Torsion invariants in symplectic Floer theory *

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1 Arnold's conjecture and Floer homology

A motivating problem in symplectic topology is the Arnold's conjecture on estimating the symplectic fixed point numbers. In the long history of this problem, the biggest breakthrough was arguably due to Floer, via the introduction of the "Floer homology".

Theorem 1.1 *[10] Let (M, ω) be a monotone symplectic manifold, ϕ be a Hamiltonian symplectomorphism of M. Then the number of fixed points of ϕ is at least $\sum_i b_i(M)$, where b_i can be the betti number of M in any coefficient.*

In the above, a symplectic manifold M is a manifold with a "symplectic form" ω, which is a closed 2-form that is nondegenerate everywhere. A symplectomorphism ϕ is a diffeomorphism of M preserving the symplectic form: $\phi^*\omega = \omega$. It is Hamiltonian if it is isotopic to the identity via a path of generating vector fields X_t defined by $i(X_t)\omega = dH_t$, $t \in [0, 1]$. H_t is a path of smooth functions on M. Monotonicity means $c_1(M)\Big|_{\pi_2(M)} = k\omega\Big|_{\pi_2(M)}$ for $k > 0$ as homomorphisms $H_2(M) \to \mathbb{R}$. This assumption guarantees the compactness of the relevant moduli spaces.

The idea of proof of Floer's theorem is to do formal Morse theory on the infinite-dimensional free loop space $\mathcal{L}M$ (more specifically, the component of contractible loops). Given ω, a path of compatible almost complex structure J_t on M, and H_t generating the symplectomorphism ϕ, one may write down a L^2-metric on $\mathcal{L}M$, and a function

*written for the proceedings of ICCM 2001 on $\mathcal{L}M$–the action functional $\mathcal{A}_H$. Formally, the critical points of $\mathcal{A}_H$ correspond to periodic orbits of H_t, which in turn may be identified with the fixed points of ϕ; the gradient flow lines correspond to (perturbed) pseudo-holomorphic cylinders ending at Hamiltonian orbits, and the Morse complex constructed from all these is the so-called Floer complex: The chain groups of this complex are free modules generated by the Hamiltonian orbits, and the boundary maps are defined by counting pseudo-holomorphic cylinders connecting two Hamiltonian orbits with index difference 1. The homology of the Floer complex is the Floer homology $HF_*(M)$.

Of course, it is highly nontrivial to make this heuristic picture actually work; one needs to establish rather technical transversality and compactness results. For details, the reader is referred to Floer's original papers and many good survey articles on this subject by Salamon, Schwarz, et al. Here we only point out that the key to Floer's proof is the *invariance* of the Floer homology under Hamiltonian isotopies (i.e. varying H_t in the definition of Floer homology). Via this invariance one is able to reduce the computation of the Floer homology to small, t-independent Hamiltonian H, and in this case the Floer homology reduces to the Morse homology of H, namely, the usual homology of M.

Naively, one would expect such an invariance proof to done by varying H_t, and see how the corresponding Floer complex changes (i.e. bifurcation analysis of the corresponding Morse theory). However, this is *not* the usual way it is done in the current Floer-theory literature. The typical method is the so-called "continuation method" : given a Hamiltonian isotopy H_s between two (t-dependent) Hamiltonians H_0, H_1, there is a natural way of defining a chain map $\mathcal{H}$ between the Floer complexes of H_0 and H_1 via moduli spaces associated to H_s. One then shows that $\mathcal{H}$ is a homotopy equivalence.

A frequent question raised in lectures on Floer theory is "why continuation method instead of bifurcation analysis"? We shall answer this question in section 7 below. However, we emphasize here that this continuation method does not work in situations we'd like to consider below.

The technical monotonicity assumption in Floer's theorem has subsequently been weakened through contributions from many people; recently, the completely general version of Arnold's conjecture has been established via the "virtual moduli technique" by Li-Tian, Ruan, Liu, Fukaya-Ono, Hofer-Salamon, and Siebert.

2 Variants of Floer theory and more refined invariants of the Floer complex

Floer homology, as described above, is useful for estimating the number of symplectic fixed points, because $HF_*(M) = H_*(M)$ and the usual homology $H_*(M)$ is always nonvanishing for compact manifolds. This is however no longer true in many variants of Floer theory.

Example 2.1 There are various ways of defining Floer homologies for symplectic manifolds with contact-type boundaries [54, 5]. Floer homology may be trivial in this situation. E.g. $HF_*(D^2) = 0$ in Viterbo's definition.

Example 2.2 To find *noncontractible* Hamiltonian orbits, one might use the space of noncontractible loops (in a fixed homotopy class) to define Floer homology. However, this version of Floer homology vanishes by its invariance under Hamiltonian isotopy, because for small Hamiltonians, there is obviously no closed orbit [4].

Example 2.3 There is a "relative" version of Floer homology, the Floer homology of Lagrangian intersections, where one considers the space of paths ending at two different Lagrangian submanifolds instead of the loop space. (This version of Floer homology is not always defined for arbitrary Lagrangian submanifolds.) There are many examples where the Lagrangian intersection Floer homology vanishes. For example, the Floer homology of compact Lagrangian submanifolds in $\mathbb{C}^n$ vanishes once it is defined and invariant under Hamiltonian isotopy, because it is easy to find Hamiltonian isotopies disengaging one from the other.

Example 2.4 One often needs to consider twisted versions of Floer homology, in particular because the action functional is in general not globally defined; the usual construction only defines a closed 1-form on the loop space. A typical example is the Floer homology of *non-Hamiltonian* symplectomorphisms ϕ. To avoid the technicalities associated with the definition of Floer homologies, here we illustrate the idea of "twisting" in the finite dimensional case. More details of the Floer-theoretic version will be provided in §3.1.

Given a closed 1-form θ and a metric g on a finite dimensional manifold M, consider the flow of $-\check{\theta}^g$, where $\check{\theta}^g$ is the vector field dual to θ with respect to g. When $\theta = df$, this is just the gradient flow of the function f. When $[\theta]$ is rational, this is the flow of a circle-valued function. It is not hard to perturb θ or the metric to make the flow Morse-Smale (such a θ is called a Morse 1-form), but it requires more care to define a Morse complex that makes sense, because there may be infinitely many flow lines between two critical points of index difference 1, for lack of energy bounds. However, if one considers the induced flow on the universal abelian covering $\pi : \tilde{M} \to M$—the regular cover of M with covering group $H_1(M)$, there *are* finitely many flow lines between two critical points of index difference 1 in $\tilde{M}$, because $\pi^*\theta = d\tilde{f}$ is exact. Note that two flow lines in M with the same starting and end points but in different homology class lift to flow lines in $\tilde{M}$ with the same starting points and different end points related by a deck transformation in $H_1(M)$. One idea is thus to "count flows with homology class" (i.e.

work with homology with twisted coefficients). However, the correct coefficient ring is not the group ring $\mathbb{Z}[H_1(M)]$, but a suitable completion of it, called the "Novikov ring" $\mathrm{Nov}(H_1(M), \mathbb{Z}; -[\theta])$. The cohomology class $[\theta] \in H^1(M;\mathbb{R})$ specifies the direction of completion. More specifically, this Novikov ring consists of formal sums $\sum_{g\in H_1(M)} a_g \cdot g$, with $a_g \in \mathbb{Z}$, such that for every $C \in \mathbb{R}$, the set $\{g \in H_1(M) \mid [\theta](g) < C \text{ and } a_g \neq 0\}$ is finite. (See e.g. [20].)

