Heegaard Splittings and Seiberg-Witten Monopoles

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Abstract. This is an expansion on my talk at the Geometry and Topology conference at McMaster University, May 2004.

We outline a program to relate the Heegaard Floer homologies of Ozsvath-Szabo, and Seiberg-Witten-Floer homologies as defined by Kronheimer-Mrowka. The center-piece of this program is the construction of an intermediate version of Floer theory, which exhibits characteristics of both theories.

1 The conjecture

This is a preliminary report on a long program aiming at a proof of the conjectural equivalence between the Heegaard Floer homologies of Ozsvath-Szabo, and the monopole Floer homologies as defined by Kronheimer-Mrowka.

Besides giving an overall picture, we give in §5, 6, 8, and 9.1 a survey of some partial results towards this goal, with details deferred to papers in preparation [17, 18].

Throughout this article, we shall work with an unspecified coefficient ring R, which may be $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}, \mathbb{Q}$, or $\mathbb{R}$. Take $R = \mathbb{Z}/2\mathbb{Z}$ if desired, since we ignore the orientation issue in this article.

Let Y be a compact oriented 3-manifold and $\mathfrak{s}$ be a spin-c structure on Y. In [19], Ozsvath-Szabo defined four versions of Heegaard Floer homologies associated to $(Y, \mathfrak{s})$, $\mathrm{HF}^-(Y, \mathfrak{s})$, $\mathrm{HF}^\infty(Y, \mathfrak{s})$, $\mathrm{HF}^+(Y, \mathfrak{s})$, and $\widehat{\mathrm{HF}}(Y, \mathfrak{s})$.

Let HM denote the Seiberg-Witten-Floer homologies defined by Kronheimer-Mrowka [22]. They come in three flavors, $\hat{\mathrm{HM}}$, $\bar{\mathrm{HM}}$, $\check{\mathrm{HM}}$. Based on Kronheimer-Mrowka's construction, we shall introduce in §5 below a fourth version, HM^{tot}, in parallel to $\widehat{\mathrm{HF}}$. In addition to the pair $(Y, \mathfrak{s})$, these Seiberg-Witten-Floer homologies depend on the cohomology class of perturbation two form, $[\omega]$, and we denote them by $\mathrm{HM}(Y, \mathfrak{s}; [\omega])$.

In spite of their very different origins, these Floer homologies have identical formal properties. They are both $R[U]$-modules, where U is a degree -2 chain map,

supported in part by NSF grant DMS #0333163.

and the first three flavors of both Floer homologies fit into long exact sequences, which we call the *fundamental exact sequences*:

$$\cdots \to \mathrm{HF}^- \to \mathrm{HF}^\infty \to \mathrm{HF}^+ \to \cdots$$
$$\cdots \to \hat{\mathrm{HM}} \to \bar{\mathrm{HM}} \to \check{\mathrm{HM}} \to \cdots$$

Conjecture 1.1 *Let $(Y, \mathfrak{s})$ be as the above, and let*

$$[w] = 2\pi c_1(\mathfrak{s}). \tag{1.1}$$

Then there are isomorphisms of $R[U]$-modules

$$\begin{aligned} \mathrm{HF}^-(Y,\mathfrak{s}) &\simeq \hat{\mathrm{HM}}(Y,\mathfrak{s};[w]), \\ \mathrm{HF}^\infty(Y,\mathfrak{s}) &\simeq \bar{\mathrm{HM}}(Y,\mathfrak{s};[w]), \\ \mathrm{HF}^+(Y,\mathfrak{s}) &\simeq \check{\mathrm{HM}}(Y,\mathfrak{s};[w]), \\ \widehat{\mathrm{HF}}(Y,\mathfrak{s}) &\simeq \mathrm{HM}^{tot}(Y,\mathfrak{s};[w]), \end{aligned}$$

which are natural with respect to the fundamental exact sequences of Heegaard and Seiberg-Witten Floer homologies.

This conjecture has been verified for all known computations of both sides. In addition, since both Heegaard Floer homologies and Seiberg-Witten Floer homologies satisfy surgery exact sequences, if there is a map between two theories natural with respect to the surgery exact sequence, then the conjecture holds.

