overall strategy of [14] and [15] fairly closely. The key technical step is the proof of the *regularity in time* of the particle radii *near their vanishing moments*. This is obtained through the construction of appropriate sub- and super-solutions by use of a maximum principle (Lemma 6.1). This is where our paper differs most from the cited works of Niethammer: we need to dynamically adjust the ansatz in the construction in a careful manner (see Section 6). In addition, due to the combined effects of parabolicity of the equation and the kinetic undercooling, additional terms involving the particle radius regularity already appear in the derivation of the global energy estimates (see Section 5). This is not the case in previous works.

The contents of this paper are as follows. Section 2 explains heuristically the origin of the mean-field model. Section 3 sets up the rescaling regime for the spatial domain and particle sizes. After this, the local in time existence of weak solution and global energy estimate are obtained in Sections 4 and 5. Section 6 provides the crucial construction of sub- and super-solutions for the heat distribution which are used to prove the $W^{1,p}$-regularity in time for the particle radii. This then leads to the global in time existence of a solution, even after the vanishing of some particles. The next two sections then provide accurate approximations for the heat distribution (Section 7) and the particle radius dynamics (Section 8). The final Section 9 proves the limiting mean field description.

2. Mean field approximation

To simplify the analysis, it is convenient to non-dimensionalize system (6). Let

$$y \to \frac{H}{\sigma} y, \qquad t \to \frac{\theta_M K H}{\sigma^2} t, \qquad v := \frac{\theta_M - \theta}{\theta_M}, \qquad \lambda := \frac{C\theta_M}{H}, \quad \text{and} \quad \beta := \frac{K}{MH\sigma}.$$

With the addition of some inhomogeneous driving force $g(t)$ acting on the interface Γ, system (6) can be written as

$$\begin{aligned} \lambda \partial_t v &= \Delta v \quad \text{in } \Omega_L, \\ V &= \nabla v \cdot n \quad \text{on } \Gamma, \\ v + g(t) &= k + \beta V \quad \text{on } \Gamma. \end{aligned} \tag{9}$$

We will construct an approximate solution by making use of the idea that in the vicinity of a particle the solution should look approximately like the one for a single particle. Hence, we first consider the single particle problem in which the particle is a ball B_R of radius R centered at the origin:

$$\begin{aligned} \lambda \partial_t v &= \Delta v \quad \text{in } \mathbb{R}^3 \backslash B_R, \\ \dot{R} &= \nabla v \cdot n \quad \text{on } \partial B_R, \\ \beta \dot{R} &= -\frac{1}{R} + v + g(t) \quad \text{on } \partial B_R, \\ \lim_{r\to\infty} v(r,t) &= v_\infty(t). \end{aligned} \tag{10}$$

Note that the *far-field value $v_\infty(t)$ is imposed as a boundary condition at infinity*.

In the elliptic (quasi-static) case $\lambda = 0$, the solution of problem (10) at any time $t > 0$ can be explicitly given by

$$v(r,t) = v_\infty(t) + \frac{R(t)(1 - R(t)v_\infty(t) - R(t)g(t))}{r(\beta + R(t))}, \tag{11}$$

and

$$\dot{R}(t) = -\frac{1 - R(t)v_\infty(t) - R(t)g(t)}{R(t)(\beta + R(t))}. \tag{12}$$

From the above formula, we see that the positivity of β indeed has a profound effect on the dynamics of particles, in particular near the time when the radius is about to vanish:

- when $R \ll 1$, if $\beta > 0$, Eq. (12) becomes

$$\dot{R} \approx -\frac{1}{R\beta} \quad \text{and hence} \quad R(t) \approx \left(C - \frac{2t}{\beta}\right)^{\frac{1}{2}}, \tag{13}$$

- while for $\beta = 0$, it becomes

$$\dot{R} \approx -\frac{1}{R^2} \quad \text{and hence} \quad R(t) \approx (C - 3t)^{\frac{1}{3}}. \tag{14}$$

Even though the solution forms (11) and (12) are for the single particle case in the quasi-static situation, we expect them to be still a good approximation for multiple particles if $\lambda \ll 1$ and all the particles are far away from each other. In this case, the overall solution v of (10) is roughly given by the linear combination of the individual solutions:

$$v(y,t) \approx v_\infty(t) + \sum_i \frac{R_i(t)(1 - R_i(t)v_\infty(t) - R_i(t)g_i(t))}{(\beta + R_i(t))|y - y_i|}, \tag{15}$$

where i is the index of the particle with center at y_i and radius R_i.

To complete the picture, we need to specify the quantity $v_\infty(t)$ and its dynamics. Note that it is a spatially constant variable describing the heat distribution far away from the solid–liquid interfaces. This justifies the terminology *mean-field description*. Due to the assumption of small volume fraction (to be prescribed later), the overall background domain Ω is very close to the region Ω_L occupied by the liquid phase. Hence, we have

$$v_\infty \approx \frac{1}{|\Omega|}\int_{\Omega_L} v.$$

We now compute

$$\partial_t \int_{\Omega_L} v = \int_{\Omega_L} \partial_t v - \int_{\partial\Omega_L} \dot{R}v = \int_{\Omega_L} \frac{1}{\lambda}\Delta v - \int_{\partial\Omega_L} \dot{R}v = -\int_{\partial\Omega_L} \frac{1}{\lambda}\nabla v \cdot n - \int_{\partial\Omega_L} \dot{R}v = -\frac{1}{\lambda}\int_{\partial\Omega_L} \dot{R} - \int_{\partial\Omega_L} \dot{R}v,$$

so that

$$\partial_t v_\infty \approx -\frac{1}{|\Omega|\lambda}\int_{\partial\Omega_L} \dot{R} - \frac{1}{|\Omega|}\int_{\partial\Omega_L} \dot{R}v.$$

Since λ is small, the second term is negligible. Note that $\partial\Omega_L = \bigcup_i \partial B(y_i, R_i)$, by (12) we then get

$$\partial_t v_\infty \approx \frac{1}{|\Omega|\lambda}\sum_i \left(\frac{1 - R_i v_\infty - R_i g_i(t)}{R_i(\beta + R_i)}\right) 4\pi R_i^2. \tag{16}$$

The purpose of the current work is to derive rigorously the solution formulae (12), (15) and (16) from the free boundary value problem (9) and give a *limiting homogenized description when the number of particles is large.*

3. Rescaling of the problem

In this section, we introduce a spatial rescaling of the Stefan problem (9) as a preparation for the derivation of a limiting description for a large number of particles.

Recall that the domain of the liquid phase is denoted by Ω_L. We consider the case that the solid phase $\Omega_S = \Omega \backslash \Omega_L$ consists of a collection of N disjoint balls, i.e.

$$\Omega_S = \bigcup_{i=1}^{N} B(y_i, R_i) \quad \text{and} \quad \Gamma = \bigcup_{i=1}^{N} \partial B(y_i, R_i). \tag{17}$$

We further assume that the centers of the balls do not move and the spherical shapes are preserved during the evolution Strictly speaking, there is no solution satisfying the above assumptions. As in [14,15], we replace the second condition of (9) by the following integral condition:

$$V_i := V|_{\partial B_i} = \frac{1}{|\partial B_i|} \int_{\partial B_i} \nabla v(y,t) \cdot n \, ds$$
$$\big(\text{where } ds \text{ is the area element and } B_i = B(y_i, R_i)\big). \tag{18}$$

Since $V_i = \dot{R}_i$, $k_i := k|_{\partial B_i} = \frac{1}{R_i}$, and $g_i := g|_{\partial B_i}$, the third condition of (9) is transformed into

$$v = \beta \dot{R}_i(t) + \frac{1}{R_i(t)} - g_i(t) \quad \text{on } \partial B\big(y_i, R_i(t)\big). \tag{19}$$

Note that now v is constant on each of $\partial B(y_i, R_i(t))$. (See Remark 3.1(5) for a discussion.)

To model the facts that the volume occupied by the solid phase is very small compared to the vessel's volume (i.e. $\mathrm{Vol}(\bigcup_i B_i) \ll \mathrm{Vol}(\Omega)$) while the inter-particle distances are very large compared with the particle size, we apply the same spatial rescaling as in [14,15]. We use δ and δ^a to denote the typical length scales for the inter-particle distance and the particle radii and consider the regime $0 < \delta^a \ll \delta$ i.e. $a > 1$. Now introduce the following change of variables

$$x = \delta^a y \quad \text{and} \quad u(x,t) = v(y,t); \tag{20}$$

$$R_i^\delta(t) := \frac{R_i(t)}{\delta^a} \quad \text{and} \quad B_i^\delta(t) := B\big(x_i, \delta^a R_i^\delta(t)\big) = B\big(y_i, R_i(t)\big). \tag{21}$$

Let further

$$\mathcal{N}(t) := \big\{i\colon\, R_i^\delta(t) > 0\big\}, \qquad N(t) = \big|\mathcal{N}(t)\big|, \quad \text{and} \quad t_i^\delta := \sup\big\{t\colon\, R_i^\delta(t) > 0\big\}, \tag{22}$$

be the collection and number of indices of particles at time t and the maximum existence time of B_i^δ. With the above, we define the following domains

$$\Omega^\delta := \delta^a \Omega; \qquad \Omega_T^\delta := \Omega^\delta \times (0,T);$$
$$\Omega_S^\delta(t) := \bigcup_{i \in \mathcal{N}(t)} \overline{B_i^\delta(t)}; \qquad \Omega_{S,T}^\delta := \bigcup_{t \in (0,T)} \big(\Omega_S^\delta(t) \times \{t\}\big);$$
$$\Omega_L^\delta(t) := \Omega^\delta \backslash \Omega_S^\delta(t); \qquad \Omega_{L,T}^\delta := \bigcup_{t \in (0,T)} \big(\Omega_L^\delta(t) \times \{t\}\big), \tag{23}$$

where T is some finite fixed time instant.

Now using the variables x and R_i^δ's, upon choosing $\delta^a = \delta^4$ (see Remark 3.1(6)), the system of Eqs. (9), adjoined with the Neumann condition on $\partial\Omega^\delta$ leads to the following initial boundary value problem (IBVP):

$$\begin{aligned}
&\lambda u_t = \delta^8 \Delta u \quad \text{in } \Omega^\delta_{L,T},\\
&u(x,t) + g_i(t) = \frac{1}{R_i^\delta(t)} + \frac{\beta}{4\pi\delta^4 (R_i^\delta(t))^2} \int\limits_{\partial B_i^\delta(t)} \nabla u \cdot n\, ds, \quad x \in \partial B_i^\delta(t),\ t \in \big(0, t_i^\delta\big),\\
&\dot{R}_i^\delta(t) = \frac{1}{4\pi\delta^4 (R_i^\delta(t))^2} \int\limits_{\partial B_i^\delta(t)} \nabla u \cdot n\, ds, \quad t \in \big(0, t_i^\delta\big),\\
&R_i^\delta(t) = 0, \quad t > t_i^\delta,\\
&\nabla u \cdot n = 0 \quad \text{on } \partial\Omega^\delta,\\
&u(x,0) = u_0(x) \quad \text{in } \Omega_L^\delta(0),\\
&R_i^\delta(0) = R_{i0}^\delta \quad \text{for } i \in \mathcal{N}(0).
\end{aligned} \tag{24}$$

The main purpose of this paper is to give a limiting description of the system as δ converges to zero. The following are some remarks about the scalings and assumptions used in our problem.

Remark 3.1 *(Assumptions).*

1. For simplicity, the underlying ambient domain Ω^δ is bounded with smooth boundary $\partial\Omega^\delta$.
2. With the current spatial rescaling, we are working in the regime that the particles are separated from each other by distances of at least $O(\delta)$, i.e. $|x_i - x_j| \geqslant C\delta$ for all $i \neq j$. Hence $N(t) = O(\delta^{-3})$. A simple such setting is to have the particles located on a regular three-dimensional lattice of lattice length δ although this is not absolutely necessary.
3. Motivated by the approximate solution (15), the initial data u_0 takes the following "well-prepared" form:

$$u_0^\delta = u_{0\infty}^\delta + \sum_i \frac{(1 - R_{i0}^\delta u_{0\infty}^\delta - R_{i0}^\delta g_{i0})\delta^4 R_{i0}^\delta}{(R_{i0}^\delta + \beta)|x - x_i|} \eta\left(\frac{|x - x_i|}{\delta}\right). \tag{25}$$

   In the above, $u_{0\infty}^\delta$ is some constant, and η is a smooth cut-off function such that $\eta(r) \equiv 1$ for $0 \leqslant r \leqslant \frac{1}{8}$ and $\eta(r) \equiv 0$ for $r \geqslant \frac{1}{4}$. Furthermore, the initial radii R_{i0}^δ's satisfy

$$\sup_i R_{i0}^\delta \leqslant R_0^\delta < \infty. \tag{26}$$

4. The inhomogeneous driving forces g_i's satisfy

$$\sup_i \sup_{t \geqslant 0} \left\{ \left|g_i(t)\right|, \left|R_i^\delta(t)\dot{g}_i(t)\right| \right\} \leqslant M < \infty. \tag{27}$$

   The above are sufficient to derive some *a priori* estimates. However, in order to have a limit equation in a *closed* form, we do need to make the assumption that each g_i is a function of the radius R_i^δ. This is stated as follows *there exist a function $G \in C^1(R_+ \times R_+)$ and a function $h \in C^1(R_+)$ such that*

$$g_i(t) = G\big(t, R_i^\delta(t)\big) + h(t). \tag{28}$$

(See Remark 9.3 for further discussion.)

5. As the typical size of the solid grains (δ^a) is assumed small compared with the mean distance between them (δ), the direct interactions between the particles is negligible and the particles thus behave as if they were isolated. The only interaction is through the mean field quantity u_∞^δ. Hence it is natural to assume that they remain spherical and their centers do not move in space. On the other hand, models incorporating non-spherical shapes and the particle motions have been considered, cf. [1,2,8], in which it is shown that these additional features only constitute to higher order effects and hence they do not affect the mean field limit.
6. In order to have a well-defined limiting description, we need to work in the setting of small volume fraction for the particles. A quantitative requirement is that the *capacity* needs to vanish. With the current rescaling regime, the order of magnitude of the capacity of a particle in Ω is given by δ^{a-3}. Hence, we take $a = 4$. In this case the capacity is of order δ and the volume fraction is of order δ^9. (See [5] for a model when the capacity does not vanish.) The choice of the scaling $\lambda = \delta^9$ is to ensure that the system is close to being quasi-static. This will be clear from the energy type identities derived in Section 5. (See the discussion in p. 4697 and Corollary 5.5.)

The main theorems proved in this paper are:

- Theorem 6.3: existence of a global solution for (24) and regularity of particle radii near their vanishing moment.
- Theorem 9.2: mean field description of the system as the number of particles goes to infinity ($\delta \to 0$). This is given by (12), (15) and (16) which govern the dynamics of the particle size, the mean field variable and the profile of the heat distribution.