This twisted version of Morse complex is called the Novikov complex, and the homology is the Novikov homology $HN(M, \theta)$. The frequent occurrence of the word "Novikov" here is of course due to the pioneering work of Novikov in this type of generalized Morse theory [39].

For any symplectic isotopy connecting the identity with a non-Hamiltonian symplectomorphism ϕ, Le-Ono [32] defined a twisted version of Floer homology and showed that when M is monotone, its total betti number equals the total betti number of the Novikov homology $HN(M, \theta_\phi)$ by pursuing Floer's strategy. (Le-Ono actually had a slightly weaker assumption on M, and they worked with a smaller covering group.) θ_ϕ above is the "flux" or "Calabi invariant" of ϕ, defined as

$$\theta_\phi = \int_0^1 i(X_t)\omega\, ds,$$

where X_t is a path of symplectic vector fields generating ϕ.

Le-Ono's result guarantees the existence of symplectic fixed points in many cases; however, there are also many examples where this version of Floer homology vanishes. The reason is that *the "twisting" procedure above often reduces the rank of the homology*; the larger the covering group is, the smaller the rank is.

As a most basic example of this phenomenon, consider the twisted cell homology of S^1. (It is well known that the Morse homology is equivalent to the cell-homology; similarly, the Novikov homology is equivalent to twisted versions of the usual homology.) Choose the simplest cell-decomposition of S^1—one 0-cell and one 1-cell. This lifts to an equivariant cell decomposition on the universal abelian covering of S^1–namely $\mathbb{R}$. Let t be a generator of the deck transformation, then the Novikov ring $\mathrm{Nov}(H_1(S^1), \mathbb{Z}; -t) = \mathbb{Z}((t))$. The chain groups C_i are generated by a single generator for $i = 0, 1$, and are trivial otherwise. The boundary operator between C_1, C_0 is $1-t$, which is an isomorphism with respect to the coefficient $\mathbb{Z}((t))$, and hence the homologies are trivial. [1] Note in contrast that the non-twisted version is obtained by setting $t = 1$, and hence there the boundary map is trivial and the homology is nontrivial. The same computation generalizes to mapping tori, yielding many examples of trivial Novikov homology.

[1] Of course, in this particular example, this triviality is most easily seen from the fact that there is a nowhere vanishing Morse 1-form on S^1.

What can one do when the Floer homology is trivial? A natural idea is to look for more refined invariants of the Floer complex, since the homology is only a very rough invariant of a chain complex. This idea has been exploited in the literature. Fukaya proposed to study the A_∞ structure of Floer theory which will capture the full rational homotopy type of the Floer complex. In [15], Fukaya and Oh showed that the A_∞ category of the Lagrangian intersection Floer theory is equivalent to the A_∞ category of the Morse theory for cotangent bundles.

As a first *non-homotopic* invariant, the torsion is the obvious next candidate to consider. Fukaya was again the first person to propose studying torsion in symplectic Floer theories [13]. His motivation was slightly different: trying to address the converse side of the Arnold-type conjectures. If one regards the Floer homology as an obstruction to canceling symplectic fixed points or Lagrangian intersection points, it is natural to ask whether this is the complete obstruction—if the Floer homology vanishes, can all the symplectic fixed points/Lagrangian intersection points be canceled by Hamiltonian isotopies? As a finite dimensional analogue, recall that the h-cobordism theorem says that on cobordisms between high dimensional simply connected manifolds, all critical points of the Morse function can be canceled as long as the homology vanishes. For non-simply connected manifolds, there is an additional obstruction—the Whitehead torsion. The s-cobordism theorem says that in this case, if the homology and the Whitehead torsion both vanish, then the critical points of the Morse function can be canceled. Fukaya sketched the definitions of the Whitehead torsions for Floer theories of symplectomorphisms and Lagrangian intersections, claimed that they are invariant under Hamiltonian isotopies, and conjectured that the Floer homology together with the torsion give complete obstructions of Hamiltonian isotopying a symplectomorphism to one without fixed points, or a Lagrangian submanifold to one without intersection with another fixed Lagrangian submanifold (the "symplectic s-cobordism conjecture").

It is very hard to imagine how an analogue of the Morse lemma may be established in the infinite-dimensional setting of Floer theory, and we shall hence limit ourselves to a more modest goal—understanding the basic properties of torsions in symplectic Floer theory.

3 Reidemeister torsion of the Floer complex

In this section we concentrate on Example 2.4 above. We can not but be rather sketchy here, and the reader is referred to [25] for details and proofs.

3.1 The Floer complex

Let ϕ be a (possibly non-Hamiltonian) symplectomorphism of a symplectic manifold (M,ω). Let T_ϕ be the *mapping torus* of ϕ:

$$T_\phi = \Big\{(x,t) : x \in M, t \in [0,1]\Big\}/(x,0) \sim (\phi(x),1).$$

It is a M-bundle over the circle of unit length S^1_1:

$$\pi_S : T_\phi \to S^1_1.$$

Let K be the subbundle of TT_ϕ consisting of tangent vectors to the fibers. The *ϕ-twisted loop space* $\mathcal{L}_\phi M$ is the space of smooth sections of T_ϕ. Given a base point $\gamma_0 \in \mathcal{L}_\phi M$, let $\mathcal{C} = \mathcal{L}_\phi M_{\gamma_0}$ be the path component of $\mathcal{L}_\phi M$ containing γ_0.