However, the difficulty in proving the above conjecture lies precisely in finding such a natural map. A quick look at the construction of both theories finds them very different both in geometric contents and abstract frameworks. We shall return to this subject in §3.

There are many variants and extensions of this conjecture which we omit in this article. For example, twisted versions of both Floer homologies are conjectured to relate in a similar fashion. In addition, Floer homologies are the building block for the definition of 4-manifold invariants and contact invariants in both theories, and these invariants are also conjectured to be equal. Overall, the Seiberg-Witten theory is more closely related to the geometry of underlying manifolds (e.g. scalar curvature), while the Ozsvath-Szabo theory, more combinatorial in flavor, is in general more computable.

1.2 Basic ingredients of Seiberg-Witten-Floer theory. Due to limitation of space, we shall not explain the construction of either theories, but refer the reader to the original literature. Here we shall only recall some basic notions for the sake of fixing notation and terminology.

A Seiberg-Witten *configuration* is a pair (A, ψ), where ψ is a section of the spinor bundle S over 3- or 4-dimensional spin-c manifold, and A is a connection on $\det S$. Because both the 3-dimensional and the 4-dimensional contexts appear in this article, we shall reserve the unhatted notation (A, ψ) for 3-dimensional configurations, and the hatted version $(\hat{A}, \hat{\psi})$ for 4-dimensional Seiberg-Witten configurations.

By a *Seiberg-Witten-Floer theory on* $(Y, \mathfrak{s})$ *perturbed by* ω, we mean the following. As a formal Morse theory, its chain groups are generated by what we call "*Seiberg-Witten critical points*", which are (gauge-equivalence classes of) solutions to the 3-dimensional (perturbed) Seiberg-Witten equations:

$$\begin{cases} \partial\!\!\!/_A \psi = 0 + \cdots \\ F_A - i\sigma(\psi,\psi) = i\omega + \cdots , \end{cases} \qquad (SW_3)$$

where $\sigma(\psi,\psi)$ is certain quadratic function of ψ, and ω is a closed 2-form.

The boundary map of the Floer complex is defined by counting what we call "*Seiberg-Witten flow lines*" between two Seiberg-Witten critical points, which are (gauge-equivalence classes of) solutions to the 4-dimensional (perturbed) Seiberg-Witten equations on $(\mathbb{R}\times Y, \mathfrak{s})$:

$$\begin{cases} \partial\!\!\!/_{\hat{A}} \hat{\psi} = 0 + \cdots \\ \mathrm{SD}(F_{\hat{A}}) - i\sigma(\hat{\psi},\hat{\psi}) = i\,\mathrm{SD}(\omega) + \cdots , \end{cases}$$

where SD denotes the self-dual part of a 2-form, and ω means the pull-back of the two form ω on Y to $\mathbb{R}\times Y$, and $\mathfrak{s}$ now denotes a spin-c structure on $\mathbb{R}\times Y$ via the identification of the spaces of spin-c structures on Y and $\mathbb{R}\times Y$.

The dots in the Seiberg-Witten equations above indicate that additional perturbation is needed to achieve transversality. These perturbations will not change the cohomology class of the CSD functional (or, in the terminology of §3, the class of the homomorphism PR). This technical point will be omitted in this article; for the precise form of these perturbations, see [22].

The 3-dimensional Seiberg-Witten invariant, denoted Sw_3, is the Euler characteristic of the Seiberg-Witten-Floer homology. When $b_1(Y)\geq 1$, it may be obtained by a straightforward signed count of solutions to (SW_3). It is independent of the perturbation when $b_1(Y)>1$. When $b_1=1$, it depends on the chamber containing $[w]$. In this article, Sw_3 will always mean the invariant in the "Taubes chamber" .

1.3 Basic ingredients of Heegaard Floer theory. Let

$$Y = H_- \cup_\Sigma H_+,$$

be a Heegaard splitting of Y, namely the 3-manifold Y is separated into the two handlebodies H_+, H_- by a Heegaard surface Σ of genus g.