The overall strategy is briefly explained here. First we extend the local in time solution to globally existing solution, i.e. beyond the times when some balls disappear. This is established by the *a priori* estimates coming from integral inequalities (Section 5) and maximum principle (Section 6). When both λ and β are positive, we need to control the appearing terms involving $R_i^\delta \dot{R}_i^\delta$ uniformly in δ and globally in time, even after some balls have vanished. This makes it necessary to estimate the growth and decay in time for the radii $R_i^\delta(t)$'s. First we analyze the single particle case. The important issue is to investigate the solution as $R \to 0^+$ for $\delta \ll 1$. The main conclusion is that $|R\dot{R}| < C < \infty$ and $\lim_{R\to 0^+} R\dot{R} = -\frac{1}{\beta}$ (these results state the regularizing effect of kinetic undercooling) and thus $R \in W^{1,p}([0,T])$ for any $1 \leqslant p < 2$. This is established by constructing proper sub- and super-solutions. It is first done for the case $R \ll 1$ and $\dot{R} < 0$. If $R > O(1)$, we show that $|\dot{R}|$ is uniformly bounded. Moreover, we prove that once $R(t)$ reaches below some small value, $\dot{R}$ will become *negative* and will *stay negative* until the extinction time of $R(t)$. We then employ the previous analysis to analyze the multiple particle case. The extension of solution beyond vanishing time follows by the energy estimates from Proposition 5.3 and standard parabolic theory.

In the second step, we derive the limiting equation for the dynamics of the mean field variable and radii as $\delta \to 0$. We produce a first order approximation for the heat distribution u^δ in Section 7. In particular, we prove that far away from the particle boundaries, the heat distribution $u^\delta(\cdot,t)$ is close to the mean field variable $u_\infty^\delta(t)$ which satisfies the following form:

$$\partial_t u_\infty^\delta(t) = 4\pi\delta^3 \sum_i \big(1 - R_i^\delta(t) u_\infty^\delta(t) - R_i^\delta(t) g_i(t)\big) \frac{R_i^\delta(t)}{R_i^\delta(t) + \beta}.$$

We then establish in Section 8 the following result which gives the dynamics of the radii as $\delta \to 0$:

the radii satisfy the following dynamical equation in some weak sense:

$$\dot{R}_i^\delta \approx -\frac{1 - u_\infty^\delta R_i^\delta - g_i R_i^\delta}{R_i^\delta (R_i^\delta + \beta)}.$$

Finally in Section 9, we provide a limiting description of u^δ and R_i^δ's as $\delta \to 0$. In order to obtain an equation which is closed in the limit, we do need to invoke the assumption (28) on the form of the inhomogeneous forces g_i's.

A note on notation. For the next few sections, we will only work with the rescaled variables x and the function u. Hence for simplicity, we suppress the super-script δ from all the symbols: Ω^δ, Ω_L^δ, B_i^δ, R_i^δ, $u_{0\infty}^\delta$, u_∞^δ and so forth. They will be recovered in Section 9. Recall that the number of particles $N(t)$ is of order $O(\delta^{-3})$. In the following, n refers to the outward normal to the *solid phase* $\Omega_S(t) = \bigcup_i B_i(t)$. We will use M or $M(T, \Omega)$ to denote general constants that might depend on the time interval $[0, T]$ and the domain Ω but not on δ.

4. Local in time existence and uniqueness

In order to formulate the existence and uniqueness result for the system of Eqs. (24), we first introduce some function spaces. In the following, T is some fixed positive time (which does not depend on δ).

- For $f : [0, T] \to \mathbb{R}$, denote $\|f\|_{L^2(0,T)} := (\int_0^T |f(t)|^2\, dt)^{1/2}$. Consider the following usual L^p and Sobolev spaces on $(0, T)$:

$$L^2(0,T) = \big(f\colon \|f\|_{L^2(0,T)} < \infty\big), \qquad L^\infty(0,T) = \Big(f\colon \sup_{t\in[0,T]} |f| < \infty\Big),$$
$$H^1(0,T) := \left\{f\colon \|f\|_{L^2(0,T)} + \left\|\frac{df}{dt}\right\|_{L^2(0,T)} < \infty\right\}$$

  and

$$\|f\|_{H^1(0,T)} = \left(\|f\|_{L^2(0,T)}^2 + \left\|\frac{df}{dt}\right\|_{L^2(0,T)}^2\right)^{\frac{1}{2}}.$$

- Let $\mathcal{D}(t)$ be a time dependent domain with smooth boundary. We define $\mathcal{D}_T = \bigcup_{t\in[0,T]} \mathcal{D}(t)$, while for $(u(\cdot,t) : \mathcal{D}(t) \to \mathbb{R})_{0\leqslant t\leqslant T}$, we denote: $\|u\|_{L^2(\mathcal{D}(t))} := (\int_{\mathcal{D}(t)} |u(\cdot,t)|^2\, dx)^{\frac{1}{2}}$.

$$H^1\big(\mathcal{D}(t)\big) = \big\{u\colon \|u\|_{L^2(\mathcal{D}(t))} + \|\nabla u\|_{L^2(\mathcal{D}(t))} < \infty\big\},$$
$$\|u\|_{H^1(\mathcal{D}(t))} := \big(\|u\|_{L^2(\mathcal{D}(t))} + \|\nabla u\|_{L^2(\mathcal{D}(t))}\big)^{\frac{1}{2}};$$
$$L^\infty\big(0,T; L^2\big(\mathcal{D}(\cdot)\big)\big) := \Big\{u\colon \sup_{t\in[0,T]} \|u\|_{L^2(\mathcal{D}(t))} < \infty\Big\},$$
$$\|u\|_{L^\infty(0,T,L^2(\mathcal{D}(t)))} := \sup_{t\in[0,T]} \|u\|_{L^2(\mathcal{D}(t))};$$
$$L^2\big(0,T; H^1\big(\mathcal{D}(\cdot)\big)\big) := \left(u\colon \int_0^T \big[\|u(\cdot,t)\|_{L^2(\mathcal{D}(t))}^2 + \|\nabla u(\cdot,t)\|_{L^2(\mathcal{D}(t))}^2\big]\, dt < \infty\right);$$

and

$$\|u\|_{L^2(0,T,H^1(\mathcal{D}(\cdot)))} := \left(\int_0^T \left[\|u(\cdot,t)\|^2_{L^2(\mathcal{D}(t))} + \|\nabla u(\cdot,t)\|^2_{L^2(\mathcal{D}(t))} \right] dt \right)^{\frac{1}{2}}.$$

The usual inner product on $L^2(\mathcal{D}(t))$ is denoted by $(\cdot,\cdot)_{\mathcal{D}(t)}$. For the later usage in this paper, the domain $\mathcal{D}(\cdot)$ in the above definition is usually taken to be $\Omega_L(\cdot)$ or simply the ambient domain Ω. For simplicity in notation, we will often omit the subscript in the norms if it does not cause any confusion.

Definition of weak solution. Now we derive the weak formulation of the solution for our governing system (24). Let $\xi = \xi(\cdot,t) : \Omega_L \to \mathbb{R}$ be such that for all $t > 0$, ξ equals a constant on all the ∂B_i's. Multiplying the parabolic equation of (24) by ξ and integrating over $\Omega_L(t)$, then by means of the boundary condition on $\partial\Omega$ and ∂B_i's, we have

$$0 = (\lambda u_t, \xi)_{\Omega_L(t)} - \delta^8 (\Delta u, \xi)_{\Omega_L(t)} = (\lambda u_t, \xi)_{\Omega_L(t)} + \delta^8 (\nabla u, \nabla \xi)_{\Omega_L(t)} + \delta^8 \sum_{i=1}^N \int_{\partial B_i} \xi \nabla u \cdot n\, ds$$

$$= (\lambda u_t, \xi)_{\Omega_L(t)} + \delta^8 (\nabla u, \nabla \xi)_{\Omega_L(t)} + \delta^8 \sum_{i=1}^N \xi|_{\partial B_i} \int_{\partial B_i} \nabla u \cdot n\, ds. \tag{29}$$

Next multiply the second equation of (24) by $\xi|_{\partial B_i}$ and integrate on ∂B_i to get

$$\int_{\partial B_i} \left(u(x,t) + g_i(t) - \frac{1}{R_i(t)} \right) \xi\, ds - \beta \delta^4 \int_{\partial B_i} \xi \nabla u \cdot n\, ds = 0. \tag{30}$$

Replacing the term $\xi|_{\partial B_i} \int_{\partial B_i} \nabla u \cdot n\, ds$ in (29) by (30) leads to

$$\lambda(u_t, \xi)_{\Omega_L(t)} + \delta^8 (\nabla u, \nabla \xi)_{\Omega_L(t)} + \frac{\delta^4}{\beta} \sum_{i=1}^N \int_{\partial B_i} \xi \left(u - \frac{1}{R_i(t)} \right) ds + \frac{\delta^4}{\beta} \sum_{i=1}^N g_i(t) \int_{\partial B_i} \xi\, ds = 0.$$

The above leads to the following definition of solution before the first vanishing moment (t_*) of any ball. Let $T > 0$ be a positive number. *A collection of functions* $u : \Omega_L(\cdot) \to \mathbb{R}$ *and* $\{R_i : [0,T] \to \mathbb{R}_+\}_{i\in\mathcal{N}}$ *is called a* weak solution *of* (24) *with initial data* u_0 *and* $\{R_{i0}\}_{i\in\mathcal{N}}$*'s if* (i) $u \in L^2(0,T;H^1(\Omega_L(\cdot)))$; (ii) $u_t \in L^2(0,T;H^{-1}(\Omega_L(\cdot)))$; (iii) $u|_{\partial B_i}$ *is a constant*; (iv) $R_i \in H^{-1}([0,T]) \cap L^2([0,T])$ *and they satisfy the following identity*:

$$\int_0^{T\wedge t_*} \left[-\lambda(u, \xi_t)_{\Omega_L(t)} + \delta^8 (\nabla u, \nabla \xi)_{\Omega_L(t)} + \frac{\delta^4}{\beta} \sum_{i=1}^N \int_{\partial B_i} u \xi\, ds \right] dt - \big(u_0, \xi(0)\big)_{\Omega_L(0)}$$

$$= \int_0^{T\wedge t_*} \frac{\delta^4}{\beta} \sum_{i=1}^N \left(\frac{1}{R_i(t)} - g_i(t) \right) \int_{\partial B_i} \xi\, ds\, dt, \tag{31}$$

and

$$\int_0^{T\wedge t_*}\left[-R_i(t)\dot{\varphi}(t)-\frac{\varphi(t)}{4\pi\delta^4(R_i(t))^2}\int_{\partial B_i(t)}\nabla u\cdot n\,ds\right]dt-R_{i0}\varphi(0)=0 \tag{32}$$

for all $\xi\in C_0^\infty([0,T\wedge t_*),C^\infty(\Omega_L(\cdot)))$ *with* $\xi|_{\partial B_i}$ *equal to a constant for all* $i\in\mathcal{N}$ *and* $\varphi\in C_0^\infty([0,T\wedge t_*))$.

With the above definition, we now present the following local existence and uniqueness result.

Theorem 4.1. *For any* $u_0\in H^1(\Omega_L(0))$ *and* $\{R_i(0)\}_{i\in\mathcal{N}}$ *satisfying* (26), *there exists a* $T>0$ *such that* (24) *has a unique weak solution* $\{u,R_i\colon i\in\mathcal{N}\}$. *Furthermore,* $u\in L^\infty(0,T;L^2(\Omega_L(\cdot)))\cap L^2(0,T;H^1(\Omega_L(\cdot)))$ *and* $R_i\in H^1([0,T])$.

Proof. The proof consists of two steps. The first is to prove the existence of solution for the parabolic problem with given R_i's (without taking into account of the conservation of heat flux at the particle boundary); the second is to use fixed point theorem to find the correct R_i's which do satisfy the conservation of heat flux. The overall procedure is more or less standard. But we include it here for self-containedness. The strategy is also used in [13] for a related problem without the kinetic undercooling.

In the following, we use R to denote the collection $\{R_i\colon i\in\mathcal{N}\}$. The notation $\|R\|_X$ refers to $\sup_i\|R_i\|_X$. In addition, any operation on R is performed in a component-wise manner: $f(R)=\{f(R_i)\colon i\in\mathcal{N}\}$.

Step I. *Given* $R\in H^1([0,T])^N$ *with* $\|R\|_{L^\infty([0,T])}$ *and* $\|R^{-1}\|_{L^\infty([0,T])}<\infty$. *We claim that there exists a unique weak solution u satisfying*:

$$\begin{aligned}
\lambda u_t&=\delta^8\Delta u\quad \text{in } \Omega_{L,T},\\
u(x,t)-\frac{\beta}{4\pi\delta^4R_i^2(t)}\int_{\partial B_i(t)}\nabla u\cdot n\,ds&=\frac{1}{R_i(t)}-g_i(t),\quad x\in\partial B_i(t),\\
\nabla u\cdot n&=0\quad \text{on } \partial\Omega,\\
u(x,0)&=u_0(x)\quad \text{in } \Omega_L(0),\\
R_i(0)&=R_{i0}\quad \text{for } i\in\mathcal{N}(0).
\end{aligned}$$

Furthermore, if $u_0\in H^1(\Omega_L(0))$, *then*

$$\begin{aligned}
&\|u\|_{L^\infty(0,T,H^1(\Omega_L(\cdot)))},\ \|u\|_{L^2(0,T;H^2(\Omega_L(\cdot)))},\ \|u_t\|_{L^2(0,T;L^2(\Omega_L(\cdot)))}\\
&\quad\leqslant C\big(\|R\|_{L^\infty([0,T])},\ \big\|R^{-1}\big\|_{L^\infty([0,T])},\ \|R\|_{H^1([0,T])}\big).
\end{aligned}\tag{33}$$

First we related the domain $\Omega_L(0)$ to $\Omega_L(t)$ by means of some diffeomorphism:

$$\phi(\cdot,R):\Omega_L(0)\to\Omega(t).$$

Define

$$\Phi(y,t):=\phi\big(y,R(t)\big)\quad\text{and}\quad\tilde{v}(y,t):=u\big(\Phi(y,t),t\big).$$

Differentiating in space we get

$$\nabla u = D\Phi^{-T}\nabla\tilde{v} \quad \text{and} \quad \frac{1}{|\partial B_i(t)|}\int\limits_{\partial B_i(t)} \nabla u\cdot n = \frac{1}{|\partial B_i(0)|}\int\limits_{\partial B_i(0)} D\Phi^{-T}\nabla\tilde{v}\cdot n,$$

while taking the derivative in time gives

$$\tilde{v}_t = u_t + \nabla u\cdot\partial_t\Phi.$$

In the above we have used the notation:

$$D\Phi^{-T} = \big((D\Phi)^T\big)^{-1} \quad \text{and} \quad \partial_t\Phi = \frac{\partial\phi}{\partial R_1}\dot{R}_1 + \cdots + \frac{\partial\phi}{\partial R_N}\dot{R}_N = (\partial_R\phi)\cdot(\partial_t R).$$