Let J be complex structure of K; it may be regarded as a path of almost complex structures J_t satisfying the appropriate matching conditions at the ends $t = 0,1$. If J_t are all compatible with ω, namely, $g_t := \omega(\cdot, J_t\cdot)$ gives a metric on M, then J defines a metric on $\mathcal{C}$:

$$\langle \xi, \eta\rangle := \int_{S^1_1} \omega(\xi(t), J_t(\gamma(t))\eta(t))\,dt \quad \text{for } \xi,\eta \in T_\gamma\mathcal{C} = L^p_1(\gamma^* K).$$

Let $\theta \in \Gamma(K^*)$ be a path of closed 1-forms θ_t on M satisfying the appropriate matching conditions at $t = 0,1$. θ together with ω define a closed 1-form on $\mathcal{C}$:

$$d\tilde{\mathcal{A}}(\gamma)(\xi) = -\int_{S^1_1} \omega(\partial_t\gamma(t), \xi(t))\,dt + \int \theta_t(\xi(t))\,dt.$$

Formally, the flow of the vector field dual to $d\tilde{\mathcal{A}}$ (with respect to the metric given by J) satisfies a perturbed Cauchy-Riemann equation

$$\frac{\partial u}{\partial s} + J_t(u)\frac{\partial u}{\partial t} + \check{\theta}_t^{\,g_t}(u) = 0. \tag{3.1}$$

In particular, the critical points of this flow are ϕ-periodic orbits of length 1 of the path of symplectic vector fields $J_t\check{\theta}_t^{\,g_t}$. Let $\mathcal{P}$ be the set of these critical points, and let $\tilde{\mathcal{P}}$ be the set of their lifts in the universal abelian covering $\tilde{\mathcal{C}}$.

One may now try to define a twisted version of Floer theory according to the scheme sketched in Example 2.4 above. There is however one important feature of the infinite dimensional version that is different from the finite dimensional picture, namely, different lifts of the same element in $\mathcal{P}$ to the universal abelian covering may have different indices due to spectral flow [1]. We now describe this phenomenon in more detail.

Since an element in $\mathcal{C}$ is a map from S^1 to T_ϕ, there is a natural map

$$\pi : H_1(\mathcal{C}) \to H_2(T_\phi)$$

sending each representative of the former, $\gamma : S^1 \to \mathcal{C}$, to the corresponding map from $S^1 \times S^1 \to T_\phi$. Let $c_1 := c_1(K)$, and let

$$\psi_c := \pi^* c_1.$$

Two elements in $\tilde{P}$ related by a deck transformation $g \in H_1(\mathcal{C})$ have indices differing by $2\psi_c(g)$. Our coefficient ring should therefore *not* be the Novikov ring of the full homology group $H_1(\mathcal{C})$, but of $\ker \psi_c \subset H_1(\mathcal{C})$. Let

$$\Lambda_F^\phi := \mathrm{Nov}(\ker \psi_c, \mathbb{Z}; -d\tilde{\mathcal{A}}).$$

Now the chain groups of the Floer complex $CF(\phi)$ is the free Λ_F^ϕ-module generated by elements in $\tilde{\mathcal{P}}$, on which $\ker \psi_c \subset H_1(\mathcal{C})$ acts by deck-transformation. $CF(\phi)$ is graded by the so-called Conley-Zehnder index,

$$CF(\phi) = \bigoplus_{k\in\mathbb{Z}} CF_k(\phi);$$

each $CF_k(\phi)$ is of finite rank: it is generated by elements in $\mathcal{P}$ of Conley-Zehnder index $k \mod (2N)$, where N is the minimal positive value of ψ_c.

The boundary operator $\partial_F(\phi) = \oplus_k \partial_{F,k}(\phi)$ is defined as follows. Let $[x, w], [y, v] \in \tilde{\mathcal{P}}$ with $\mathrm{ind}([x, w]) = k$, $\mathrm{ind}([y, v]) = k - 1$.

$$\langle [y, v], \partial_{F,k}(\phi)[x, w] \rangle := \sum_{u \in \hat{\mathcal{M}}_P([x,w],[y,v])} \mathrm{sign}(u),$$

where $\hat{\mathcal{M}}_P([x, w], [y, v])$ is the moduli space of perturbed pseudo-holomorphic cylinders (i.e. solutions of (3.1)) connecting $[x, w], [y, v]$. For generic J, θ, these moduli spaces are smooth, compact 0-dimensional manifolds which may be given coherent orientations. (Assuming M is weakly monotone for simplicity.) Such (J, θ) are said to be *regular*. By typical gluing theorems, $\partial_F(\phi)^2 = 0$, and thus $(CF(\phi), \partial_F(\phi))$ indeed defines a chain complex.

Note that

$$(CF_k(\phi), \partial_{F,k}(\phi)) = (CF_{k+2N}(\phi), \partial_{F,k+2N}(\phi)).$$

We say that $CF(\phi)$ is a *periodic* chain complex with period $2N$. Equivalently, we may regard the grading as taking values in $\mathbb{Z}/(2N\mathbb{Z})$.

3.2 The Reidemeister torsion

We consider a refined version of Reidemeister torsion which is an abelianized version of the Whitehead torsion. This version of Reidemeister torsion was first introduced by Turaev.

Assume first for simplicity that the coefficient ring is a field F. Let (C_*, ∂_*) be a finite or periodic complex of even period, of finite dimensional vector spaces. The standard short exact sequences $0 \to Z_i \to C_i \to B_{i-1} \to 0$ and $0 \to B_i \to Z_i \to H_i \to 0$ induce a canonical isomorphism

$$\mathcal{T} : \bigotimes_i \det(C_i)^{(-1)^i} \to \bigotimes_i \det(H_i)^{(-1)^i}; \tag{3.2}$$

the latter being F when (C_*, ∂_*) is acyclic. Note that when (C_*, ∂_*) is periodic with period $2N$, i takes value in $\mathbb{Z}/(2N\mathbb{Z})$; $(-1)^i$ makes sense here as the period is even.

Let e be an ordered basis for C_*, i.e. an ordered basis e_i for each C_i. It induces a $[e] \in \bigotimes_i \det(C_i)^{(-1)^i}$. We define the *Reidemeister torsion*

$$\tau(C_*; e) := \begin{cases} \mathcal{T}(e) & \text{if } H_* = 0, \\ 0 & \text{otherwise.} \end{cases}$$

In our case, the coefficient ring R will not be a field, but its quotient ring $Q(R)$ will be finite direct sum of fields. In this case, when $Q(R) = \oplus_j F_j$, we define

$$\tau(C_*, e) := \sum_j \tau(C_* \otimes_R F_j, e \otimes 1) \in \bigoplus_j F_j = Q(R).$$

Since we always work in the context of (twisted) Morse complexes or cell complexes, there are natural choices of the basis e—namely, e should consist of lifts of critical points or cells in the universal abelian covering. For example, for a regular (J, θ), the *Reidemeister torsion of the Floer complex* is

$$\tau_F(\phi) := \tau(CF(\phi), e) \in Q(\Lambda_F^\phi)/(\pm \ker \psi_c);$$

where e is a basis of CF given by lifts of elements in $\mathcal{P}$. Different ordering of these elements results in a possible change of sign for $\tau(CF(\phi), e)$, and different lifts result in a multiplication of τ by an element in $\ker \psi_c$; so by quotienting out $\pm \ker \psi_c$ (as a multiplicative subgroup of the ring $Q(\Lambda_\phi)$) in the definition, we obtain an invariant independent of these choices.