A Morse function $f: Y\to\mathbb{R}$ is said to *adapt to the Heegaard splitting* if $f^{-1}(0)=\Sigma$ is the Heegaard surface, and

$$H_+ = f^{-1}\mathbb{R}_{\geq 0}, \quad H_- = f^{-1}\mathbb{R}_{\leq 0}$$

contain respectively one minimum p_+ and g index 1 critical points, and one maximum p_- and g index 2 critical points.

Let α_i, $i=1,\ldots,g$ denote the descending cycles on Σ from the g index 2 critical points. Similarly, let β_i, $i=1,\ldots,g$ denote the ascending cycles on Σ from the g index 1 critical points. Let

$$\mathbb{T}_\alpha := \alpha_1\times\cdots\times\alpha_g, \quad \mathbb{T}_\beta := \beta_1\times\cdots\times\beta_g \subset \mathrm{Sym}^g\,\Sigma.$$

Suppose $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ intersects transversely.

As a formal Morse theory, the chain groups in the Heegaard Floer theory is generated by what we call the "*Heegaard critical points*", which are intersection points of $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$. The boundary map is defined by counting what we call "*Heegaard flow lines*", which are holomorphic disks

$$\mu: \mathbb{R}\times[0,1]\to \mathrm{Sym}^g\,\Sigma, \quad \text{with } \mu(\cdot,0)\in\mathbb{T}_\alpha,\ \mu(\cdot,1)\in\mathbb{T}_\beta.$$

Of paramount importance in Heegaard Floer theory is the choice of a *base point*: Let $z\in\Sigma$ be a point avoiding the descending and ascending cycles, and let $\gamma_z\subset Y$

be the flow line of f from p_+ to p_- through z. We shall explain the role of z and γ_z in Heegaard Floer homologies in §4.

2 First motivation: Taubes's work on Sw = Gr

In fact, it is not surprising that the Seiberg-Witten-Floer homologies should be related to curve-counting invariants. Since Taubes's seminal work, the relation between Seiberg-Witten theory and Gromov's theory of pseudo-holomorphic curves has been well-known.

Theorem 2.1 (Taubes) *Let (X, ϖ) be a closed, oriented, symplectic 4-manifold, and $\mathfrak{s}$ be a spin-c structure on X. Then*

$$\mathrm{Sw}_4(X, \mathfrak{s}) = \mathrm{Gr}(X, \mathfrak{s}),$$

where Sw_4 is the Seiberg-Witten invariant for 4-manifolds, and Gr is a variant of Gromov invariant that counts embedded, possibly disconnected, pseudo-holomorphic curves (with multiplicity) in the homology class determined by $\mathfrak{s}$.

In the case where X is an algebraic manifold, this is just a simple analog of the correspondence between line bundles and divisors, and has been known since the first discovery of Seiberg-Witten theory. We shall briefly explain some of Taubes's ideas, as they will be central to our program.

First, choose a metric on X with respect to which the symplectic form ϖ is self dual (hence harmonic). The metric, together with ϖ, determines an almost complex structure on X. Observe that the Clifford action by ϖ splits the spinor bundle into a direct sum of eigenspaces:

$$S = E \oplus E \otimes K^{-1}, \tag{2.1}$$

where K^{-1} is the anti-canonical bundle. We shall therefore write

$$\hat{\psi} = (\hat{\alpha}, \hat{\beta})$$

in accordance with this splitting. As the Levi-Civita connection determines a connection of K^{-1}, specifying a spin-c connection is equivalent to specifying a connection $\hat{A}^E$ of E. We shall therefore denote a Seiberg-Witten configuration in this context as

$$(\hat{A}^E, (\hat{\alpha}, \hat{\beta})).$$

Taubes considered perturbation to the Seiberg-Witten equations on X by the two-form

$$\omega = r\varpi + \text{insignificant terms}, \quad \text{where } r \gg 1 \text{ is a constant.}$$

He showed that as $r \to \infty$, the zero locus $\alpha^{-1}(0)$ approaches a holomorphic curve, which is also the support of the current $\lim_{r\to\infty} F_{A^E}$. We shall call this a "*Taubes curve*".