Let $M = \|R\|_{L^\infty(0,T)} + \|R^{-1}\|_{L^\infty(0,T)}$. Note the following estimates:

$$\big\|D\Phi, D^2\Phi, D\Phi^{-T}, D\big(D\Phi^{-T}\big)\big\|_{L^\infty(0,T;L^\infty(\Omega_L(t)))} \leqslant C(M),$$

and

$$\|\partial_t\Phi\|_{L^2(0,T;L^\infty(\Omega_L(\cdot)))} \leqslant C(M)\|R\|_{H^1(0,T)}. \tag{34}$$

Let $A = D\Phi^T D\Phi$. Then the function $\tilde{v}$ solves

$$\begin{gathered}
\lambda\sqrt{\det A}\,\partial_t\tilde{v} - \delta^8\operatorname{div}\big(\sqrt{\det A}\,A^{-1}\nabla\tilde{v}\big) = \lambda\sqrt{\det A}\,D\Phi^{-T}\nabla\tilde{v}\cdot\partial_t\Phi \quad \text{in } \Omega_L(0)\times(0,T),\\
\tilde{v} - \frac{\beta}{4\pi\delta^4R_i^2(0)}\int\limits_{\partial B_i(0)} D\Phi^{-T}\nabla\tilde{v}\cdot n\,ds = \frac{1}{R_i} - g_i, \quad x\in\partial B_i(0),\\
\nabla\tilde{v}\cdot n = 0 \quad \text{on } \partial\Omega_T,\\
\tilde{v}(x,0) = u_0(x) \quad \text{in } \Omega_L(0).
\end{gathered} \tag{35}$$

To handle the inhomogeneous boundary condition on the $\partial B_i(0)$'s, we consider the solution $w(y,t)$ of the problem

$$\begin{gathered}
\operatorname{div}\big(\sqrt{\det A}\,A^{-1}\nabla w\big) = 0 \quad \text{in } \Omega_L(0),\\
w - \frac{\beta}{4\pi\delta^4R_i^2(0)}\int\limits_{\partial B_i(0)} D\Phi^{-T}\nabla w\cdot n\,ds = \frac{1}{R_i} - g_i, \quad x\in\partial B_i(0),\\
w\cdot n = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{36}$$

Setting $v := \tilde{v} - w$, then v satisfies

$$\begin{gathered}
\lambda\partial_t v - \frac{\delta^8}{\sqrt{\det A}}\operatorname{div}\big(\sqrt{\det A}\,A^{-1}\nabla v\big) = \lambda D\Phi^{-T}\nabla v\cdot\partial_t\Phi - \lambda\partial_t w + \lambda D\Phi^{-1}\nabla w\cdot\partial_t\Phi,\\
v - \frac{\beta}{4\pi\delta^4R_i^2(0)}\int\limits_{\partial B_i(0)} D\Phi^{-T}\nabla v\cdot n\,ds = 0, \quad x\in\partial B_i(0).
\end{gathered} \tag{37}$$

Applying elliptic regularity to (36), we get

$$\|w, \partial_R w\|_{L^\infty(0,T;H^1(\Omega_L(t)))} \leqslant C(M). \tag{38}$$

Note that $\partial_t w = \partial_R w \cdot \partial_t R$, and we also have

$$\|\partial_t w\|_{L^2(0,T;H^1(\Omega_L(\cdot)))} \leqslant C(M)\|R\|_{H^1(0,T)}. \tag{39}$$

Combining the above together with estimate (34), Eq. (37) can be written as

$$\lambda\partial_t v - \frac{\delta^8}{\sqrt{\det A}}\operatorname{div}\bigl(\sqrt{\det A}A^{-1}\nabla v\bigr) = f_1 \cdot \nabla v + f_2,$$

for some

$$f_1 \in \bigl(L^2\bigl(0, T; L^\infty\bigl(\Omega_L(0)\bigr)\bigr)\bigr)^3 \quad \text{and} \quad f_2 \in L^2\bigl(0, T; L^\infty\bigl(\Omega_L(0)\bigr)\bigr).$$

If $u_0 \in H^1(\Omega(0))$, then by standard theory for parabolic problems [9], it follows that there exists a unique solution v of (37) leading to the solution $\tilde{v} = v + w$ of (35) in the class $L^\infty(0, T; L^2(\Omega_L(0))) \cap L^2(0, T; H^1(\Omega_L(0)))$. By (34), it follows that u also belongs to $L^\infty(0, T; L^2(\Omega_L(\cdot))) \cap L^2(0, T; H^1(\Omega_L(\cdot)))$. The improved regularity statement (33) also follows from standard theory.

Step II. *This step shows that for T small enough, there exists an $R \in H^1([0, T])$ such that the following condition is satisfied*

$$\dot{R}_i(t) = \frac{1}{4\pi\delta^4 R_i^2(t)} \int\limits_{\partial B_i(t)} \nabla u \cdot n\, ds, \quad t \in (0, T) \textit{ for all } i \in \mathcal{N},$$

where u is from Step I.

Using the kinetic under-cooling condition, the above can be written as

$$\dot{R}_i(t) = \frac{1}{\beta|\partial B_i(t)|} \int\limits_{\partial B_i(t)} \left(u - \frac{1}{R_i(t)} + g_i(t)\right) = \frac{1}{\beta|\partial B_i(0)|} \int\limits_{\partial B_i(0)} \left(v + w - \frac{1}{R_i(t)} + g_i(t)\right).$$

For this, we define the function space:

$$\mathcal{M}_T = \bigl\{R \in H^1\bigl([0, T]\bigr)^N\colon\ R(0) = R_0,\ \|R\|_{H^1([0,T])} \leqslant D\bigr\},$$

where D is some fixed number and the operator $\mathcal{F}(R) : \mathcal{M}_T \to H^1([0, T])^N$:

$$\mathcal{F}(R)_i(t) = R_{i0} + \int\limits_0^t \left[\frac{1}{\beta|\partial B_i(0)|} \int\limits_{\partial B_i(0)} \left(v + w - \frac{1}{R_i(\tau)} + g_i(\tau)\right)\right] d\tau, \quad i \in \mathcal{N}.$$

The goal is to prove that $\mathcal{F}$ has a fixed point in $\mathcal{M}_T$ if $T = T(D)$ is small enough.

For this, let $R, S \in \mathcal{M}_T$ with $R(0) = S(0)$ and let w_1, w_2, v_1, v_2 be the solutions of (36) and (37) with the radius function given by R and S. Then,

$$\mathcal{F}(R)_i(t) - \mathcal{F}(S)_i(t) = \int_0^t \Bigg[\frac{1}{\beta|\partial B_i(0)|} \int\limits_{\partial B_i(0)} \Bigg((v_1 - v_2) + (w_1 - w_2) - \left(\frac{1}{R_i(\tau)} - \frac{1}{S_i(\tau)} \right) + \big(g_i(\tau, R_i) - g_i(\tau, S_i) \big) \Bigg) \Bigg] d\tau.$$

Consider the equation satisfied by $v_1 - v_2$:

$$\begin{aligned}
&\lambda \partial_t (v_1 - v_2) - \frac{\delta^8}{\sqrt{\det A_1}} \operatorname{div}\big(\sqrt{\det A_1} A_1^{-1} \big(\nabla (v_1 - v_2) \big) \big) \\
&\quad = \frac{\delta^8}{\sqrt{\det A_1}} \operatorname{div}\big(\sqrt{\det A_1} A_1^{-1} \nabla v_2 \big) - \frac{\delta^8}{\sqrt{\det A_2}} \operatorname{div}\big(\sqrt{\det A_2} A_2^{-1} \nabla v_2 \big) \\
&\qquad - \lambda (\partial_t w_1 - \partial_t w_2) + \lambda D\Phi_1^{-1} (\nabla w_1 - \nabla w_2) \cdot \partial_t \Phi_1 + \lambda D\Phi_1^{-1} \nabla w_2 \cdot \partial_t \Phi_1 \\
&\qquad - \lambda D\Phi_2^{-1} \nabla w_2 \cdot \partial_t \Phi_2,
\end{aligned} \tag{40}$$

and

$$\begin{aligned}
&(v_1 - v_2) - \frac{\beta}{4\pi \delta^4 R_i^2(0)} \int\limits_{\partial B_i(0)} D\Phi_1^{-T} \nabla (v_1 - v_2) \cdot n\, ds \\
&\quad = -\frac{\beta}{4\pi \delta^4 R_i^2(0)} \int\limits_{\partial B_i(0)} \big(D\Phi_1^{-T} - D\Phi_2^{-T} \big) \nabla v_2 \cdot n\, ds \quad \text{for } i \in \mathcal{N},
\end{aligned} \tag{41}$$

and that for $w_1 - w_2$:

$$\operatorname{div}\big(\sqrt{\det A_1} A_1^{-1} \nabla (w_1 - w_2) \big) = -\operatorname{div}\big(\big(\sqrt{\det A_1} A_1^{-1} - \sqrt{\det A_2} A_2^{-1} \big) \nabla w_2 \big), \tag{42}$$

and

$$\begin{aligned}
&(w_1 - w_2) - \frac{\beta}{4\pi \delta^4 R_i^2(0)} \int\limits_{\partial B_i(0)} D\Phi_1^{-T} \nabla (w_1 - w_2) \cdot n\, ds \\
&\quad = -\frac{\beta}{4\pi \delta^4 R_i^2(0)} \int\limits_{\partial B_i(0)} \big(D\Phi_1^{-T} - D\Phi_2^{-T} \big) \nabla w_2 \cdot n\, ds \\
&\qquad + \left(\frac{1}{R_i} - \frac{1}{S_i} \right) - \big(g_i\big(t, R_i(t)\big) - g_i\big(t, S_i(t)\big) \big) \quad \text{for } i \in \mathcal{N},
\end{aligned} \tag{43}$$

where A_1, Φ_1 and A_2, Φ_2 are the A and Φ for the radius functions R and S respectively. The estimates (34), (38), (39) lead to

$$\|A_1 - A_2\|_{L^\infty(0,T;L^\infty(\Omega_L(0)))}, \|w_1 - w_2\|_{L^\infty(0,T;H^1(\Omega_L(0)))} \leqslant C(M) \|R - S\|_{L^\infty(0,T)},$$

and

$$\|\partial_t \Phi_1 - \partial \Phi_2\|_{L^2(0,T;L^\infty(\Omega_L(0)))}, \|\partial_t w_1 - \partial_t w_2\|_{L^2(0,T;H^1(\Omega_L(0)))} \leqslant C(M) \|R - S\|_{H^1(0,T)}. \tag{44}$$

Using the above together with the fact $\|v_2\|_{L^\infty(0,T;H^1(\Omega_L(0)))} < \infty$, we see that the right-hand side f for Eq. (40) satisfies

$$\|f\|_{L^2(0,T;L^2(\Omega_L(0)))} \leqslant M_1\|R-S\|_{L^\infty(0,T)} + M_2\|\dot{R}-\dot{S}\|_{L^2(0,T)}.$$

We are then led to the following estimate:

$$\|v_1 - v_2\|_{L^2(0,T,H^1(\Omega_L(0)))} \leqslant C(D)\|R-S\|_{H^1(0,T)}.$$

As $\|v_1 - v_2\|_{L^2(\partial\Omega_L(0))} \leqslant \|v_1 - v_2\|_{H^1(\Omega_L(0))}$, we have that

$$\int_0^T \left|\mathcal{F}(R)_i(t) - \mathcal{F}(S)_i\right|^2 dt \leqslant C(D)T\|R-S\|^2_{H^1(0,T)}.$$

In the above, we have also used the assumption (27) about the g_i's. Finally if T is chosen small enough, Banach Fixed Point theorem can be employed, leading to the existence of a fixed point for $\mathcal{F}$ in $\mathcal{M}_T$ and hence a solution of (24). $\square$

In order to extend the local in time solution to globally existing solution, in particular beyond the times when some balls disappear, we would need *a priori* estimates. They will be established by means of integral inequalities and maximum principle. The overall strategy is as follows. First the weak solution obtained above exists up to the first time t_i some ball vanishes ($R_i = 0$). From the global energy estimates derived in Section 5 together with the temporal particle radius regularity proved in Section 6, the limit $u(\cdot, t_i) = \lim_{t\to t_i^-} u(\cdot, t)$ exists. We can then use $u(\cdot, t_i)$ as the *new initial data* for (24). In this manner, a solution is constructed between any two times some radii vanish and hence the existence of a solution up to any finite time (independent of δ) is established.

5. Integral identities

In this section, we will present some integral identities in line of energy type estimates. As the domain Ω_L is time dependent, we find it convenient to extend u to the whole domain $\Omega \supset \Omega_L$ by means of

$$u|_{B_i} = u|_{\partial B_i} \quad \text{for all } i.$$

The extended function is still denoted by u. Furthermore, we introduce the notation $f_i(t) = R_i(t)\dot{R}_i(t)$.

Proposition 5.1. *Let u be the solution of* (24). *Then we have*

$$\begin{aligned}
&\lambda\int_\Omega u(t) + \frac{\lambda 2\pi\delta^{12}}{3}\sum_{i=1}^N R_i^2(t) + \frac{4\pi\delta^{12}}{3}\sum_{i=1}^N R_i^3(t) + \lambda 4\pi\delta^{12}\beta\sum_{i=1}^N\int_0^t f_i^2(r)\,dr \\
&\quad = \lambda\int_\Omega u(0) + \frac{\lambda 2\pi\delta^{12}}{3}\sum_{i=1}^N R_i^2(0) + \frac{4\pi\delta^{12}}{3}\sum_{i=1}^N R_i^3(0) \\
&\qquad + \frac{\lambda 4\pi\delta^{12}\beta}{3}\left(\sum_{i=1}^N R_i^2(t)f_i(t) - \sum_{i=1}^N R_i^2(0)f_i(0)\right)
\end{aligned}$$

$$-\frac{\lambda 4\pi\delta^{12}}{3}\sum_{i=1}^{N}R_i^3(t)g_i(t)+\frac{\lambda 4\pi\delta^{12}}{3}\sum_{i=1}^{N}R_i^3(0)g_i(0)$$
$$+\lambda 4\pi\delta^{12}\int_0^t\sum_{i=1}^{N}R_i(r)f_i(r)g_i(r)\,dr. \qquad \square \tag{45}$$

Proof. We integrate (24) on Ω to get

$$\lambda\int_{\Omega}u_t-\lambda\int_{\Omega\setminus\Omega_L}u_t=\delta^8\int_{\partial\Omega_L}\frac{\partial u}{\partial n}.$$

Note that the part of $\partial\Omega^\delta$ on solid–liquid interfaces, we use the outward normal to the B_i's. Hence

$$\lambda\frac{d}{dt}\int_{\Omega}u-\lambda\sum_i\frac{4\pi}{3}\left(\delta^4R_i\right)^3\left(\frac{1}{R_i}-g_i+\beta\dot{R}_i\right)_t=-\sum_i 4\pi\delta^{12}R_i^2\dot{R}_i,$$
$$\lambda\frac{d}{dt}\int_{\Omega}u-\frac{\lambda 4\pi\delta^{12}}{3}\sum_i R_i^3\left(-\frac{\dot{R}_i}{R_i^2}-\dot{g}_i+\beta\ddot{R}_i\right)+4\pi\delta^{12}\sum_i R_i^2\dot{R}_i=0,$$
$$\lambda\frac{d}{dt}\int_{\Omega}u+\frac{\lambda 4\pi\delta^{12}}{3}\sum_i\left(R_i\dot{R}_i+R_i^3\dot{g}_i-\beta R_i^3\ddot{R}_i\right)+\frac{4\pi\delta^{12}}{3}\sum_i\frac{d}{dt}R_i^3=0,$$
$$\lambda\frac{d}{dt}\int_{\Omega}u+\frac{\lambda 4\pi\delta^{12}}{3}\sum_i\left(\frac{1}{2}\frac{d}{dt}R_i^2+R_i^3\dot{g}_i-\beta R_i^3\ddot{R}_i\right)+\frac{4\pi\delta^{12}}{3}\sum_i\frac{d}{dt}R_i^3=0.$$

Upon integrating in time from 0 to t and employing integration by parts, we obtain (45). $\square$

Remark 5.2. For conceptual understanding and in order to compare with known results, we simplify the above identity for the case $g_i(t)\equiv 0$.