4 Invariance results

Surprisingly enough, $\tau_F(\phi)$ is *not* invariant under Hamiltonian isotopies (i.e. varying θ by exact forms). During a Hamiltonian isotopy, there might be flow lines coming from a critical point back to itself, and $\tau_F(\phi)$ jumps when this happens. (This happens at a "handle-slide bifurcation" or a "death-birth bifurcation" in the terminology of [25].)

This might seem like a disaster, since the invariance under Hamiltonian isotopies, as we saw before, was the key in Floer's proof of Arnold conjecture. However, we showed in [25] that this may be remedied by introducing a "correction term"—the zeta function. This zeta function is defined as a counting invariant of perturbed pseudo-holomorphic tori, which are closed orbits in the flow (3.1). At a handle-slide or death-birth bifurcation, new closed orbits will be created or destroyed, changing the zeta function in a way that compensates the change in the Reidemeister torsion.

4.1 The zeta function

Let $\mathcal{M}_O$ be space of solutions to (3.1) for $s \in S^1_T$ (circle of length T) for arbitrary $T \in \mathbb{R}^+$, namely, the space of closed orbits of the flow. For generic (J, θ), this is a disjoint union of finite dimensional S^1 manifolds, where S^1 acts simply by translation in s. Let $\hat{\mathcal{M}}_O = \mathcal{M}_O / S^1$. This quotient is *not* a manifold, but an orbifold, because the S^1 action is not free at "multiple covers". Regarding an element $u \in \mathcal{M}_O$ as a map $S^1_T \to \mathcal{C}$, the *multiplicity* $m(u)$ is the largest integer k such that u factors through a k-fold covering $S^1_T \to S^1_{T/k}$. An $u \in \mathcal{M}_O$ is a *multiple cover* if $m(u) > 1$. The isotropy of the S^1 action at a multiple cover u is $\mathbb{Z}/m(u)\mathbb{Z}$.

The dimension of $\hat{\mathcal{M}}_O$ at u is $2\psi_c([u])$ by Riemann-Roch, where $[u] \in H_1(\mathcal{C})$ is the homology class of u. We shall only be interested in closed orbits in $\hat{\mathcal{M}}^0_O \subset \hat{\mathcal{M}}_O$, the 0-dimensional subset; their homology classes are all in $\ker \psi_c$ by the above index computation.

The Floer-theoretic *zeta function* is defined as

$$\zeta_F(\phi) := \exp\left(\sum_{u \in \hat{\mathcal{M}}^0_O} \frac{\operatorname{sign}(u)}{m(u)} [u] \right). \tag{4.1}$$

Computation of the energy functional leads to a compactness theorem, guaranteeing that the exponent on the right hand side above is in $\mathrm{Nov}^+(\ker \psi_c, \mathbb{Q}; -d\tilde{\mathcal{A}})$, where $\mathrm{Nov}^+(G, \mathbb{Q}; N) \subset \mathrm{Nov}(G, \mathbb{Q}; N)$ denote the subset consisting of formal sums $\sum_g a_g \cdot g$ for which $a_g \neq 0$ only when $N(g) > 0$. $\exp : \mathrm{Nov}^+(G, \mathbb{Q}; N) \to \mathrm{Nov}(G, \mathbb{Q}; N) \hookrightarrow Q(\mathrm{Nov}(G, \mathbb{Z}; N))$ is well defined via the usual power series. Thus the above definition gives an element in $Q(\Lambda^\phi_F)$.

Note that coefficients in the exponent in (4.1) are "orbifold Euler numbers", which are invariant under orbifold cobordisms.

Remarks 4.1 **(a)** We take exponential in the definition of the zeta function because we are doing everything multiplicatively.

(b) Note that the definition of $\zeta_F(\phi)$ above much resembles the Gromov-Taubes invariant. The perturbed pseudo-holomorphic tori here has the interpretation as the perturbed pseudo-holomorphic sections of the symplectic mapping tori $\Big(M \times [0,1]/(x,0) \sim (f(x),1)\Big) \times S^1$. In [28], Ionel and Parker defined a very similar version of zeta function, which computes an averaged version of the Gromov-Taubes invariant. The difference is this: regard the tori as $S^1_s \times S^1_t$; the subscripts s and t denote the parameters. Our pseudo-holomorphic tori are parameterized in S^1_t with fixed length, while the zeta function in [28] counts unparameterized pseudo-holomorphic tori of arbitrary conformal structure. In fact, the resemblance with the Gromov invariant was one of our main motivations for working with the abelian version of torsion (Reidemeister torsion), instead of the non-abelian Whitehead torsion, which counts curves with *homotopy* class.
(c) For (4.1) to be well defined, one needs to check the relevant transversality and compactness results. These are mostly routine, though the transversality for multiple covers is a little more delicate. The proof in [25] was inspired by the work of Taubes in [50]. It depends on translating the problem of transversality for the multiple covers into the problem of simultaneous surjectivity of a sequence of differential operators over simple orbits, and the property of the kernel and cokernel of these operators as representations of $\mathbb{Z}/m\mathbb{Z}$.

4.2 The invariant I_F

Define the product

$$\tau_F(\phi)\zeta_F(\phi) =: I_F(\phi).$$

Our main result concerns the invariance property of $I_F(\phi)$.

Theorem 4.2 *[25] Let (M,ω) be a weakly monotone symplectic manifold and let ϕ be a symplectomorphism of M and $\gamma_0 \in \mathcal{L}_\phi M$ be chosen as in §3.1. Let the pairs $(J_\lambda, \theta_\lambda)$ $\lambda \in [0,1]$ be a path of almost complex structures and closed 1 forms described in §3.1, such that (J_0,θ_0) and (J_1,θ_1) are both regular. Then $I_F(\phi)$ defined using either (J_0,θ_0) or (J_1,θ_1) are the same.*

We assumed weakly monotonicity only for technical simplicity; presumably this condition may be removed via the virtual moduli technique of [33, 16, 41].

Under stronger conditions, I_F is in fact an invariant under symplectic isotopy. Let $\mathrm{Symp}_0(M)$ denote the identity component of the symplectomorphism group of M consisting of the path component containing the identity. There is an obvious action of $\pi_1(\mathrm{Symp}_0(M))$ on $\pi_0(\mathcal{L}_\phi M)$.

Theorem 4.3 *[25] If M is monotone, then $I_F(\phi)$ only depend on the symplectic mapping class of ϕ, and the class of $[\gamma_0]$ in $\pi_0(\mathcal{L}_\phi M)/\pi_1(\mathrm{Symp}_0(M))$.*

The monotonicity condition is used to guarantee that the Novikov ring is unchanged when varying the parameters used in the definition of I_F, so that we have the necessary energy bounds for compactness results. We do not know if I_F is invariant under symplectic isotopy in general.