His proof also gives a local description of the large r Seiberg-Witten solutions: Away from the Taubes curve, the solutions approximate the simple form

$$(\hat{A}^E, (\hat{\alpha}, \hat{\beta})) = (0, (\sqrt{r}, 0)), \quad \text{with respect to a trivialization of } E,$$

while near the Taubes curve, it approximates a family of vortex solutions parameterized by the Taubes curve.

A heuristic way to understand this result is as follows. Locally, varying r has the effect of rescaling the coordinates by $\sqrt{r}$. Thus, as $r \to \infty$, the Nijenhuis tensor can be ignored, reducing to the simpler Kähler situation mentioned above.

As we shall need several variants of Taubes's settings, we shall refer to them in general as "*Seiberg-Witten-Taubes*" theories, abbreviated as SWT.

2.2 Taubes's picture in 3 dimensions: Morse-Novikov theory.

(i) SWT theory on closed 4-manifolds. Instead of a symplectic form, one may consider any self-dual harmonic 2-form for ϖ. For generic metric, such ϖ vanishes along a set of circles in X, and K^{-1} and the splitting (2.1) makes sense away from these circles. Taubes proved in [26] an extension of the convergence theorem in part I of [25] to this setting. Here, a Taubes curve is a pseudo-holomorphic subvariety with boundary at $\varpi^{-1}(0)$; more precisely, the intersection number of the Taubes curve with any linking 2-sphere of $\varpi^{-1}(0)$ is 1.

However, lack of understanding on the local behaviors of the Seiberg-Witten solutions and the corresponding pseudo-holomorphic subvarieties deters progress on establishing a full equivalence generalizing Theorem 2.1. In fact, even the desired generalization of the Gromov invariant is undefined.

(ii) SWT theory on closed 3-manifolds. The 3-dimensional story is considerably simpler. Consider perturbation to the 3-dimensional Seiberg-Witten equations on a closed spin-c 3-manifold $(Y, \mathfrak{s})$ by $r\varpi + \cdots$, where ϖ is now a harmonic 2-form. The Hodge dual $*\varpi =: \vartheta$ is a harmonic 1-form; for a generic metric, it is Morse. This brings us to the realm of Morse-Novikov theory, as follows.

The zero locus $\varpi^{-1}(0)$ is now the critical points of ϑ: they come in g pairs of index 2 and index 1 critical points. Let $\check{\vartheta}$ denote the dual vector field of ϑ. The counterpart of a Taubes curve is a union of finite-length flow lines of $-\check{\vartheta}$, such that the union of their boundary is precisely the critical set $\varpi^{-1}(0)$. We call this a "*Taubes orbit*", and the flow lines constituting a Taubes orbit the "*constituent flow lines*".

Basing on Taubes's picture, we wrote down a counting invariant of Taubes orbits, I_3, and conjectured:

Conjecture [12] *Let $(Y, \mathfrak{s})$ be a closed spin-c 3-manifold with $b_1(Y) > 0$. Then*

$$\mathrm{Sw}_3(Y, \mathfrak{s}) = I_3(Y, \mathfrak{s}).$$

In fact, I_3 is just a special case of an invariant I in general Morse-Novikov theory. This invariant I takes the form of a product of the Reidemeister torsion of the Morse-Novikov complex, with a dynamical zeta function that counts closed orbits of the flow of $-\check{\vartheta}$. Moreover, we showed:

Theorem [12, 13] *For an oriented, closed manifold M,*

$$I(M) = \tau(M),$$

$\tau(M)$ being the combinatorially defined Reidemeister torsion of the manifold M. In particular, $I_3(Y) = T(Y)$, the Turaev torsion.

It is known that Sw_3 is equivalent to the Turaev torsion, either by the surgery formulae of Meng-Taubes, or by the TQFT arguments of Donaldson-Mark. Thus, the above Conjecture is proven indirectly.