1. For the quasi-static problem $\lambda=0$ with $\beta\geqslant 0$ the following volume conservation condition is obtained

$$\delta^3\sum_{i=1}^{N}R_i^3(t)=\delta^3\sum_{i=1}^{N}R_i^3(0),$$

   as in [14,15].
2. For the parabolic case $\lambda>0$ and $\beta=0$: setting $\lambda:=\delta^9$ as in (45), we obtain the result of [14]:

$$\int_{\Omega}u(t)+\frac{4}{3}\pi\sum_{i=1}^{N}\delta^3R_i^3(t)+\frac{2}{3}\pi\sum_{i=1}^{N}\delta^{12}R_i^2(t)=\int_{\Omega}u(0)+\frac{4}{3}\pi\sum_{i=1}^{N}\delta^3R_i^3(0)+\frac{2}{3}\pi\sum_{i=1}^{N}\delta^{12}R_i^2(0).$$

Our result extends the above to the case when λ and β are both positive.

Next we derive the identity for $\|u\|_{L^2(\Omega)}$.

Proposition 5.3. *Let u be the solution of* (24)*. Then we have*

$$\begin{aligned}
&\frac{\lambda}{2}\int_\Omega u^2(t) + \delta^8\int_0^t\int_\Omega |\nabla u|^2(s)\,ds + 2\pi\delta^{12}\sum_i R_i^2(t) + \lambda\frac{4\pi\delta^{12}}{3}\sum_i R_i(t)\\
&\quad + 4\pi\delta^{12}\beta\int_0^t\sum_i f_i^2(s)\,ds + \lambda 4\pi\delta^{12}\beta\int_0^t\sum_i\frac{f_i^2(s)}{R_i(s)}\,ds\\
&= \frac{\lambda}{2}\int_\Omega u^2(0) + \frac{4\pi\delta^{12}}{2}\sum_i R_i^2(0) + \lambda\frac{4\pi\delta^{12}}{3}\sum_i R_i(0) + \lambda\frac{4\pi\delta^{12}}{3}\beta\sum_i R_i(t)f_i(t)\\
&\quad - \lambda\frac{4\pi\delta^{12}}{3}\beta\sum_i R_i(0)f_i(0) + \lambda\frac{2\pi\delta^{12}}{3}\beta^2\sum_i R_i f_i^2(t) - \lambda\frac{2\pi\delta^{12}}{3}\beta^2\sum_i R_i f_i^2(0)\\
&\quad - \lambda 2\pi\delta^{12}\beta^2\int_0^t\sum_i\frac{f_i^3(s)}{R_i(s)}\,ds + 4\pi\delta^{12}\int_0^t\sum_i R_i f_i g_i(s)\,ds - \lambda\frac{4\pi\delta^{12}}{3}\sum_i R_i^2(t)g_i(t)\\
&\quad + \lambda\frac{4\pi\delta^{12}}{3}\sum_i R_i^2(0)g_i(0) + \lambda 4\pi\delta^{12}\sum_i\int_0^t f_i(s)g_i(s)\,ds - \lambda\frac{4\pi\delta^{12}}{3}\beta\sum_i R_i^2(t)f_i(t)g_i(t)\\
&\quad + \lambda\frac{4\pi\delta^{12}}{3}\beta\sum_i R_i^2(0)f_i(0)g_i(0) + \lambda 4\pi\delta^{12}\beta\sum_i\int_0^t f_i^2 g_i\,ds + \lambda\frac{2\pi\delta^{12}}{3}\sum_i R_i(t)^3 g_i^2(t)\\
&\quad - \lambda\frac{2\pi\delta^{12}}{3}\sum_i R_i(0)^3 g_i^2(0) - \lambda 2\pi\delta^{12}\int_0^t\sum_i R_i f_i g_i^2(s)\,ds. \qquad (46)
\end{aligned}$$

Proof. Multiplying (24) by u and integrating on Ω_L, we get

$$\lambda\int_{\Omega_L} u_t u = \delta^8\int_{\Omega_L}\triangle u u,$$

$$\lambda\int_\Omega u_t u - \lambda\int_{\Omega\setminus\Omega_L} u_t u = -\delta^8\int_{\partial\Omega_L}\frac{\partial u}{\partial n}u - \delta^8\int_{\Omega_L}|\nabla u|^2.$$

Using the boundary conditions in (24), it follows that

$$\begin{aligned}
&\frac{\lambda}{2}\frac{d}{dt}\int_\Omega u^2 + \delta^8\int_\Omega|\nabla u|^2 - \lambda\frac{4\pi\delta^{12}}{3}\sum_i R_i^3\left(\frac{1}{R_i} - g_i + \beta\dot{R}_i\right)_t\left(\frac{1}{R_i} - g_i + \beta\dot{R}_i\right)\\
&\quad + 4\pi\delta^{12}\sum_i R_i^2\dot{R}_i\left(\frac{1}{R_i} - g_i + \beta\dot{R}_i\right) = 0,
\end{aligned}$$

$$\frac{\lambda}{2}\frac{d}{dt}\int_\Omega u^2 + \delta^8\int_\Omega |\nabla u|^2 - \lambda\frac{4\pi\delta^{12}}{3}\sum_i R_i^3\left(-\frac{\dot R_i}{R_i^2} - \dot g_i + \beta \ddot R_i\right)\left(\frac{1}{R_i} - g_i + \beta \dot R_i\right)$$
$$+ 4\pi\delta^{12}\sum_i\left(R_i\dot R_i - R_i^2\dot R_i g_i + \beta R_i^2 \dot R_i^2\right) = 0,$$
$$\frac{\lambda}{2}\frac{d}{dt}\int_\Omega u^2 + \delta^8\int_\Omega |\nabla u|^2 + 4\pi\delta^{12}\sum_i\left(\frac{d}{dt}\frac{R_i^2}{2} + \beta R_i^2\dot R_i^2 - R_i^2\dot R_i g_i\right)$$
$$+ \lambda\frac{4\pi\delta^{12}}{3}\sum_i\left(R_i\dot R_i + R_i^3\dot g_i - \beta R_i^3\ddot R_i\right)\left(\frac{1}{R_i} - g_i + \beta\dot R_i\right) = 0.$$

Expanding the above, and integrating in time from 0 to t together with integration by parts gives the stated identity. □

Remark 5.4. Again, we give the simplified form of the above in the case $g_i(t) \equiv 0$ and compare with known results.

1. $\lambda = 0$, $\beta \geqslant 0$:

$$\delta^3\sum_i R_i^2(t) + \frac{1}{2\pi\delta}\int_0^t\int_\Omega |\nabla u|^2\,ds + 2\beta\int_0^t\sum_i \delta^3 f_i^2\,ds = \delta^3\sum_i R_i^2(0),$$

   as in [15].
2. $\lambda > 0$, $\beta = 0$:

$$\frac{1}{2}\int_\Omega u^2(t) + \frac{1}{\delta}\int_0^t\int_\Omega |\nabla u|^2\,ds + 2\pi\delta^3\sum_i R_i^2(t) + \frac{2}{3}\pi\delta^{12}\sum_i R_i(t)$$
$$= \frac{1}{2}\int_\Omega u^2(0) + 2\pi\delta^3\sum_i R_i^2(0) + \frac{2}{3}\pi\delta^{12}\sum_i R_i(0),$$

   where we have set $\lambda := \delta^9$ (in accordance to [14]).

Note that when both λ and β are positive, as in the current case with kinetic undercooling, extra terms involving f_i appear on the right-hand side of (46). This causes the need to estimate $|R_i\dot R_i|$. This is the main goal of Section 6.

Here we explain in more detail the usage of the above result and the choice of $\lambda = \delta^9$. Since we are aiming at a limiting *mean field* description – the particles interact mainly through the quantity $u_\infty(t)$, we expect that the heat distribution $u(\cdot,t)$ will become roughly spatially constant (but still time dependent), i.e. $\nabla u \approx 0$ as $\delta \to 0$.

In view of the estimates of Remark 5.4, if either one of λ or β equals zero (as in [15] and [14]), the term $\int_0^t\int_\Omega |\nabla u|^2(s)\,ds$ is estimated easily by the initial data. This is not the case for the parabolic setting with kinetic undercooling and inhomogeneous driving forces presented in this paper (where *both λ and β are non-zero*). In the current case, upon solving for $\int_0^t\int_\Omega |\nabla u|^2(s)\,ds$ in the identity of Proposition 5.3, we observe that if g_i, R_i, and $f_i = R_i\dot R_i$ are uniformly bounded in time for any i, then

$$\int_0^t \int_\Omega |\nabla u|^2(s)\,ds \leqslant -\lambda 4\pi \delta^{12-8}\beta \int_0^t \sum_i \frac{f_i^2(s)}{R_i(s)}\,ds - \lambda 2\pi \delta^{12-8}\beta^2 \int_0^t \sum_i \frac{f_i^3(s)}{R_i(s)}\,ds + C\frac{\lambda}{\delta^8} + C\delta^4 \sum_{i\in\mathcal{N}} O(1).$$

Hence as long as $R_i \geqslant O(1) > 0$ for any i, we will have

$$\int_0^t \int_\Omega |\nabla u|^2(s)\,ds \leqslant C\lambda\delta^4 \sum_{i\in\mathcal{N}} O(1) + C\frac{\lambda}{\delta^8} + C\delta^4 \sum_{i\in\mathcal{N}} O(1) \leqslant C\lambda\delta + C\frac{\lambda}{\delta^8} + C\delta.$$

The problem arises if time is approaching some extinction time t_i ($\dot{R} < 0$ and $R_i \to 0$), so that the term $\frac{f_i^3}{R_i}$ (which appears in the estimate only when λ and β are non-zero) will have a large negative value. In fact it will blow up to $-\infty$. However, we will prove that $f_i = R_i \dot{R}_i \to -\frac{1}{\beta}$ as $t \to t_i^-$. With this observation, near t_i, this term can be controlled in the following way:

$$-\lambda 4\pi \delta^{12-8}\beta \int_0^t \sum_i \frac{f_i^2(s)}{R_i(s)}\,ds - \lambda 2\pi \delta^{12-8}\beta^2 \int_0^t \sum_i \frac{f_i^3(s)}{R_i(s)}\,ds = -4\lambda\delta^4\pi\beta \int_0^t \sum_i \frac{f_i^2(s)}{R_i(s)}\left[1 + \frac{\beta}{2} f_i\right] ds \leqslant 0,$$

and thus it follows that

$$\int_0^t \int_\Omega |\nabla u|^2(s)\,ds \leqslant C\frac{\lambda}{\delta^8} + C\delta^4 \sum_{i\in\mathcal{N}} O(1) \leqslant C\frac{\lambda}{\delta^8} + C\delta.$$

(Alternatively, upon solving $\dot{R}_i R_i \approx -\frac{1}{\beta}$, we have $R_i \approx C(t_i - t)^{\frac{1}{2}}$ so that $\int_0^{t_i} \frac{1}{R_i(t)}\,dt < \infty$. Hence the term $\int_0^{t_i} \frac{f_i^3}{R_i}\,dt$ will also be bounded.) The above leads us to set $\lambda = \delta^9$ as mentioned in Remark 3.1. With this we have that

$$\int_0^t \int_\Omega |\nabla u|^2(s)\,ds \leqslant C\delta,$$

so that in the limit $\delta \to 0$, u will indeed converge to a spatial constant (in some weak sense).

We summarize the above observation in the following statement.

Corollary 5.5. *Let $\lambda = \delta^9$. Let further $t_* = t_i$ be the first extinction time. Suppose there is an $M > 0$ (independent of δ) such that $\sup_i \sup_{t\leqslant t_*}\{|R_i(t)\dot{R}_i(t)|, R_i(t)\} < M$, and if $\lim_{t\to t_*^-} R_i\dot{R}_i = -\frac{1}{\beta}$, then we have*

$$\sup_{t<t_*} \|u\|^2_{L^2(\Omega)} + \frac{1}{\delta}\|\nabla u\|^2_{L^2(\Omega_{t_*})} \leqslant M.$$

(In the above we have also used the assumption (27) for the g_i's.)

The next section is to prove the validity of the assumptions used in the above corollary. This will be shown using maximum principle by means of sub- and super-solutions. Then the result of Corollary 5.5 will be used to extend the solution u of (24) to even after the moment some balls have vanished.

6. Regularity of the radii R_i's

6.1. Preliminaries

We first record the following lemma on a maximum principle suitable for our problem. It is the parabolic version of Lemma 4.2 in [15].

Lemma 6.1. *Let $\{\Omega(t)\}_{t\geqslant 0}$ be a time dependent Lipschitz domain and $\bigcup_i\{B_i(t)\}_{t\geqslant 0}$ be a finite collection of disjoint balls such that $\bigcup_i B_i(t)\subset\Omega(t)$ for all $t\geqslant 0$.*

Let u be a function which is constant on each ∂B_i and satisfy for all $t\geqslant 0$ the following statements

$$u_t-\triangle u\geqslant(\leqslant)\,0\quad\text{in }\Omega(t)\setminus\bigcup_i B_i(t),$$

$$u-c_i\int\limits_{\partial B_i(t)}\nabla u\cdot n\geqslant(\leqslant)\,0\quad\text{on }\partial B_i(t),\text{ for all }i,$$

$$\nabla u\cdot n\geqslant(\leqslant)\,0\quad\text{on }\partial\Omega(t),$$

where $c_i\geqslant 0$ for all i. If $u(x,0)\geqslant(\leqslant)\,0$, then $u\geqslant(\leqslant)\,0$ in $\Omega(t)\setminus\bigcup_i B_i(t)$ for $t>0$.