In [25], the last two theorems are proved in a slightly more general setting. We shall briefly discuss the strategy of proof in section 7 below.

5 Sample computations, and relation to mirror symmetry?

I_F can be computed in some cases.

5.1 Reduction to the classical version

Theorem 4.2 is actually not so surprising if one knew the finite dimensional analogue. This finite dimensional result was established previously in the author's joint works with M. Hutchings, which we now briefly describe.

Given a finite dimensional, closed oriented manifold M and a Morse 1-form on it, let Λ_N^θ denote the Novikov ring $\mathrm{Nov}(H_1(M); \mathbb{Z}, [\theta])$. The Morse-theoretic torsion $\tau_N(\theta) := \tau(CN, e) \in Q(\Lambda_N^\theta)/\pm H_1(M)$ is defined from the Morse-Novikov complex in a entirely parallel fashion to the Floer-theoretic version in §4.1, and as in the Floer-theoretic version, it does not depend on the ordering or the lifts of the critical points in our choice of the basis.

Let $\zeta_N(\theta) \in \Lambda_N^{\theta\times}$ be defined by the same formula (4.1), except that u here denotes orbits of the flow along $-\check{\theta}$.

Given a cell-decomposition of M, it induces an $H_1(M)$-equivariant cell decomposition on $\tilde{M}$. Let $C_*(\tilde{M})$ be the $\mathbb{Z}[H_1(M)]$-coefficiented equivariant cell chain complex. A basis e_M of $C_*(\tilde{M})$ is obtained by choosing a lift of each cell in M to $\tilde{M}$.

$$\tau(M) := \tau(C_*(\tilde{M}), e_M) \in Q(\mathbb{Z}[H_1(M)])/\pm H_1(M)$$

is the so-called *"Milnor torsion"* of M; again it does not depend on the ordering and the lifts of the cells in the choice of e_M.

Theorem 5.1 *[21, 22] $\tau_N(\theta)\zeta_N(\theta) = \iota\tau(M)$, where $\iota : Q(\mathbb{Z}[H_1])/\pm H_1 \hookrightarrow Q(\Lambda_N^\theta)/\pm H_1(M)$ is the natural inclusion.*

This result is natural because in a sense it is a combination of two old results of Milnor's: when M is a mapping torus (in other words, M admits a circle-valued function without critical points), Milnor showed that $\tau(M)$ may be computed by an analogue of the Weil zeta function [36]. (Obviously $\tau_N = 1$ in this case, since there is no critical point.) On the other hand, Milnor also showed that the torsion of a real-valued Morse function is equal to $\tau(M)$ [37]. (The zeta function is trivial in this case, since there is no closed orbit.) However, we discovered this result from a rather unexpected origin: While trying to compute the Seiberg-Witten invariant for 3-manifold, Taubes's philosophy led us to define a counting invariant of gradient flow lines of a circle-valued harmonic Morse function, which should be equivalent to the 3-dimensional Seiberg-Witten invariant, according to a conjectural generalization of Taubes's proof of the equivalence of the 4-dimensional Seiberg-Witten invariant and Gromov invariant. (Some progress towards such a proof was made in [24].) Our computation showed that such an counting invariant is in fact the classical Reidemeister torsion, and hence surprisingly, the Seiberg-Witten invariant, which has a purely analytic definition by counting solutions to a PDE, is equivalent to the combinatorially defined topological invariant–the well-known torsion.

Now, Theorem 4.3 implies that in the monotone case, one may compute the Floer-theoretic I_F using a small t-independent θ, which reduces to the finite dimensional version as in the case of Floer homology. Thus by the previous theorem, we have:

Corollary 5.2 *[25] Suppose M is monotone and ϕ is connected to the identity via the symplectic-isotopy ϕ_t, $t \in [0,1]$; $\phi_0 = \mathrm{Id}$; $\phi_1 = \phi$. Let the base point in $\mathcal{L}_\phi M$, γ_0, be the path $\gamma_0(t) = \phi_t(p_0)$ for a base point $p_0 \in M$. Then*

$$I_F(\phi) = \iota'\iota\tau(M),$$

where $\tau(M)$ is the Milnor torsion of M, and $\iota'\iota$ is the natural inclusion $Q(\mathbb{Z}[H_1(M)])/\pm H_1(M) \hookrightarrow Q(\mathrm{Nov}(H_1(M) \times \pi_2(M), \mathbb{Z}; -\theta \oplus 0))/\pm H_1(M)$.

5.2 Generalization to Stein manifolds

Unfortunately, the requirement for M to be *closed and symplectic* places very strong topological constraints on M. Practically all known closed symplectic 4-manifolds have uninteresting Milnor torsion (i.e. its value in each field component of $Q(\Lambda_\phi)$ is either 0 or 1). It is an open problem to find higher dimensional closed symplectic manifolds with interesting torsion.

On the other hand, from the work of Eliashberg [6], it is known that Stein manifolds (which are open symplectic manifolds) have rather flexible topology—any $2n$ dimensional almost complex manifold with boundary admitting a Morse function with Morse indices $\leq n$ is Stein for $n > 2$. There are therefore plenty of Stein manifolds with interesting Milnor torsion. We believe that a version of I_F may be defined for Stein manifolds following the construction of Floer complex in [5] and that in the monotone case it reduces to the Milnor torsion. For example, a cylinder $S^1 \times [0,1]$ may be endowed with the structure of a Stein manifold (regarding it as one half of a torus). I_F in this case should be equal to $1-t$, t being a generator of $H_1(S^1)$ [27].

5.3 The Lagrangian intersection version

Suppose L, L' are oriented, spin[2], monotone Lagrangian submanifolds in M with minimal Chern numbers > 2, then a $I_F(L, L')$ may be defined [26] using the Floer homology of Lagrangian intersections of L, L', and a zeta function counting pseudo-holomorphic annuli with boundaries on L, L'. When $M = T^*X$ is a cotangent bundle, and L, L' are symplectic isotopic to the zero-section, $I_F(L, L')$ is always defined. The argument for Theorem 4.3 proves:

Theorem 5.3 *[26] Let X be a closed orientable manifold. Let L be the zero section of T^*X, and L' be a symplectic isotopic to L. Then $I_F(L, L') = \tau(L)$.*

Since X can be quite arbitrary, this result yields many examples of interesting I_F, especially when X is odd-dimensional. When X is a mapping torus, the Milnor torsion of X can be computed by a Lefschetz trace formula (see e.g. [22]). When X is 3-dimensional, its Milnor torsion can be computed via a surgery formula (see e.g. [53]).