(iii) SWT Floer theory on closed 3-manifolds. It is also possible to generalize this picture to Floer theory. Consider the Seiberg-Witten-Floer theory with the same perturbation ω. Its chain group will be generated by 3-dimensional SWT solutions, which corresponds to the Taubes orbits described above. The boundary map is defined via the moduli spaces of Seiberg-Witten solutions on $\mathbb{R} \times Y$, with perturbation $\mathrm{SD}(\omega) + \cdots$. The latter now vanishes along lines $\mathbb{R} \times \varpi^{-1}(0)$. The corresponding Taubes curves are now pseudo-holomorphic curves in $\mathbb{R} \times Y \backslash (\mathbb{R} \times \varpi^{-1}(0))$, with boundary at the vanishing locus $\mathbb{R} \times \varpi^{-1}(0)$. Thus, the analog of Theorem 2.1 states that the Seiberg-Witten-Floer homology is equivalent to a symplectic version of Floer homology, analogous to the contact homology.

In the simplest special case, when Y is a mapping torus, it is possible to choose ϑ such that $\varpi^{-1}(0) = \emptyset$. Hutchings established some foundations of the proposed symplectic version of Floer homology in this case, which he calls "periodic Floer homology" [11].

While this picture potentially offers an interesting connection with contact homology, it has so far not offered great help on the understanding of Seiberg-Witten-Floer homologies, as its construction and computation is no simpler than Seiberg-Witten-Floer homologies.

2.3 Heegaard-splittings and real-valued Morse theory. Instead of the Morse-Novikov theory of closed 1-forms, it is desirable to have a geometric interpretation of 3-dimensional Seiberg-Witten theory in terms of real-valued Morse theory, for the following two reasons:

- The important case of rational homology spheres ($b_1(Y) = 0$) is excluded from the discussion in §2.2.
- Via the Heegaard splitting associated to a real-valued Morse function, it is possible to reduce the computation to 2-dimensions, namely, to the Heegaard surface. This would entail great simplifications.

At the first sight, this does not appear possible. Suppose $\vartheta = df$ for a real valued function f. The harmonicity of ϑ implies the harmonicity of f, but there is no non-constant harmonic function on a closed manifold.

This, however, does not constitute a serious obstacle. Real-valued harmonic functions do exist on *non-compact* 3-manifolds. For instance, one may consider

1. **(3-d Euclidean SWT theory)** $Y\#\mathbb{R}^3$ with Euclidean metric at infinity. Seiberg-Witten-Taubes theories on such 3-manifolds are considered in [16].
2. **(3-d cylindrical SWT theory)** Deleting two points p_+, p_- from Y, then choosing a complete metric on the resulting open manifold, so that it has two cylindrical ends $\mathbb{R}_\pm \times S^2$. There exist harmonic functions f on such cylindrical manifolds which are asymptotic to

$$\tau + C_\pm \tag{2.2}$$

on the two cylindrical ends, where τ parameterizes $\mathbb{R}_\pm$. Consider spin-c structures such that at the spheres at infinity, $E|_{S^2_{\pm\infty}}$ has nonnegative degree. In particular, when the degree is 0, denote the corresponding spin-c structure by $\mathfrak{s}^\circ$. More generally, one may consider any 3-manifolds with a finite number of cylindrical ends, with a harmonic function with asymptotic condition (2.2) on the "negative" and "positive" ends as $\tau \to -\infty, \infty$ respectively.

Taubes's pictures for both settings are similar; so we shall take the second situation for example. Let f be a Morse function adapted to a Heegaard splitting, as described in §1.3. Then the constituent flow lines of a Taubes orbit of spin-c structure $\mathfrak{s}^\circ$ may be described as intersection points of descending cycles α_i and ascending cycles β_j, and the Taubes orbit corresponds precisely to a intersection point of $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ in $\mathrm{Sym}^g\,\Sigma$. Notice that the fact that f is real-valued excludes the possibility of closed orbits from constituent flow lines; in particular, all constituent flow lines in Taubes orbits have multiplicity 1. This represents another advantage of real-valued Morse theory over the Morse-Novikov theory.