The rigorous proof of the above can be produced following the steps in [15]. Hence it is omitted. It can be intuitively understood as follows. If $u\geqslant 0$ at $t=0$, then by strong maximum principle, it cannot reach zero *inside* the domain $\Omega(t)\setminus\bigcup_i B_i(t)$. By means of the Hopf lemma, the boundary conditions also prevent the occurrence of zero on $\partial\Omega(t)$ and $\partial B_i(t)$. Hence u will be strictly positive for all $t>0$.

Equipped with the above result, we are ready to construct sub- and super-solutions which will be used to control the growth and decay of the radii $R_i(t)$'s. First we present an *a priori* bound using the above maximum principle.

Lemma 6.2. *There exist two constants $M_1(T,\Omega)$ and $M_2(T,\Omega)$ such that for any solution u of* (24) *with initial data* (25)*, we have*

$$M_1(T,\Omega)\leqslant u(x,t)\leqslant M_2(T,\Omega)+u_{\infty 0}+\sum_{i\in N}\frac{\delta^4}{|x-x_i|}.\tag{47}$$

(*In general, M_1 might be negative.*) *The above leads to that for some constant $M>0$,*

1. *at any particle boundary*: *for x such that $|x-x_i|=\delta^4R_i$,*

$$u|_{\partial B_i}\leqslant M+\frac{1}{R_i};\tag{48}$$

2. *away from any of particle boundary*: *for x such that $|x-x_i|\geqslant\frac{\delta}{4}$ for all i,*

$$|u|\leqslant M.\tag{49}$$

Proof. The fact that statements (48) and (49) follow from (47) is due to the assumptions on the bound on the number of particles and their spatial separation by at least of distance $O(\delta)$ (see Remark 3.1(2)).

The proof of the lower bound in (47) is simply due to the fact that a negative constant with large magnitude $(-M)$ satisfies

$$(-M) \leqslant -g_i(t) + \frac{1}{R_i} + \frac{\beta}{4\pi\delta^4 R_i^2}\int_{\partial B_i} \nabla(-M)\cdot n,$$

and hence is a sub-solution.

The proof of the upper bound in (47) is similar to [14, Lemma 17]. It turns out that the function $\overline{V}$ denoting the right-hand side of (47) is automatically a super-solution for large enough $M_2(T,\Omega)$. The reasoning is as follows.

1. For any $i \in \mathcal{N}(t)$,

$$\overline{V}|_{\partial B_i} = M_2 + u_{\infty 0} + \frac{1}{R_i} + \sum_{j\neq i}\frac{\delta^4}{|x_j - x_i|} \geqslant M_2 + u_{\infty 0} + \frac{1}{R_i} + O(1)\sum_{j\neq i}\frac{\delta^4}{\delta}$$
$$\geqslant M_2 + u^\delta_{\infty 0} + \frac{1}{R_i} + O(1)\sum_{j\neq i}\delta^3 \geqslant M_2 + O(1) + \frac{1}{R_i}.$$

   In the above, we have used the fact that $N(t) = O(\delta^{-3})$ and $|x_i - x_j| \geqslant c\delta$ for any $i \neq j$.
2. Next we compute the gradient term: again for any $i \in \mathcal{N}(t)$,

$$\frac{\beta}{4\pi\delta^4 R_i^2}\int_{\partial B_i}\nabla\overline{V}\cdot n = \frac{\beta}{4\pi\delta^4 R_i^2}\int_{\partial B_i}\nabla\Bigg[\sum_{j\in N^\delta(t)}\frac{\delta^4}{|x - x_j|}\Bigg]\cdot n$$
$$= \frac{\beta}{4\pi\delta^4 R_i^2}\int_{\partial B_i}\nabla\bigg[\frac{\delta^4}{|x - x_i|}\bigg]\cdot n + \frac{\beta}{4\pi\delta^4 R_i^2}\int_{\partial B_i}\nabla\Bigg[\sum_{j\neq i}\frac{\delta^4}{|x - x_j|}\Bigg]\cdot n$$
$$\geqslant \frac{\beta}{4\pi\delta^4 R_i^2}\bigg[-\frac{\delta^4}{\delta^8 R_i^2}\bigg]4\pi\delta^8 R_i^2 + \frac{O(1)}{\delta^3}\frac{\beta}{4\pi\delta^4 R_i^2}\bigg[\frac{\delta^4}{\delta^2}\bigg]4\pi\delta^8 R_i^2$$
$$= -\frac{\beta}{R_i^2} + O\big(\delta^3\big).$$

   Hence with M_2 chosen big enough and δ being small, we always have

$$\overline{V} \geqslant -g_i + \frac{1}{R_i} + \frac{\beta}{4\pi\delta^4 R_i^2}\int_{\partial B_i}\nabla\overline{V}\cdot n.$$

3. In order for $\overline{V}$ to satisfy the Neumann boundary condition on $\partial\Omega$, we consider a modification function w similar to [14, Lemma 17]. Let $h = \sum_{i\in N^\delta}\frac{\delta^4}{|x - x_i|}$ and w be the solution of the following equation:

$$\delta w_t = \triangle w \quad \text{in } \Omega_T,$$
$$\nabla w \cdot n = -\nabla h \cdot n \quad \text{on } \partial\Omega_T,$$
$$w(0,\cdot) = w_0 \quad \text{in } \Omega,$$

where w_0 solves

$$-\triangle w_0 = \int_{\partial\Omega} \nabla h \cdot n,$$
$$\nabla w_0 \cdot n = -\nabla h \cdot n,$$
$$\int_{\Omega} w_0 = 0.$$

By [14, Lemmas 17, 20], w_0 and w satisfy the following estimates

$$\|w_0\|_\infty \leqslant M\sqrt{\delta}, \qquad \|w\|_\infty \leqslant M, \quad \text{and} \quad \|\nabla w\|_\infty \leqslant M\delta^\gamma \quad \text{for any } \gamma < \tfrac{1}{2}.$$

With the above, $\nabla(\overline{V} + w) \cdot n = 0$ on $\partial\Omega$ and upon choosing M_2 large enough, we have

$$(\overline{V} + w)|_{\partial B_i} \geqslant g_i - \frac{1}{R_i} + \frac{\beta}{4\pi\delta^4 R_i^2} \int_{\partial B_i} \nabla(\overline{V} + w) \cdot n,$$

so that the desired result is still true with $\overline{V}$ replaced by $\overline{V} + w$. $\square$

Now we proceed to construct sub- and super-solutions so as to control the growth and decay rates of the particle radii.

6.2. Single particle case

We first consider the case of a single particle which forms the building block for the general multiple particle scenario. In the following, we will use the notation $B_{\delta^4 R}$ to emphasize the radius of the rescaled ball. In this case, problem (24) is formulated in the following form:

$$\delta u_t = \triangle u \quad \text{on } \{|x| \geqslant \delta^4 R(t)\},$$
$$u = \frac{1}{R} - g(t) + \frac{\beta}{4\pi\delta^4 R^2(t)} \int_{\partial B_{\delta^4 R}} \nabla u \cdot n \quad \text{on } \{|x| = \delta^4 R(t)\},$$
$$\dot{R} = \frac{1}{4\pi\delta^4 R^2} \int_{\partial B_{\delta^4 R}} \nabla u \cdot n. \tag{50}$$

The key is to investigate the solution as $R \to 0^+$ in the regime $\delta \ll 1$. The main conclusion is that $|R\dot{R}| < C < \infty$ and hence $R \in W^{1,p}([0,T])$ for any $1 \leqslant p < 2$. As a by-product, we get $\lim_{R\to 0^+} R\dot{R} = -\frac{1}{\beta}$. This will be established by constructing sub- and super-solutions. It is first done for the case $R \ll 1$ and $\dot{R} < 0$. If $R > O(1)$, we will show that $|\dot{R}|$ is uniformly bounded. However, once $R(t)$ reaches below some small value, $\dot{R}$ will become *negative* and will *stay negative* until the extinction time of $R(t)$.

Construction of sub-solution under the assumption: $\dot{R} \leqslant 0$, $R \ll 1$. Let $R(t)$ be given. Then $U(x,t)$ is a sub-solution of (50) if

$$\delta U_t \leqslant \triangle U, \quad \text{on } \{|x| \geqslant \delta^4 R(t)\},$$

and

$$U \leqslant \frac{1}{R(t)} - g(t) + \frac{\beta}{4\pi\delta^4 R^2(t)} \int_{\partial B_{\delta^4 R}} \nabla U \cdot n \quad \text{on } \{|x| = \delta^4 R(t)\}.$$

For any constant C, consider the function

$$U_{C,R}(x) = C + \left(\frac{1 - RC - Rg}{R + \beta}\right)\frac{\delta^4 R}{|x|}. \tag{51}$$

By simple computations, $U_{C,R}$ satisfies the following properties:

$$\begin{aligned}
U_{C,R}(x) &> 0 \quad \text{for } |x| \geqslant \delta^4 R, \\
U_{C,R}(x) &\geqslant C \quad \text{for } |x| \geqslant \delta^4 R \quad \text{and} \quad R(C+g) \leqslant 1, \\
U_{C,R}(\delta^4 R) &= \frac{1 + \beta C - Rg}{R + \beta}, \\
U_{C,R}(\delta^4 R) &= \frac{1}{R} - g + \frac{\beta}{4\pi\delta^4 R^2} \int_{\partial B_{\delta^4 R}} \nabla U_{C,R} \cdot n, \\
\lim_{R \to 0^+} U_{C,R}(\delta^4 R) &= C + \frac{1}{\beta}, \\
\lim_{|x| \to \infty} U_{C,R}(x) &= C.
\end{aligned}$$

Note that $|U_{C,R}|$ is uniformly bounded by some constant $M(C,G) < \infty$. Furthermore,

$$\frac{\partial U_{C,R}}{\partial C} = 1 - \frac{\delta^4 R^2}{(R+\beta)|x|} \geqslant 1 - \frac{R}{R+\beta} = \frac{\beta}{R+\beta} > 0 \quad \text{if } |x| \geqslant \delta^4 R, \tag{52}$$

so that we can use the constant C to adjust the far-field value in order to ensure that at $t = 0$, $U_{C,R}$ is smaller than the initial data.

Now let $R = R(t)$ be given from the solution of (50) and $C = C(t)$ be some time dependent function (to be specified). Then $\Delta U_{C,R} = 0$ and

$$\begin{aligned}
\frac{\partial U_{C(t),R(t)}(x)}{\partial t} &= \frac{\delta^4 \dot{R}}{(R+\beta)^2|x|}\left[(R+\beta)(1 - 2RC - 2Rg) - R + R^2 C + R^2 g\right] \\
&\quad + \left[1 - \frac{\delta^4 R^2}{(R+\beta)|x|}\right]\dot{C} - \frac{\delta^4 R^2}{(R+\beta)|x|}\dot{g} \\
&= \frac{\delta^4 \dot{R}}{(R+\beta)^2|x|}\left[\beta - R^2 C - 2R\beta C - R^2 g - 2Rg\beta\right] \\
&\quad + \left[1 - \frac{\delta^4 R^2}{(R+\beta)|x|}\right]\dot{C} - \frac{\delta^4 R^2}{(R+\beta)|x|}\dot{g}.
\end{aligned}$$

Using the standing assumptions that $\dot{R} \leqslant 0$ and $R \ll 1$ and also (27) on g, the above can be made negative by choosing $C(t)$ such that $\dot{C}(t)$ is much bigger that $|R\dot{g}|$. Thus $U_{C,R}$ is a sub-solution. So if $C(0)$ is chosen small enough (possibly with negative value), we have $u_0 \geqslant U_{C(0),R(0)}$ and hence $u \geqslant U_{C,R}$ for $t > 0$. This leads to

$$\dot{R} = \frac{1}{4\pi\delta^4 R^2} \int\limits_{\partial B_{\delta^4 R}} \nabla u \cdot n = \frac{1}{\beta}\left[u - \frac{1}{R} + g\right] \geqslant \frac{1}{\beta}\left[U_{C,R}(R) - \frac{1}{R} + g\right]$$
$$= \frac{1}{\beta}\left[\frac{1 + C\beta - Rg}{R+\beta} + g - \frac{1}{R}\right] \gtrsim -\frac{1}{\beta R}. \tag{53}$$

Construction of super-solution under the assumption: $\dot{R} < 0$, $R \ll 1$. Again let $R(t)$ be taken from the solution of (50), then $V(x,t)$ is a super-solution if

$$\delta V_t \geqslant \Delta V \quad \text{on } \{|x| \geqslant \delta^4 R(t)\}, \tag{54}$$

and

$$V \geqslant \frac{1}{R(t)} - g + \frac{\beta}{4\pi\delta^4 R^2(t)} \int\limits_{\partial B_{\delta^4 R}} \nabla V \cdot n \quad \text{on } \{|x| = \delta^4 R(t)\}. \tag{55}$$

Consider the function

$$V_{C(t),R(t)}(x) = \frac{\delta^4 a(t)}{|x|} + C(t) + \frac{(1 - RC(t) - Rg)\delta^4 R}{(R+\beta)|x|}, \tag{56}$$

where $a(t)$ and $C(t)$ are to be determined. Note that $\Delta V_{C(t),R(t)} = 0$ and

$$\frac{\partial V_{C(t),R(t)}}{\partial t} = \frac{\delta^4 \dot{a}}{|x|} + \frac{\delta^4 \dot{R}}{(R+\beta)^2|x|}\left[\beta - R^2 C - 2R\beta C - 2R^2 g - 2Rg\beta\right]$$
$$+ \left[1 - \frac{\delta^4 R^2}{(R+\beta)|x|}\right]\dot{C} - \frac{\delta^4 R^2}{(R+\beta)|x|}\dot{g}$$
$$\approx \frac{\delta^4 \dot{a}}{|x|} + \frac{\delta^4 \dot{R}}{\beta|x|} + \left[1 - \frac{\delta^4 R^2}{(R+\beta)|x|}\right]\dot{C} - \frac{\delta^4 R^2}{(R+\beta)|x|}\dot{g}.$$

To make (54) hold, we choose $a(t)$ and $C(t)$ such that

$$\dot{a} + \frac{\dot{R}}{\beta} \geqslant 0 \quad \text{or} \quad a(t) = a_0 - \frac{R(t)}{\beta} > 0, \quad \text{and} \quad \dot{C} \text{ is much bigger than } |R\dot{g}| \quad \big(\text{recall again (27)}\big).$$

As $\dot{R} < 0$, a convenient choice is

$$a(t) = \frac{R(0)}{\beta} - \frac{R(t)}{\beta}.$$

Condition (55) is then equivalent to

$$\frac{a(t)}{R(t)} > \beta\delta^4\delta^4 a(t)(-1)\frac{1}{\delta^8 R^2(t)}$$

which is always true as long as $a(t) > 0$. Thus V is a super-solution. So if $C(0)$ is chosen big enough, we have $u_0 \leqslant V_{C(0),R(0)}$ and hence $u \leqslant V_{C(t),R(t)}$ for $t > 0$.