5.4 An S^1-equivariant version.

Inspired by the analogy of the zeta function with the elliptic Gromov invariant (cf. Remarks 4.1 (b)), we ask when it is possible to interprete the elliptic Gromov invariant as a zeta function of flows on a loop space. This is indeed possible in some cases. An example is the symplectic mapping torus $M = \Sigma_f \times S^1_1$, where $f : \Sigma \to \Sigma$ is a symplectomorphism. M is endowed with a standard symplectic form and a standard almost complex structure. The zeta function of Ionel-Parker (which computes the elliptic Gromov invariant in certain classes) may be interpreted as an S^1-equivariant version of zeta function of flows as follows. Let $\mathcal{C}$ be the space of loops of unit length on M in the homology class of $\{p\} \times S^1 \subset M$. Since this homology class is nontrivial, there is a *free* S^1 action on $\mathcal{C}$. Now consider the flow given by the perturbed Cauchy-Riemann equation (3.1), *with t-independent J, θ.* This is the flow of a S^1-equivariant Morse 1-form on $\mathcal{C}$, and one may define correspondingly an S^1-equivariant version of I_F in this situation. In particlar, when $\theta = 0$, there are no critical points, as is already pointed out in Example 2.2. On the other hand, the S^1-equivariant zeta function for $\theta = 0$ is precisely the Ionel-Parker zeta function. Hence the S^1-equivariant I_F in this case should equal the elliptic Gromov invariant discussed in [28], which is computed by the Lefschetz numbers of f.

[2] According to [17], it suffices to be “relatively spin”.

5.5 General formula from Mirror symmetry?

In their famous paper in 1993, the four physicists Bershadsky, Cecotti, Ooguri, Vafa extended mirror symmetry to higher dimensional curves. At the 1-loop level, this says that the genus 1 Gromov-Witten invariant from the A-model side should correspond to a holomorphic analytic torsion from the B-model side. Our zeta function is a Gromov-Witten type invariant—in fact, in some cases we saw that it is precisely the elliptic Gromov-Witten invariant. Our construction might thus be the starting point of a mathematical explanation of this 1-loop mirror symmetry according the general philosophy of mirror symmetry being a correspondence between symplectic geometry and complex geometry on mirror manifolds. (genus 1 Gromov-Witten=torsion on symplectic side; holomorphic analytic torsion=torsion on the complex side). In fact, according to Kontsevitch's formulation of the homological mirror symmetry conjecture [30], tree level mirror symmetry is an equivalence between the Fukaya-Floer category from the A model side, and the category of coherent sheaves on the B model side. In particular, the Floer cohomologies should correspond to certain sheaf cohomologies on the B model side. Recall that the holomorphic analytic torsion is the "correction term" to the standard metric on the determinant line bundle of sheaf cohomologies to construct the invariant Quillen metric; our construction is the analogous story on the symplectic side—the zeta function=genus 1 Gromov-Witten as a correction term to the determinant line of Floer homologies on the other side. (This was observed by Fukaya [14, 17].) Conversely, this 1-loop mirror symmetry might give a general formula for I_F, since the B-model side is easier to compute.

6 Sample applications

6.1 using τ_F: existence of symplectic fixed points, Lagrangian intersection points, or closed orbits

Since the critical points in the Floer theory correspond to symplectic fixed points, Lagrangian intersection points, or periodic orbits of symplectic vector fields depending on the version of Floer theory used, nontriviality of the Reidemeister torsion τ_F will guarantee the existence of symplectic fixed points/Lagrangian intersection points/periodic orbits. Here is an example of such results.

Corollary 6.1 *[25] When M is monotone and ϕ is symplectic isotopic to the identify, ϕ has a fixed point if $\tau(M) \neq \zeta_F(\phi)$, with γ_0 being the standard base point in $\mathcal{L}_\phi M$ associated the symplectic isotopy.*

This corollary has its full strength only if we can compute the zeta function, which is usually quite nontrivial except in special cases. For example, we know that the zeta function is 1 if the action functional is globally defined.

If one doesn't know what the zeta function is, one can always look at the less refined version of the torsion. Let $\bar{\tau}(M)$ be the image of $\tau(M)$ under the canonical map:

$$Q(\mathbb{Z}[H_1(M)])/\pm H_1(M) \to Q(\mathrm{Nov}(H_1(M), \mathbb{Z}; -\theta))/\pm \mathrm{Nov}(H_1(M), \mathbb{Z}; -\theta)^{\times},$$

where $\mathrm{Nov}(H_1(M), \mathbb{Z}; -\theta)^{\times}$ denotes the group of units in the Novikov ring, in which the zeta function takes value.

(*) If M is monotone, and ϕ is symplectic isotopic to the identity, and $\bar{\tau}(M) \neq 1$, then ϕ has a fixed point.

In particular, under the same conditions on M and ϕ, if the Floer homology is nontrivial, then one of component of $\tau(M)$ is zero (note that $Q(\mathbb{Z}[H_1(M)])$ is a direct sum of fields), and thus ϕ has a fixed point. (This is a weaker version of the result of [32].)

Unfortunately one can not obtain quantitative results in this direction using the abelian version of torsion we consider. An non-abelian version of (*) can actually be established via the same arguments:

(**) If M is monotone, and ϕ is symplectic isotopic to the identity, then the number of fixed point of ϕ is larger or equal to the minimal rank of representations of the Whitehead torsion of M.

This is quantitative, through in practice it is usually difficult to compute. See [19] for some examples in the Lagrangian intersection case.

6.2 using ζ_F: existence of *noncontractible* closed orbits

Though it was our original motivation, the type of applications described in the last subsection is not really a very effective way of utilizing the invariant I_F. By nature, I_F is a "flow"-counting invariant instead of a "critical point"-counting invariant like the homology. Nontriviality of I_F implies the existence of (perturbed) pseudo-holomorphic tori, and existence of pseudo-holomorphic curves is typically very useful in symplectic topology [18]. Here are some examples exploiting this.

Both examples below concern the existence of noncontractible periodic orbits of symplectic vector fields. So far very little is known about the existence of *noncontractible* periodic orbits due to lack of available tools; most results are for contractible orbits. Notice for instance that the Floer homology is computable and nonvanishing only in the contractible case.

Example 6.2 Let M be a closed oriented manifold with a nowhere vanishing closed 1-form θ. θ then corresponds to a section of T^*M, which is a Lagrangian submanifold $L' \subset T^*M$ and does not intersect the zero section L—another Lagrangian submanifold of T^*M. (Let T^*M be endowed with the usual exact symplectic form, and the standard almost complex structure.) Furthermore, there is a symplectic isotopy moving L to L'.

Remark 6.3 The problem of the existence of such nowhere vanishing closed 1-forms is a classical problem; it is solved by Latour for dimensions ≥ 6; by Stallings, Thurston for 3 dimensions. It in particular implies that the Novikov homology and the torsion $\bar{\tau}(M)$ are both trivial. Notice that this is precisely the situation where the type of applications in §6.1 do not work.