Moreover, as explained in §2.2, the Taubes curves corresponding to SWT flow lines now correspond to holomorphic curves with boundary along the lines $\mathbb{R} \times \mathrm{Crit}(f)$. Analogy with the Atiyah-Floer conjecture (cf. §7.2, 9.2) leads to the expectation that these corresponds to holomorphic disks in $\mathrm{Sym}^g\,\Sigma$ with boundary along $\mathbb{T}_\alpha$, $\mathbb{T}_\beta$, and that the Seiberg-Witten-Floer homology corresponds to the Floer homology of "Lagrangian" intersections of $(\mathrm{Sym}^g\,\Sigma; \mathbb{T}_\alpha, \mathbb{T}_\beta)$. This is precisely what Ozsvath-Szabo call "HF^∞".

Independent of its relation with Seiberg-Witten theory, this Morse-theoretic picture defines certain topological invariants. However, more serious problems of these ideas are:

- These are decidedly different from the ordinary Seiberg-Witten invariant or Floer homologies of Y.
- In fact, they are not interesting invariants, since they only depend on homological information about Y.

The first point is not unexpected, since this story supposedly corresponds to Seiberg-Witten theory on *noncompact* manifolds, while gauge-theoretic invariants on noncompact manifolds are typically very sensitive to asymptotic conditions. In fact, viewing the cylindrical manifolds in situation 2 as the closed manifold Y minus two points, the choice of spin-c structures and perturbations in the Seiberg-Witten-Taubes theory are those which do not extend across the two points.

The second problem is, of course, a lot worse. To illustrate it, notice that

$$\begin{aligned}
&\#\,(\mathbb{T}_\alpha \cap \mathbb{T}_\beta)\\
&\quad = \det\left(\#(\alpha_i \cap \beta_j)\right)\\
&\quad = \begin{cases} 0 & \text{as } b_1(Y) > 0;\\ |H_1(Y;\mathbb{Z})| & \text{as } b_1(Y) = 0,\end{cases}
\end{aligned}$$

which contains a lot less information than the ordinary Seiberg-Witten invariants of 3-manifolds.

This is where people abandoned this approach. It took the advent of Ozsvath-Szabo's amazing idea–filtration– to revive this program. (See §4).

3 The Floer-theory framework

We now examine the abstract frameworks of both Floer theories.

3.1 Basic ingredients of a Floer theory. A typical Floer theory is modeled on a formal Morse theory over an infinite dimensional space $\mathcal{C}$. Usually, one has a possibly non-exact Morse 1-form over $\mathcal{C}$. Furthermore, in contrast to the finite-dimensional Morse theory, the (relative) index of the critical points is only defined modulo an integer N. There is a minimal abelian covering $\tilde{\mathcal{C}} \to \mathcal{C}$ such that the

Morse 1-form lifts to a differential of a $\mathbb{R}$-valued Morse function, and indices of lifts of the critical points are well-defined in $\mathbb{Z}$ *for all parameters of the theory* (e.g perturbations, metrics, spin-c structures). Let G be the covering group, and let

$$\mathrm{PR}: G \to \mathbb{R}; \quad \mathrm{SF}: G \to \mathbb{Z}$$

denote homomorphisms defined by the change in the values of the Morse function and the indices under deck transformation. (PR stands for "period", and SF stands for "spectral flow").

These are important basic ingredients of the Floer theory, because Floer complexes with different local coefficients are modeled on the Morse complexes on sub-covers $\hat{\mathcal{C}}$ of $\tilde{\mathcal{C}}$, on which the Morse 1-form lifts to be exact. Thus, $G, \mathrm{SF}, \mathrm{PR}$ encode the module structure, coefficient rings, and grading of these various Floer complexes.

Unless otherwise specified, a "Floer complex" means the version corresponding to a minimal $\hat{\mathcal{C}}$. It has ordinary R-coefficients when:

$$\mathrm{PR}\,|_{\mathrm{Ker\,SF}} = \emptyset. \tag{3.1}$$

The finite dimensional Morse-Novikov theory also leads one to expect the Floer homology to be invariant under variation of metric and perturbations which satisfy

$$\text{PR remains on a positive ray from 0.} \tag{3.2}$$

3.2 Comparing the frameworks of two theories. The following table compares the formal framework of the Seiberg-Witten and Heegaard Floer theories in the above notation. It gives a first indication of the many fundamental differences between the two theories.