Now considering the dynamics of $R(t)$, we have

$$\begin{aligned}
\dot{R} &= \frac{1}{4\pi\delta^4 R^2}\int_{\partial B_{\delta^4 R}} \nabla u\cdot n = \frac{1}{\beta}\left[u - \frac{1}{R} + g\right] \leqslant \frac{1}{\beta}\left[V - \frac{1}{R} + g\right] \\
&= \frac{1}{\beta}\left[\frac{a}{R} + \frac{1 + C\beta}{R+\beta} - \frac{1}{R} + g\right] = \frac{1}{\beta R}\left[a(t) - 1 + \frac{R(1 + C\beta) + gR(R+\beta)}{(R+\beta)}\right] \\
&= \frac{1}{\beta R}\left[a_0 - \frac{R(t)}{\beta} - 1 + \frac{R(1 + C\beta) + gR(R+\beta)}{(R+\beta)}\right] \\
&= \frac{1}{\beta R}\left[-1 + \frac{R(0)}{\beta} - \frac{R(t)}{\beta} + \frac{R(1 + C\beta) + gR(R+\beta)}{(R+\beta)}\right] \lesssim -\frac{1}{\beta R}. \qquad (57)
\end{aligned}$$

Combining (53) and (57), we finally have,

$$-\frac{1}{\beta R} \leqslant \dot{R} \leqslant -\frac{1 - O(1)}{\beta R}. \qquad (58)$$

Construction for balls with big radius. This section considers the case when R is not small. The idea is to modify the previous construction of sub- and super-solutions by a term with small L^∞-norm but large Laplacian value (see [14, Lemma 18]).

Let (R, u) be the solution of (24). In addition, we assume for some fixed constants δ_0, A_1, A_2 and B such that

$$\begin{gathered}
\delta \leqslant \delta_0; \\
A_1 < R(t) < A_2; \\
\dot{R} \text{ is uniformly bounded by } \frac{B}{\delta}. \qquad (59)
\end{gathered}$$

To produce a super-solution, we consider the following function:

$$\widetilde{V}_{C,R}(x,t) = C + \frac{(1 - RC - Rg)\delta^4 R}{(R+\beta)|x|} - \frac{1}{2}|x - x_i|^2 + \epsilon, \qquad (60)$$

where $\epsilon \gg \delta$. It holds that

$$\begin{aligned}
\delta\frac{\partial \widetilde{V}_{C(t),R(t)}}{\partial t} - \triangle \widetilde{V}_{C(t),R(t)} = \delta\Bigg\{&\frac{\delta^4\dot{R}}{(R+\beta)^2|x|}\left[\beta - R^2C - 2R\beta C - 2R^2 g - 2Rg\beta\right] \\
&+ \left[1 - \frac{\delta^4 R^2}{(R+\beta)|x|}\right]\dot{C} - \frac{\delta^4 R^2}{(R+\beta)|x|}\dot{g}\Bigg\} + 3, \qquad (61)
\end{aligned}$$

and

$$\widetilde{V}_{C,R} \geqslant -g + \frac{1}{R} + \frac{\beta}{4\pi\delta^4 R^2} \int\limits_{\partial B_{\delta^4 R}} \nabla \widetilde{V} \cdot n. \tag{62}$$

Under the assumption (59), the right-hand side of (61) is positive. Hence V is a *super-solution*. As before we obtain that

$$\dot{R} \leqslant \frac{1}{\beta R}\left[-1 - \frac{R}{\beta} + \frac{R(1 + C\beta) + gR(R+\beta)}{R+\beta}\right] < M, \tag{63}$$

for some constant M independent of δ.

For sub-solution, we similarly consider

$$\widetilde{U}_{C,R}(x,t) = C + \frac{(1 - RC - Rg)\delta^4 R}{(R+\beta)|x|} + \frac{1}{2}|x - x_i|^2 - \epsilon. \tag{64}$$

Again by (59), $\widetilde{U}_{C,R}$ will be a *sub-solution*. So we have

$$\dot{R} \geqslant \frac{1}{\beta R}\left[-1 - \frac{R}{\beta} + \frac{R(1 + C\beta) + gR(R+\beta)}{R+\beta}\right] > -M. \tag{65}$$

Hence we obtain

$$|\dot{R}| < M. \tag{66}$$

6.3. *Multi-particle case: existence beyond vanishing of some balls*

Now we employ the above single particle analysis to prove *a priori* bounds for the multiple particle case. Consider the initial data u_0 given by (25). By Theorem 4.1, the solution exists locally in time. The key is to extend the solution globally in time, beyond the vanishing times of some balls.

Let T be some fixed constant. By the uniform estimate (49), on the set $K = \{x\colon\ |x - x_i| \geqslant \frac{\delta}{4}$ for all $i\}$ (i.e. away from each ∂B_i), $|u|_{0\leqslant t\leqslant T}$ is bounded uniformly by some fixed constant. Hence if $\widetilde{C}_i^-(0)$ and $\widetilde{C}_i^+(0)$ are chosen sufficiently small and large respectively, using (60) and (64), we have $\widetilde{U}_{\widetilde{C}_i^-(0),R_i(0)} \leqslant u_0 \leqslant \widetilde{V}_{\widetilde{C}_i^+(0),R_i(0)}$ and hence

$$\widetilde{U}_{\widetilde{C}_i^-(t),R_i(t)} \leqslant u \leqslant \widetilde{V}_{\widetilde{C}_i^+(t),R_i(t)},$$

for as long as $A_1 \leqslant R_i \leqslant A_2$ and $|\dot{R}_i| \leqslant \frac{B}{\delta}$. On the other hand, by (66), it follows that $|\dot{R}_i| \leqslant M$. Now given any finite time interval $[0,T]$, choose $A_2 = R_0 + 2MT$. Then the upper bounds $R_i \leqslant A_2$ are always true for time interval $[0,T]$ (independent of δ).

If some $R_i(t)$ ever reaches some small value A_1, by (63), $\dot{R}_i$ will be negative. Similarly choose C_i^- and C_i^+ to be sufficiently small and large such that $U_{C_i^-(t),R_i(t)}$ and $V_{C_i^+(t),R_i(t)}$ from (51) and (56) satisfy

$$U_{C_i^-(t),R_i(t)} \leqslant \widetilde{U}_{\widetilde{C}_i^-(t),R_i(t)} (\leqslant u) \quad \text{and} \quad (u \leqslant) \widetilde{V}_{\widetilde{C}_i^+(t),R_i(t)} \leqslant V_{C_i^+(t),R_i(t)}.$$

Now by (57), $\dot{R}$ will *stay negative* and hence $U_{C_i^-(t),R_i(t)}$ and $V_{C_i^+(t),R_i(t)}$ remain to be sub- and super-solutions up to the vanishing moment t_i of R_i. Finally estimates (58) hold.

Now let t_* be the first vanishing time of some ball (t_i). We then have

$$\sup_i \sup_{t<t_*} |R_i \dot{R}_i| \leqslant M < \infty, \quad \text{and} \quad \sup_i \sup_{t<t_*} R_i(t) \leqslant M < \infty. \tag{67}$$

Upon integrating the ODE $|\dot{R}R| \leqslant M$, we conclude that if R_i vanishes at t_*, then

$$\left|R_i(t)\right| \leqslant C(t - t_*)^{\frac{1}{2}} \quad \text{and} \quad \int_0^{t_*} \frac{1}{R_i(t)}\, dt \leqslant C. \tag{68}$$

In particular, we have that $R_i \in W^{1,p}([0, t_*])$ for all $1 \leqslant p < 2$.

With the above, the extension of solution beyond t_* follows as in [14, pp. 158–159, 165]. We briefly outline the procedure here for completeness. By Corollary 5.5, we have that $\sup_{t<t_*} \|u\|_{L^2(\Omega)}$ and $\|\nabla u\|_{L^2(\Omega_{t_*})}$ are bounded independently of δ. Hence standard parabolic theory leads to the existence in L^2 of $u(\cdot, t_*) = \lim_{t\to t_*^-} u(\cdot, t)$. Next we evolve Eq. (24) from $t = t_*$ using $u(\cdot, t_*)$ as initial data. However, in general $u(\cdot, t_*)$ does not belong to $H^1(\Omega)$ so that we cannot directly invoke the local in time existence result Theorem 4.1. On the other hand, the H^1-condition is only needed near the boundary of each existing particles. Near the location where a ball has just vanished, only a regular heat equation is involved which is well-posed with L^2-initial data. A localization procedure is then used to construct the solution starting from $u(\cdot, t_*)$. By the uniform estimate from Corollary 5.5, this process can be continued after each vanishing moment of some balls. Hence, the solution exists up to any finite time T.

6.4. *Iteration step*

The purpose of this step is to improve the constant $1 - O(1)$ in the right-hand side of (58). This is not absolutely necessary for the later parts from the point of view of estimates and convergence results – all is needed is that $R \in W^{1,p}([0, T])$ and $R^{-1} \in L^1([0, T])$, but we feel it is of independent interest as it gives the limiting asymptotics of $R(t)$ near its extinction time in the strong form.

From the form of the super-solution, we need to progressively reduce a_0 in (56). The expression for the super-solution is simplified as

$$V_0(x, t) = \frac{\delta^4}{\beta|x|}\big(R(0) - R(t)\big) + A + Bt,$$

for some A and B large enough (but independent of time and δ).

Let t_1 be such that $R(t_1) = \frac{R(0)}{2}$. Then

$$V_0(x, t_1) = \frac{\delta^4 R(0)}{2\beta|x|} + A + Bt_1 \geqslant u(x, t_1) \quad \text{(where } u \text{ is the true solution).}$$

Note that

$$\frac{1}{\beta} + A + Bt_1 + \frac{\delta^4}{\beta|x|}\big(R(t_1) - R(t)\big) \geqslant \frac{\delta^4 R(0)}{2\beta|x|} + A + Bt_1 \quad \text{for all } t \geqslant t_1 \text{ and } |x| \geqslant \delta^4 R(t).$$

Hence by the similar argument as before, the function

$$
\begin{aligned}
V_1(x,t) &= \frac{1}{\beta} + A + Bt_1 + \frac{\delta^4}{\beta|x|}\big(R(t_1) - R(t)\big) + A + B(t - t_1)\\
&= \frac{1}{\beta} + 2A + Bt + \frac{\delta^4}{\beta|x|}\big(R(t_1) - R(t)\big),
\end{aligned}
$$

is again a super-solution for $t \geqslant t_1$. Now we have for $t \geqslant t_1$ that

$$
\begin{aligned}
\dot{R} &\leqslant \frac{1}{\beta}\left[V_1 - \frac{1}{R} + g\right] = \frac{1}{\beta}\left[\frac{1}{\beta} + 2A + Bt + \frac{\delta^4(R(t_1) - R(t))}{\beta\delta^4 R(t)} - \frac{1}{R} + g\right]\\
&= \frac{1}{\beta R(t)}\left[-1 + \frac{R(t_1)}{\beta} + R(t)(2A + Bt + g)\right].
\end{aligned}
$$

To continue, let t_2 be the time such that $R(t_2) = \frac{R(0)}{4}$. Set

$$
\begin{aligned}
V_2(x,t) &= \frac{1}{\beta} + \frac{1}{\beta} + 2A + Bt_2 + \frac{\delta^4}{\beta|x|}\big(R(t_2) - R(t)\big) + A + B(t - t_2)\\
&= \frac{2}{\beta} + 3A + Bt + \frac{\delta^4}{\beta|x|}\big(R(t_2) - R(t)\big).
\end{aligned}
$$

It is again a super-solution for $t > t_2$. By induction, let

$$
V_n(x,t) = \frac{n}{\beta} + (n+1)A + Bt + \frac{\delta^4}{\beta|x|}\big(R(t_n) - R(t)\big) \quad \text{where } R(t_n) = \frac{R(0)}{2^n}.
$$

Finally, let

$$
V^*(x,t) = \inf_n V_n(x,t), \tag{69}
$$

which stands as a super-solution for all $t > 0$. Therefore we obtain

$$
\dot{R} \leqslant \frac{1}{\beta R}\left[-1 + \frac{R(t_n)}{\beta} + R(t)\left(\frac{n}{\beta} + (n+1)A + Bt + g(t)\right)\right] \quad \text{for } t_n \leqslant t \leqslant t_{n+1}.
$$

The above shows that

$$
R\dot{R} \leqslant -\frac{1}{\beta} \quad \text{as } R \to 0^+.
$$

We summarize the conclusion of Sections 6.3 and 6.4, in the following existence and regularity theorem for the system (24).

Theorem 6.3. *Let the initial data u_0, R_{i0} and the inhomogeneous driving forces g_i satisfy the conditions* (25), (26) *and* (27). *Then for any time $T < \infty$ and δ small enough,*

1. *there is a solution u of* (24) *in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ satisfying*

$$
\sup_{t\in[0,T]} \big\|u(t)\big\|^2_{L^2(\Omega)} + \frac{1}{\delta}\int_0^T \big\|\nabla u(t)\big\|^2_{L^2(\Omega)}\,dt \leqslant M < \infty; \tag{70}
$$

2. *the radii R_i's satisfy* $\sup_i \sup_{t\geqslant 0} R_i(t) < \infty$ *and* $\sup_i \|R_i\|_{W^{1,p}([0,\min(t_i,T)])} \leqslant M < \infty$ *for any* $1 \leqslant p < 2$. *Furthermore, we have that*

$$|R_i\dot{R}_i| \leqslant M < \infty \quad \textit{and} \quad \lim_{t\to t_i^-} R_i\dot{R}_i = -\frac{1}{\beta}, \tag{71}$$

so that for $t < t_i$,

$$C_1(t_i - t)^{\frac{1}{2}} \leqslant R_i(t) \leqslant C_2(t_i - t)^{\frac{1}{2}} \quad \textit{and} \quad \int_0^{t_i} \frac{1}{R_i^p(t)}\,dt \leqslant C \quad \textit{for any } p < 2. \tag{72}$$

With the above existence result for our system and the regularity of the evolving radii, our approach now follows quite closely to that of [14]. The steps include: (i) construction of a first order approximation for the heat distribution (Section 7); (ii) construction of a first order approximation for the radii (Section 8); and (iii) derivation of the limit equations as $\delta \to 0$ (Section 9). We will still outline the main steps to keep the paper self-contained and to emphasize the essential features, in particular the derivation of the limit equations. On the other hand, there are some differences in the procedure which we will point out in appropriate places.

7. First order approximation for heat distribution

The goal here is to produce a good approximation for the heat distribution which is then used to derive the limiting equation for the dynamics of the mean field variable u_∞ and radii R_i's as $\delta \to 0$. This is facilitated by the following expression:

$$\zeta(x,t) = u_\infty(t) + \sum_i \left(\frac{1 - R_i(t)u_\infty(t) - R_i(t)g_i(t)}{R_i(t) + \beta}\right)\frac{\delta^4 R_i(t)}{|x - x_i|}. \tag{73}$$

Using the above, we will construct sub- and super-solutions to control the difference between the actual solution u and R_i's (from (24)) and the approximation ζ.