Note that in this case, the natural map $i_* : H_1(\Omega(T^*M, L, L')) \to H_1(L)$ is an isomorphism.

Claim 6.4 [26] Let M, L, L' be as above. Let H be a 1-periodic Hamiltonian on T^*M separating L, L'; namely,

$$H : T^*M \times S^1_1 \to \mathbb{R}; \quad \mu := \int_{S^1_1} \min H_t\Big|_{L'} - \max H_t\Big|_{L} > 0.$$

Suppose further that ∇H_t is compactly supported $\forall t \in S^1_1$, and there is a class $h \in H_1(M;\mathbb{Z})$ such that $[\theta](h) = m > 0$ and $\ln \tau(M) : H_1(M) \to \mathbb{Q}$ is nontrivial at h. Then:
(a) If h is primitive, then there exists a $\lambda \in (0, m/\mu]$ such that the Hamiltonian flow of λH has a closed periodic orbit in the homology class h.
(b) If H is autonomous (i.e. t-independent), then the same holds for any such h (however the closed periodic orbit may be a multiple cover).

This claim for example works for mapping tori. In this case, $\ln \tau(M)$ is computed by Lefschetz numbers.

The idea of proof is to use the computation of I_F in Theorem 5.3. Nontriviality of $\ln \tau(M)$ implies the existence of holomorphic annuli with boundaries at L, L' in the class of h for $\lambda = 0$. However, a simple computation of the energy implies that nonexistence of such perturbed holomorphic annuli for $\lambda \geq m/\mu$. This means that the parameterized moduli space of holomorphic annuli in this class must be non-compact somewhere between $\lambda = 0$ and m/μ, which can happen only if there is a sequence of holomorphic annuli with width going to infinity. Such a sequence implies the existence of a Hamiltonian orbit in the limit.

Remarks 6.5 **(a)** This significantly generalizes the result of Lalonde-Gatien [31]. They only consider M that is a mapping torus of an involution of a flat manifold, in which case they may work over the covering space $\mathbb{C}^n$ and use results from complex analysis

to guarantee the existence of holomorphic annuli.
(b) Similar results will possibly also hold for Lagrangian embedding of such M in closed manifolds, for small λ.

Example 6.6 Consider the S^1-equivariant version of I_F described in §5.4. If this version of I_F is invariant under symplectic isotopy, we expect:

Conjecture 6.7 Let $M = \Sigma_f \times S^1$ be the symplectic mapping torus of the symplectomorphism $f : \Sigma \to \Sigma$ of the symplectic manifold (Σ, ω). The projection $\pi : \Sigma_f = \Sigma \times [0,1]/(x,0) \sim (f(x),1) \to S^1$ gives a $q \in H^1(\Sigma_f;\mathbb{Z})$. M is endowed with a standard symplectic form $\Omega = \omega + q \wedge dt$, $t \in S^1$. Let $h \in H_1(\Sigma_f)$ be a class with nontrivial Lefschetz number and suppose $m := q(h) > 0$. Let X be a (t-independent) symplectic vector field on M with flux $\theta \in H^1(M)$. Let $i^*\theta = \mu[dt]$ for $\mu > 0$. Then:
$\forall \lambda > m/\mu$, there exists a 1-periodic closed orbit of λX.

We in particular expect this to be at least true when X is not only symplectic, but almost complex, because this is an analogue of the monotonicity condition and we expect I_F to be invariant under symplectic isotopy in this case.

Finally, we remark that some results on symplectic capacity or topological constraints on Lagrangian embedding have been obtained via the existence of holomorphic *disks*. What can the existence of holomorphic annuli/tori tell us in these directions?

7 Ideas of proof for Theorems 4.2, 4.3

Of course, the finite dimensional computations in [21, 22] do not apply in the infinite dimensional setting here. On the other hand, as we mentioned, the usual continuation method of typical invariance proofs in Floer theory does not work here either, since this method depends on the construction of chain homotopy equivalences, while the torsion invariant we consider is not even a homotopy invariant.

In [25], we therefore had to resort to the seemingly straightforward bifurcation analysis: connecting two pairs (J_1, θ_1), (J_2, θ_2) via a path, we show that this path may be perturbed into an *"admissible"* one; then analyze how the Floer complex and the zeta function change at each bifurcation of the admissible isotopy.

Roughly speaking, an admissible isotopy has three possible types of bifurcations, which occur one at a time: (1) death/births of pairs of critical points, or equivalently a degenerate critical point which we may assume to be standard in a sense; (2) type I handle-slides, or equivalently finite-energy paths from a critical point to another critical point of the same index; (3) type II handle-slides, or equivalently a finite-energy paths from a critical point back to itself.

Of course, the appeal of this direct approach is its conceptual simplicity, and (relatively easy) adaptability. Although this is also the obvious route to basically all invariance results in Floer theory, it has hitherto been avoided in the literature due to the analytic difficulties associated with the appearance of degenerate critical points. (Hence the 140 pages of [25]! This also answers the question raised in the end of section 1.)

We however believe that this direct proof of invariance should be carried out for once, even just for better understanding of the Floer theory in general, and we hope the analysis in [25] opens the door for the study of other more refined invariants of Floer theories, for which the invariance properties can not be established via conventional methods. It might even offer an alternative to the continuation method in more intricate variants of Floer homology (e.g. contact homology), when the continuation method itself is difficult to carry out.

We remark that historically this was the first strategy adopted by Floer in [9] for the proof of invariance of the Lagrangian intersection Floer homology under Hamiltonian isotopies. However, Floer's proof in [9] was incomplete and full of mistakes, but probably nobody cared about it ever since the more elegant continuation method was discovered.

In [25], we implement the program Floer sketched in [9] with the following differences: (1) There are several places where Floer used tools from complex analysis which do not apply in other situations (e.g. the Floer theory for symplectomorphisms considered in Example 2.4); instead, we provide a fairly general proof. (2) Since in general the action functional is not globally defined, there are possibly infinitely many handle-slide bifurcations during the isotopy, and this requires more delicate arguments depending on a filtration by energy throughout. (3) There are several new features in the bifurcations that do not occur in Floer's situation. For example, new orbits are created with Type II handle-slides or death/birth bifurcations, and in addition to the new paths and orbits created by gluing two paths at the bifurcation, there are also ones created by gluing three or more paths simultaneously. This is actually the hardest part of the proof.

For experts in gluing theory, we highlight some less standard parts of the proof below.

- At a death/birth bifurcation, there is a degenerate critical point, and we therefore need to find the suitable Fredholm context to describe the structure of the moduli of paths to or from the degenerate critical point. The correct Fredholm theory[3] is one with Banach spaces of different polynomial weights on the "transversal" and "longitudinal" directions. (The longitudinal direction is asymptotic to the direction of degeneracy of the degenerate critical point.) To justify that this Fredholm theory suffices to describe our moduli space, it is crucial to establish a pointwise decay property of paths to or from the degenerate critical point. In [9], this is done via complex analysis methods. Instead, we give a generic proof via the center manifold theory for flows on Banach manifolds.

[3]This type of Fredholm theory appeared in e.g. [9, 24].

- With the polynomially weighted norms chosen above, it is only natural to use a norm combining the weighted norms in the gluing theory for paths ending/starting at the degenerate critical point. This norm is commensurate with the usual Sobolev norm, but depends on $\lambda \in \Lambda$, which parameterizes the isotopy. This however introduces several difficulties. First, the usual pregluing construction is not good enough, since the "error" would then be too large due to the large longitudinal contributions. Instead, one needs to find diffeomorphisms $\{\gamma_i : I_i \to \mathbb{R}\}$ such that the proposed pregluing $w_\chi(s) := u_i(\gamma_i(s))$ approximates a true flow "nicely". Here $\{I_i\}$ is a partition of $\mathbb{R}$, and $\{u_i\}$ are flows that constitutes a broken trajectory that is to be glued. (We know that the pregluing has this form since by the Gromov compactness we know that a flow in the end of the moduli space is pointwise (in s) close to the broken trajectory.) In particular, the approximation must be particularly precise near the degenerate critical point due to the polynomially increasing weight in the norms. (Thus γ_i can not be a simple translation, which is the conventional choice.) A delicate pointwise estimate that depends on analyzing the behavior of flows near the degenerate critical point shows that this can be done; furthermore, these γ_i may be chosen in a relatively simple way: they may be *defined* by the requirement that they make the longitudinal component of the error vanish[4].

  In simpler situations (e.g. gluing two paths or less), it may also be shown that the deformation operator $\hat{D}$ has a uniformly-bounded right-inverse. Say, the bifurcation occurs at $0 \in \Lambda$. The proof follows the usual proof by contradiction framework, but the argument is a lot more intricate. The domain $\Theta = \mathbb{R} \times S^1$ is partitioned into several regions (typically with length proportional to $|\lambda|^{-1/2}$), where the weight function (in the norms) approximates the weight functions for u_i or $C|\lambda|^{-1}$ when w_χ may be approximated by u_i or is near the degenerate critical point y respectively. Suppose there is a sequence of unit vectors $(\alpha_\lambda, \xi_\lambda)$ in a codimension-ind$(\hat{D})$ subspace of the domain of $\hat{D}$ such that $\hat{D}(\alpha_\lambda, \xi_\lambda) \to 0$ as λ approaches the point bifurcation. (Here α_λ parameterizes the deformation in Λ.) The approximations mentioned above help to estimate $(\alpha_\lambda, \xi_\lambda)$ on these different regions. Finally, it is essential to show that $|\lambda|^{-1-1/(2p)}\alpha_\lambda \to 0$ as $\lambda \to 0$. (Note in contrast that in the "usual" case, such as the case with Type I handle-slides, it is only necessary to show that $\alpha_\lambda \to 0$; the extra factor $|\lambda|^{-1-1/(2p)}$ is due to the polynomial weight.) This rather technical lemma is proved using the fact that when s is in the region near the degenerate critical point, $\partial_\lambda \nabla H_\lambda(w_\chi(s))$ is almost a constant vector in the degenerate direction.

- In the cases of gluing more than two paths starting/ending at the same critical

[4] This was Floer's choice in [9]; however there it appears to be a lucky though reasonable guess.

point, degenerate or nondegenerate, (i.e. a death/birth of Type II handle-slide), a quick dimension-counting shows that the proofs of the simpler cases above can not be directly generalized to prove the uniform right-invertibility of the deformation operator with the natural Banach spaces. (In a previous attempt to solve this problem, we were able to construct exotic Banach spaces between which the deformation operators in the case of death/births are right-invertible, but then we ran into difficulties with the nonlinear aspects of the gluing theory.) In the case of a death/birth, we are able to solve this problem via a variant of Taubes's obstruction bundle technique. Typically, the moduli space is described locally by the "Kuranishi model", namely, the zero locus of an analytic map between the kernel and the cokernel of the deformation operator $\hat{D}$. However, it is often very difficult (such is the case in our situation) to obtain explicit descriptions of the cokernel and the analytic map, except in special cases. Our idea is work with "generalized kernels" and "generalized cokernels" instead, which are easier to come up with explicitly. These are defined as finite-dimensional subspaces $\ker_g$, coker_g in the domain and range of the deformation operator $\hat{D}$ respectively, such that $\hat{D}$ is injective on $\ker_g$ and we may decompose (*non-orthogonally*) the domain and range into $\ker_g \oplus \ker_g^c$ and $\hat{D}\ker_g^c \oplus \operatorname{coker}_g$ respectively. The previous arguments may be adapted to show that $\hat{D}$ has a uniform right inverse over $\hat{D}\ker_g^c$, and thus the solution space to the gluing problem has the description of the zero locus of a nonlinear map from $\ker_g$ to coker_g. Of course, the drawback is that now one has work with *non-orthogonal* projections, which in general can be very complicated. Fortunately, in our case, we found a simple way of estimating the projection to coker_g (where the dominant term is given by a simple integral, see [25] Proposition 9.14). Using it, we obtain estimates that show that the restriction of $\hat{D}$ dominates other non-constant terms in this nonlinear map, which has a simple form (up to ignorable terms) that is clearly invertible. Thus again corresponding to each set of gluing parameters, there is a unique solution to the gluing problem.

- The case of gluing more than two paths in a Type II handle-slide, on the other hand, seems hopeless to have a simple description as above. We therefore avoid this problem altogether by adopting the "finite-covering trick" in [23]. The idea is to observe that at a Type II handle-slide, I_F changes at most by a factor that is a power series in the h with leading term 1, where h is the homology class of the path from the critical point back to itself. Using the relation between the torsions or zeta functions of a space and its finite-cyclic coverings, if the coefficient of h vanishes in this power series (which is proved by gluing once), then all the higher order terms vanish too. This argument depends on the fact that there is a non-equivariant perturbation on the finite covering to the lift of the flow that makes the isotopy over the finite covering admissible. In the finite dimensional case, this is trivial since we have all the Morse functions at our disposal. However, in our infinite dimensional setting, one needs to make sure that the perturbation fits into the framework of elliptic PDE, so that we have all the required structure theory. In some cases, we found that this can be done by a local perturbation, but in general the perturbation will involve a nonlocal term. We do not appear to have a nice general theory for the solution spaces of this generalized version of (3.1); for example, we do not seem to have an analogue of Aronszajn's theorem, which is used extensively in the proof of transversality and compactness for the moduli of flows. However, if the perturbation is sufficiently small, we do have the required properties.

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