For this, we define

$$u_\pm = \zeta + w + z \pm M\delta^\gamma, \tag{74}$$

where the correction functions w and z satisfy

$$\begin{aligned} &\delta w_t = \triangle w - \delta\partial_t u_\infty(t) \quad \text{in } \Omega_T, \\ &\nabla w \cdot n = -\nabla\zeta \cdot n \quad \text{on } \partial\Omega, \\ &w(0,\cdot) = w_0(\cdot), \end{aligned} \tag{75}$$

and

$$\begin{aligned} &\delta z_t = \triangle z - \delta \sum_i \left(\frac{(1 - R_i(t)u_\infty(t) - R_i g_i(t))R_i(t)}{R_i(t) + \beta}\right)_t \frac{\delta^4}{|x - x_i|} \quad \text{in } \Omega_{L,T}, \\ &z = \frac{\beta}{4\pi\delta^4 R_i^2(t)} \int_{\partial B_i} \nabla z \cdot n \quad \text{on } \partial B_i, \end{aligned}$$

$$\nabla z \cdot n = 0 \quad \text{on } \partial\Omega,$$

$$z(0,\cdot) = z_0(\cdot), \tag{76}$$

which are used to handle the inhomogeneous boundary conditions on $\partial\Omega$ and the ∂B_i's. Their initial data are chosen as $z_0 \equiv 0$ and $w_0 = u_0 - \zeta_0$ so that all the boundary conditions are satisfied at $t = 0$. The M is chosen to be large enough so that $u_- \leqslant u_0 \leqslant u_+$ at $t = 0$.

The estimates for w are summarized by the following lemma.

Lemma 7.1. *If we choose the mean-field variable $u_\infty(t)$ according to*

$$\partial_t u_\infty(t) = 4\pi\delta^3 \sum_i \big(1 - R_i(t)u_\infty(t) - R_i(t)g_i(t)\big)\frac{R_i(t)}{R_i(t)+\beta}, \qquad u_\infty(0) = u_{\infty 0}, \tag{77}$$

then for any $0 < \gamma < \frac{1}{2}$, there exists an M_γ such that

$$\|w\|_{L^\infty(\Omega_T)} \quad \text{and} \quad \|\nabla w\|_{L^\infty(\Omega_T)} \leqslant M_\gamma \delta^\gamma. \tag{78}$$

The proof is omitted as it is exactly the same as [14, Lemma 20] using careful energy type estimates from parabolic regularity theory. But for completeness we will indicate the origin of (77). This equation is to ensure that $\int_\Omega w = 0$ so that the behavior of u far away from the interfaces is indeed captured by the mean-field variable u_∞. In addition, technically speaking, the estimate for ∇w is proved first which together with the zero mean condition then gives the estimate for w. With this in mind, we integrate (75) and obtain

$$0 = \delta\frac{d}{dt}\int_\Omega w = \int_{\partial\Omega} \triangle w - \delta\partial_t u_\infty = \int_{\partial\Omega} \nabla w \cdot n - \delta\partial_t u_\infty.$$

Hence, it follows that

$$\delta\partial_t u_\infty = \int_{\partial\Omega} \frac{\partial w}{\partial n} = -\int_{\partial\Omega} \frac{\partial \zeta}{\partial n} = -\int_{\partial\Omega} \sum_i \left(\frac{1 - R_i(t)u_\infty(t) - R_i(t)g_i(t)}{R_i(t)+\beta}\right)\delta^4 R_i(t)\nabla\frac{1}{|x-x_i|}\cdot n.$$

As $\int_{\partial\Omega} \nabla\frac{1}{|x|}\cdot n = -4\pi$, the above gives (77).

The estimates for z are stated in the next lemma.

Lemma 7.2. *In the following, M denotes some generic finite constant independent of δ.*

1. *Let t_i be the vanishing time of B_i, then*

$$\big|z(t)\big|_{\partial B_i} \leqslant M_T\big|\log(t_i - t)\big| \quad \text{for } t < t_i. \tag{79}$$

2. *Let $A = \Omega \setminus \bigcup_i B(x_i, \frac{\delta}{4})$, then*

$$\sup_{t\in[0,T]} \frac{1}{\delta^2}\int_\Omega \big(z(t)\big)^2 + \frac{1}{\delta^3}\int_0^T\int_\Omega |\nabla z|^2 + \frac{1}{\delta}\int_0^T\int_A \big|D^2 z\big|^2 \leqslant M. \tag{80}$$

By Sobolev embedding theorem, the above gives

$$\|z\|_{L^2(0,T,L^\infty(A))} \leqslant M\sqrt{\delta}. \tag{81}$$

Proof. The proof is similar to [14, Lemma 21] using energy type estimates for parabolic equation, but in the current case with the effect of kinetic undercooling in the parabolic setting, some additional terms appear in the derivation of some energy identities. This leads to the need of estimates of the type (79).

We write (76) in the following form:

$$\delta z_t = \triangle z - \delta h, \quad \text{where } h = \sum_i \left(\frac{(1 - R_i(t)u_\infty(t) - R_i g_i(t))R_i(t)}{R_i(t) + \beta} \right)_t \frac{\delta^4}{|x - x_i|}.$$

Multiplying the above equation by z and extending z from Ω_L to Ω by $z|_{B_i} = z|_{\partial B_i}$ lead to

$$\delta \int_{\Omega_L(t)} z_t z = \int_{\Omega_L(t)} \triangle z z - \delta \int_{\Omega_L(t)} hz,$$

$$\delta \int_\Omega z_t z - \delta \int_{\Omega \setminus \Omega_L(t)} z_t z = \int_{\partial \Omega_S(t)} z \frac{\partial z}{\partial n} - \int_\Omega |\nabla z|^2 - \delta \int_\Omega hz,$$

$$\delta \int_\Omega z_t z - \delta \sum_i \left(\frac{4\pi \delta^{12} R_i^3}{3} \right) \dot{z}_i z_i = -\sum_i 4\pi \delta^8 R_i^2 z_i (z_n)_i - \int_\Omega |\nabla z|^2 - \delta \int_\Omega hz,$$

where $z_i = z|_{\partial B_i}$ and $(z_n)_i = \frac{\partial z}{\partial n}|_{\partial B_i}$. As $z_i = \beta \delta^4 (z_n)_i$, the above becomes

$$\delta \int_\Omega z_t z + \sum_i \frac{4\pi \delta^8 R_i^2 z_i^2}{\beta \delta^4} + \int_\Omega |\nabla z|^2 = \delta \sum_i \left(\frac{4\pi \delta^{12} R_i^3}{3} \right) \dot{z}_i z_i - \delta \int_\Omega hz, \tag{82}$$

or

$$\delta \frac{d}{dt} \int_\Omega \frac{1}{2} z^2 + \frac{4\pi \delta^4}{\beta} \sum_i R_i^2(t) z_i^2(t) + \int_\Omega |\nabla z|^2 = \frac{4\pi \delta^{13}}{3} \sum_i R_i^3(t) \left(\frac{z_i^2}{2} \right)_t - \delta \int_\Omega hz. \tag{83}$$

Integrating in time then gives

$$\begin{aligned} &\delta \int_\Omega \frac{1}{2} z^2(t) + \frac{4\pi \delta^4}{\beta} \int_0^t \sum_i R_i^2(s) z_i^2(s)\, ds + \int_0^t \int_\Omega |\nabla z|^2 + \delta \int_0^t \int_\Omega hz \\ &= \frac{4\pi \delta^{13}}{3} \sum_i R_i^3(t) \left(\frac{z_i^2}{2} \right)(t) - \frac{4\pi \delta^{13}}{3} \int_0^t \sum_i 3R_i^2(s) \dot{R}_i(s) \left(\frac{z_i^2}{2} \right)(s)\, ds \\ &\quad + \delta \int_\Omega \frac{1}{2} z^2(0) - \frac{4\pi \delta^{13}}{3} \sum_i R_i^3(0) \left(\frac{z_i^2}{2} \right)(0). \end{aligned} \tag{84}$$

From the above, we see that the $z_i(t)$'s appear in the right-hand side which forces us to consider their estimate.

As $\sup_{t\in[0,T]}\{\sup_i R_i(t), |R_i(t)g_i(t)|\} < \infty$, we simplify Eq. (76) as

$$\delta z_t = \triangle z - \delta \sum_i \frac{\delta^4 (A_i(t) + B_i(t)\dot{R}_i(t))}{|x - x_i|}, \tag{85}$$

where A_i and B_i are some uniformly bounded smooth functions. We construct sub- and super-solutions for z by

$$z_{\text{super}}(t) = M_1 + \sum_i \frac{\delta^4 a_i(t)}{|x - x_i|} \quad \text{and} \quad z_{\text{sub}}(t) = -M_1 - \sum_i \frac{\delta^4 a_i(t)}{|x - x_i|},$$

where $\dot{a}_i(t) = M_2 + M_3|\dot{R}_i|$. M_1, M_2 and M_3 are large enough constants. (This is similar to the construction of the super-solution V in (56).) Then (79) follows from

$$\begin{aligned}
\left|z_i(t)\right| &\leqslant M_1 T + \int_0^t \frac{\dot{a}(s)}{R_i(s)}\, ds \leqslant M_1 T + \int_0^t \frac{M_2 + M_3|\dot{R}(s)|}{R_i(s)}\, ds \\
&= M_1 T + \int_0^t \frac{M_2 R_i + M_3 |R_i(s)\dot{R}_i(s)|}{R_i^2(s)}\, ds \\
&\leqslant M_1 T + M \int_0^t \frac{1}{R_i^2(s)}\, ds \leqslant M_1 T + M \int_0^t \frac{1}{(t_i^\delta - s)}\, ds \leqslant M_1 T + M\left|\log(t_i - t)\right|.
\end{aligned}$$

By Theorem 6.3(2), we see that the right-hand side of (84) is bounded by a finite constant. Then the same computations of [14, Lemma 21, pp. 172–173] can be applied. They first give

$$\int_\Omega z^2 + \frac{1}{\delta} \int_0^t \int_\Omega |\nabla z|^2 \leqslant M\delta^2,$$

and then the higher order regularity result follows

$$\sup_{t\in[0,T]} \frac{1}{\delta^2} \int_\Omega \bigl(z(t)\bigr)^2 + \frac{1}{\delta^3} \int_0^T \int_\Omega |\nabla z|^2 + \frac{1}{\delta} \int_0^T \int_A \left|D^2 z\right|^2 \leqslant M.$$

These conclude the proof of (81).

(Note here that we do not need to any give special consideration for new initial data right after some balls have vanished such as in [14, p. 167]. This is because the summands in ζ (73) corresponding to the vanishing R_i's automatically become zero.) □

Estimates (78) and (81) together with (73) and (74) give the following corollary which says that *far away* from the particles, the heat distribution u is close to the mean field variable u_∞.

Corollary 7.3. *For any $0 < \gamma < \frac{1}{2}$, there is a constant M_γ such that*

$$\left\| u - u_\infty(t) \right\|_{L^2([0,T],L^\infty(A))} \leqslant M_\gamma \delta^\gamma. \tag{86}$$

8. Approximation of the dynamics of the radii

The following is the main theorem of this paper which gives the dynamics of the radii as $\delta \to 0$.

Theorem 8.1. *Let u_∞ be given as in (77). Then for any $i \in \mathcal{N}(t)$ and $\varphi \in W^{1,1}([0,T])$, it holds that*

$$\left| \int_0^{T\wedge t_i} \varphi\left[R_i(R_i+\beta)\dot{R}_i - (u_\infty R_i + g_i R_i - 1)\right] dt \right| \leqslant C_\gamma \delta^\gamma \|\varphi\|_{W^{1,1}}. \tag{87}$$

The above means that in the weak sense, the radii satisfy the following dynamical equation:

$$\dot{R}_i = -\frac{1 - u_\infty R_i - g_i R_i}{R_i(R_i+\beta)}. \tag{88}$$

The proof is the same as [14, Theorem 2.b]. As this is the key result, we present the steps here to illustrate the main idea and estimates.

Proof of Theorem 8.1. Define

$$\psi_i(x,t) = \frac{\delta^4 R_i(t)}{|x-x_i|} \eta\left(\frac{|x-x_i|}{\delta}\right),$$

where η is a smooth function such that $\eta(s) \equiv 1$ for $0 \leqslant s \leqslant \frac{1}{8}$ and $\eta(s) \equiv 0$ for $s \geqslant \frac{1}{4}$. This function satisfies

$$\psi_i|_{\partial B_i} = 1, \qquad \frac{1}{4\pi\delta^4} \int_{\partial B_i} \nabla\psi_i \cdot n = -R_i,$$

and the identity,

$$\int_{\Omega_L} \psi_i \triangle u = -\int_{\partial B_i} \psi_i \frac{\partial u}{\partial n} + \int_{\partial B_i} u \frac{\partial \psi_i}{\partial n} + \int_{\Omega_L} u \triangle \psi_i.$$

Using the dynamics of $R_i(t)$, we have $\frac{d}{dt}(\frac{1}{3}R_i^3(t)) = \frac{1}{4\pi\delta^4}\int_{\partial B_i} \nabla u \cdot n$ from which we compute

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{3}R_i^3\right) &= \frac{1}{4\pi\delta^4}\int_{\partial B_i} \psi_i \nabla u \cdot n = \frac{1}{4\pi\delta^4}\int_{\partial B_i} u \frac{\partial\psi_i}{\partial n} - \frac{1}{4\pi\delta^4}\int_{\Omega_L} \psi_i \triangle u + \frac{1}{4\pi\delta^4}\int_{\Omega_L} u \triangle\psi_i \\
&= \frac{u_i}{4\pi\delta^4}\int_{\partial B_i} \frac{\partial\psi_i}{\partial n} + \frac{1}{4\pi\delta^4}\int_{\Omega_L} \left(u - u_\infty(t)\right)\triangle\psi_i \\
&\quad - \frac{\delta}{4\pi\delta^4}\int_{\Omega_L} \psi_i u_t + \frac{u_\infty(t)}{4\pi\delta^4}\int_{\Omega_L} \triangle\psi_i \quad (\text{as } \delta u_t = \triangle u)
\end{aligned}$$

$$= -R_i u_i - \frac{u_\infty(t)}{4\pi\delta^4}\int_{\partial B_i}\frac{\partial\psi_i}{\partial n} + \frac{1}{4\pi\delta^4}\int_{\Omega_L}\big(u-u_\infty(t)\big)\triangle\psi_i - \frac{\delta}{4\pi\delta^4}\int_{\Omega_L}\psi_i u_t$$

$$= -R_i\left(\frac{1}{R_i} - g_i + \beta\dot{R}_i\right) + u_\infty(t)R_i(t) + \frac{1}{4\pi\delta^4}\int_{\Omega_L}\big(u-u_\infty(t)\big)\triangle\psi_i$$

$$- \frac{\delta}{4\pi\delta^4}\int_{\Omega_L}\psi_i u_t \quad \left(\text{as } u_i = \frac{1}{R_i} - g_i + \beta\dot{R}_i\right).$$

Hence, we obtain

$$\begin{aligned} &R_i(R_i+\beta)\dot{R}_i - (u_\infty R_i + g_i R_i - 1) \\ &\quad = \frac{1}{4\pi\delta^4}\int_{\Omega_L}\big(u-u_\infty(t)\big)\triangle\psi_i - \frac{\delta}{4\pi\delta^4}\int_{\Omega_L}\psi_i u_t. \end{aligned} \tag{89}$$

Now let φ be a test function on $[0,T]$. Then we have

$$\begin{aligned} &\int_0^T \varphi\big[R_i(R_i+\beta)\dot{R}_i - (u_\infty R_i + g_i R_i - 1)\big]\,dt \\ &\quad = \int_0^T \varphi\left[\int_{\Omega_L}\frac{(u-u_\infty(t))\triangle\psi_i}{4\pi\delta^4}\right]dt - \delta\int_0^T \varphi\left[\int_{\Omega_L}\frac{\psi_i u_t}{4\pi\delta^4}\right]dt. \end{aligned} \tag{90}$$

The first term of the right-hand side of (90) is estimated as

$$\begin{aligned} \int_0^T \varphi\left[\int_{\Omega_L}\frac{(u-u_\infty(t))\triangle\psi_i}{4\pi\delta^4}\right]dt &\leqslant \|\varphi\|_{L^\infty([0,T])}\big\|u-u_\infty(t)\big\|_{L^\infty(\mathrm{supp}(\triangle\psi_i))} \times \frac{1}{4\pi\delta^4}\int_{\mathrm{supp}(\triangle\psi_i)}|\triangle\psi_i| \\ &\leqslant C_\gamma\delta^\gamma\|\varphi\|_{L^\infty([0,T])}. \end{aligned}$$

For the second term, we compute,

$$\begin{aligned} \int_0^T \varphi\int_{\Omega_L}\frac{\psi_i u_t}{4\pi\delta^4}\,dt &= \int_0^T \frac{\varphi}{4\pi\delta^4}\left[\int_{\Omega_L}\big((u\psi_i)_t - u\psi_{i_t}\big)\right]dt \\ &= \int_0^T \frac{\varphi}{4\pi\delta^4}\left[\int_{\Omega_L}(u\psi_i)_t - \int_{\Omega_L}u\frac{\delta^4\dot{R}_i}{|x-x_i|}\eta\left(\frac{|x-x_i|}{\delta}\right)\right]dt. \end{aligned}$$

Note that $\int_{\Omega_L}(u\psi_i)_t = (\int_{\Omega_L}u\psi_i)_t + (u\psi_i)|_{\partial B_i}(\delta^4\dot{R}_i)(4\pi\delta^8 R_i^2)$. Hence, we arrive at

$$\int_0^T \varphi \int_{\Omega_L} \frac{\psi_i u_t}{4\pi\delta^4}\,dt = -\int_0^T \varphi_t \int_{\Omega_L} \frac{u\psi_i}{4\pi\delta^4}\,dt - \varphi(0)\int_{\Omega_L} \frac{u(\cdot,0)\psi_i}{4\pi\delta^4} + \int_0^T \frac{\varphi\delta^8}{4\pi}\left(\frac{1}{R_i} - g_i + \beta\dot{R}_i\right)\dot{R}_i R_i^2\,dt$$
$$- \int_0^T \frac{\varphi\dot{R}}{4\pi}\int_{\Omega_L}\frac{u\eta}{|x-x_i|}\,dt$$
$$= -\int_0^T \varphi_t \int_{\Omega_L} \frac{u\psi_i}{4\pi\delta^4}\,dt - \varphi(0)\int_{\Omega_L} \frac{u(\cdot,0)\psi_i}{4\pi\delta^4} + \int_0^T \frac{\varphi\delta^8}{4\pi}\big(R_i\dot{R}_i - g_i R_i^2\dot{R}_i + \beta\dot{R}_i^2 R_i^2\big)\,dt$$
$$- \int_0^T \frac{\varphi\dot{R}}{4\pi}\int_{\Omega_L}\frac{u\eta}{|x-x_i|}\,dt.$$

Using the facts that

$$\|u\|_{L^\infty(0,T,L^2(\Omega))},\ \left\|\frac{1}{|x|}\right\|_{L^2(\Omega)},\ \left\|\frac{\psi_i}{4\pi\delta^4}\right\|_{L^\infty(0,T,L^2(\Omega))},\ \|R_i\dot{R}_i\|_{L^\infty(0,T)},\ \|\dot{R}_i\|_{L^1(0,T)} \leqslant M,$$

we finally have the conclusion:

$$\left|\int_0^T \varphi\big[R_i(R_i+\beta)\dot{R}_i - (u_\infty R_i + g_i R_i - 1)\big]\,dt\right| \leqslant M_\gamma \delta^\gamma \|\varphi\|_{W^{1,1}(0,T)}. \qquad \square \tag{91}$$

9. Limit problem as $\delta \to 0$

This section presents and proves the main result of this paper: the limit description of u and R_i's as $\delta \to 0$. Here for clarity, we recover the super-script δ in u^δ, u_∞^δ and R_i^δ to emphasize their dependence on δ.

With the estimates derived so far, all the results of [14,15] in principle carry over. However, in order to obtain an equation which is *closed* in the limit, we do need to invoke the assumption (28) on the form of the inhomogeneous forces g_i's. This will also motivate the incorporation of white noise in the future work so that the machinery of stochastic analysis is applicable.

Since the estimates are the same as those in [14,15], we will omit the proof of the existence of a limit which is a consequence of general compactness results. Instead, we will concentrate on the derivation of the limit equations. For this, we introduce the empirical measure $\nu^\delta \in L^1(0,T;C^0(0,K_T]))^*$ of the radii:

$$\langle \nu^\delta, \varphi\rangle = \int_0^T \frac{1}{N(t)}\sum_{i\in\mathcal{N}(t)} \varphi\big(t, R_i^\delta(t)\big)\,dt \quad \text{for } \varphi \in L^1\big([0,T];C^0[0,K_T]\big), \tag{92}$$

where $K_T = \sup_{i,\delta}\|R_i^\delta\|_{L^\infty(0,T)}$. Then we have the following convergence result:

Lemma 9.1. *Given any $T<\infty$, there exist a $\nu^* \in L^1(0,T;C^0[0,K_T])^*$ and $u_\infty^* \in W^{1,p}(0,T)$ $(1 \leqslant p < \infty)$ such that for a subsequence of $\delta \to 0$, the following hold*:

$$\nu^\delta \rightharpoonup \nu^* \quad \textit{in the weak* topology of } L^1\big(0,T;C^0[0,K_T]\big)^*, \tag{93}$$

$$u_\infty^\delta \to u_\infty^* \quad \textit{uniformly in } (0,T), \tag{94}$$

$$u^\delta \to u_\infty^* \quad \textit{in } L^2\big(0,T;H^1(\Omega)\big). \tag{95}$$

Furthermore, there exists a family of probability measures $\{\nu_t^*\}_{t\geqslant 0} \subset C^0[0,K_T]^*$ *and a non-negative function* $\alpha \in L^\infty(0,T)$ *such that*

$$\big\langle \nu^*, \varphi\big\rangle = \int_0^T \int \varphi(t,R)\, d\nu_t^*(R)\,\alpha(t)\,dt \quad \textit{for } \varphi \in L^1\big(0,T;C^0[0,K_T]\big)^*. \tag{96}$$

In the above, $\alpha(t) = \lim_{\delta\to 0} \frac{N(t)}{N(0)}$ *represents the percentage of active particles in the system.*

The proof of the above is some application of convergence of measures and L^p spaces. The specific concept used is that of Young measures. For details, see [14, Lemmas 7, 8] and [15, Lemma 5.1].

In order to have a closed equation in the limit, we state here again the assumption about the functional form for the $g_i(t)$'s:

$$\text{there exist a function } G \in C^1(R_+ \times R_+) \text{ and a function } h \in C^1(R_+) \text{ such that}$$
$$g_i(t) = G\big(t, R_i(t)\big) + h(t). \tag{28}$$

We will make some remarks about this assumption after presenting the main theorem which is stated as follows:

Theorem 9.2. *The mean field variable* u_∞^* *and the distribution* ν^* *satisfy*

$$\partial_t u_\infty^*(t) = 4\pi \int_O^\infty \big(1 - Ru_\infty^*(t) - RG(t,R) - Rh(t)\big)\frac{R}{R+\beta}\, d\nu_t^*(R)\,\alpha(t)\,dt, \tag{97}$$

and

$$\int_0^T \int \big\{\partial_t\varphi(t,R) + V(t,R)\partial_R\varphi(t,R)\big\}\, d\nu_t^*(R)\alpha(t)\,dt + \int \varphi(0,R)\,d\nu_0^*(R) = 0 \tag{98}$$

for all $\varphi \in C_0^\infty([0,T]\times R_+)$*, where*

$$V(t,R) = -\frac{1 - Ru_\infty^*(t) - RG(t,R) - Rh(t)}{R(R+\beta)}, \tag{99}$$

and ν_0^* *is the limit of the empirical measure of the initial radii* R_{i0}^δ*.*

Proof. For (97), let $\eta \in C_0^1(0,T)$. Then we get

$$\int_0^T \eta(t)\big(u_\infty^\delta\big)_t\,dt = \int_0^T \eta(t)\left[4\pi\delta^3 \sum_i \big(1 - R_i^\delta u_\infty^\delta - R_i^\delta g_i\big)\frac{R_i^\delta}{R_i^\delta + \beta}\right]dt.$$

For the left-hand side of the above, we have

$$\int_0^T \eta(t)\big(u_\infty^\delta\big)_t\,dt = -\int_0^T \eta_t(t)u_\infty^\delta\,dt \to -\int_0^T \eta_t(t)u_\infty^*\,dt = \int_0^T \eta(t)\big(u_\infty^*\big)_t\,dt.$$

Considering the right-hand side, we express it in terms of the empirical measure ν^δ:

$$\int_0^T \eta(t)\bigg[4\pi\delta^3\sum_i\big(1-R_i^\delta u_\infty^\delta - R_i^\delta g_i\big)\frac{R_i^\delta}{R_i^\delta+\beta}\bigg]\,dt = \big\langle \nu^\delta, \Phi^\delta\big\rangle,$$

where $\Phi^\delta(t,R) = 4\pi\eta(t)[1-Ru_\infty^\delta(t) - R(G(t,R)+h(t))]\frac{R}{R+\beta}$. By the strong convergence of u_∞^δ to u_∞^* and the form of g_i's, we have that

$$\big\langle \nu^\delta, \Phi^\delta\big\rangle \to \int_0^T \eta(t)\int 4\pi\big(1-Ru_\infty^* - RG(t,R) - Rh(t)\big)\frac{R}{R+\beta}\,d\nu_t^*(R)\alpha(t)\,dt,$$

which gives (97).

For (98), consider for any $\phi \in C_0^\infty([0,T],R_+)$:

$$\int_0^T \eta(t)\bigg[\frac{1}{N}\sum_{i\in\mathcal{N}}\frac{d}{dt}\phi\big(t,R_i^\delta(t)\big)\bigg]\,dt + \frac{1}{N}\sum_{i\in\mathcal{N}}\phi\big(0,R_{i0}^\delta\big)\,dt = 0.$$

The convergence of the second term is trivial. For the first term, we compute

$$\int_0^T \eta(t)\bigg[\frac{1}{N}\sum_{i\in\mathcal{N}}\frac{d}{dt}\phi\big(t,R_i^\delta(t)\big)\bigg]\,dt = \int_0^T \eta(t)\bigg[\frac{1}{N}\sum_{i\in\mathcal{N}}\phi_t\big(t,R_i^\delta(t)\big)\bigg]\,dt + \int_0^T \eta(t)\bigg[\frac{1}{N}\sum_{i\in\mathcal{N}}\phi_R\big(t,R_i^\delta(t)\big)\dot{R}_i^\delta\bigg]\,dt.$$

The first term on the right becomes

$$\int_0^T \eta(t)\bigg[\frac{1}{N}\sum_{i\in\mathcal{N}}\phi_t\big(t,R_i^\delta(t)\big)\bigg]\,dt = \big\langle \nu^\delta, \eta\partial_t\phi\big\rangle \to \big\langle \nu^*, \eta\partial_t\phi\big\rangle.$$

For the second term, we compute

$$\int_0^T \eta(t)\bigg[\frac{1}{N}\sum_{i\in\mathcal{N}}\phi_R\big(t,R_i^\delta(t)\big)\dot{R}_i^\delta\bigg]\,dt = \int_0^T \eta(t)\bigg[\frac{1}{N}\sum_{i\in\mathcal{N}}\phi_R\big(t,R_i^\delta(t)\big)\big(\dot{R}_i^\delta - V\big(t,R_i^\delta\big)\big)\bigg]\,dt + \int_0^T \eta(t)\bigg[\frac{1}{N}\sum_{i\in\mathcal{N}}\phi_R\big(t,R_i^\delta(t)\big)V\big(t,R_i^\delta\big)\bigg]\,dt.$$

As ϕ has compact support, only the values of the radii which are bounded away from zero matter in the computation. Hence a trivial modification of the proof of Theorem 8.1, in particular the steps (89) and (90) give

$$\int_0^T \eta(t)\left[\frac{1}{N}\sum_{i\in\mathcal{N}}\phi_R\big(t, R_i^\delta(t)\big)\big(\dot{R}_i^\delta - V\big(t, R_i^\delta\big)\big)\right]dt \to 0.$$

Finally we have the convergence result:

$$\int_0^T \eta(t)\left[\frac{1}{N}\sum_{i\in\mathcal{N}}\phi_R\big(t, R_i^\delta(t)\big)V\big(t, R_i^\delta\big)\right]dt \to \langle \nu^*, \eta\phi_R\rangle,$$

which all together gives (98), completing the proof of the theorem. □

Remark 9.3. Here we explain the need to impose the functional form (28) for the inhomogeneous forces. From the derivation of the limit equations, we are forced to deal with summations in the form of

$$\int_0^T \varphi(t)\sum_i F\big(t, R_i(t), \{R_j(s)\}_{j,\,0\leqslant s\leqslant t}, g_i(t)\big)\,dt \quad \text{for some nonlinear function } F.$$

The dependence on $\{R_j(s)\}_{j,\,0\leqslant s\leqslant t}$ is through the mean-field variable $u_\infty^\delta(t)$. In principle the above can all be expressed in terms of some Young measures. But it is not clear if there is any meaningful equation we can obtain to describe these Young measures. The limit equations will thus not be closed – the usual problem when dealing with weak convergence in nonlinear equations. Imposing some probabilistic independence among the g_i does not help immediately due to the non-local dependence in time. A reasonable alternative is to consider *white noise* for the g_i's so that techniques from Itô's calculus can be used to take advantage of the stochastic cancellation in time. Such an approach is used in many works deriving continuum equations from particle systems with mean-field or long range interactions. This will be investigated in some future works.

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