	Seiberg-Witten	Heegaard
$\mathcal{C}$	$\{(A,\psi)\}/\mathcal{G}$: quotient of the spaces of configurations by gauge group action	$\Omega(\mathrm{Sym}^g(\Sigma); \mathbb{T}_\alpha, \mathbb{T}_\beta)$: space of paths $\gamma: [0,1] \to \mathrm{Sym}^g(\Sigma)$, with $\gamma(0) \in \mathbb{T}_\alpha$, $\gamma(1) \in \mathbb{T}_\beta$
	$\mathcal{C}$ singular at fixed points (reducibles)	$\mathcal{C}$ smooth
G	$H^1(\mathcal{C}; \mathbb{Z}) = H^1(Y; \mathbb{Z})$	$\pi_2(\mathbf{x}, \mathbf{x}) = \mathbb{Z} \oplus H^1(Y; \mathbb{Z})$ (*)
PR	$2\pi(2\pi c_1(\mathfrak{s}) - [w])$ (†)	$-C \oplus 0$ (‡) C is a positive constant
SF	$c_1(\mathfrak{s})$	$2 \oplus c_1(\mathfrak{s})$

Remark (*) Projection to the $\mathbb{Z}$ factor is given by the intersection number of the base point z with the 2-chain in Σ associated to the 1-cycle in $\Omega(\mathrm{Sym}^g(\Sigma); \mathbb{T}_\alpha, \mathbb{T}_\beta)$.

(†) Here and below, we regard a cohomology class in $H^2(Y)$ as a homomorphism $H^1(Y) \to \mathbb{R}$ via Poincaré duality.

(‡) This assumes that $\mathbb{T}_\alpha$, $\mathbb{T}_\beta$ are Lagrangian. However, as defined by Ozsvath-Szabo, they are typically only totally real. This technical point is ignored here, as this section is for motivational purposes only.

As the covering group G of the Heegaard Floer theory contains an extra $\mathbb{Z}$-factor than that of the Seiberg-Witten theory, to put both theories on the same footing, one regards the Heegaard Floer complex (CF^∞) as an *infinitely generated* $R[H^1(Y;\mathbb{Z})]$ module. In other words, instead of a Morse theory on $\mathcal{C}$, model the Heegaard Floer theory on the Morse theory on the infinite-cyclic covering of $\mathcal{C}$ corresponding to the projection of G to $\mathbb{Z}$. We denote this infinite-cyclic covering by $\hat{\mathcal{C}}_z$, since the projection is determined by z.

On the other hand, notice that according to the table, if one chooses the class of perturbation form to be $[w] = 2\pi c_1(\mathfrak{s})$, then the homomorphism PR in Seiberg-Witten theory is 0. In this case, the homomorphisms PR and SF in Seiberg-Witten theory agree with the restriction of their Heegaard counterparts to the $H^1(Y;\mathbb{Z})$-factor. This is a first motivation for the choice (1.1) in Conjecture 1.1. Furthermore, with this choice, the condition (3.1) holds in both theories, and thus both Floer complexes are of R-coefficients.

With these explained, one may continue to observe the numerous formal parallelisms of both theories. However, these parallel aspects come from entirely different origins. In fact, both Floer theories require refinements of the basic Floer theory framework outlined in §3.1 above, but the main difference is that the refinement needed for each theory is based on different principles: the Heegaard Floer theory relies on a filtration of the complex associated to the infinite-cyclic covering, while the Seiberg-Witten theory is an S^1-equivariant theory.

We shall discuss these different refinements separately in the next two sections. For now, we continue the table of comparison that highlights the difference of the refinements.

Formal analogies	Seiberg-Witten	Heegaard
both are $R[u]$-modules	because this is an equivariant theory: U generates $H^*(BS^1)$	because this is modeled on Morse theory on a $\mathbb{Z}$-cover: U generates deck transformation
both complexes are ∞-generated R-modules	because of reducible critical points (fixed points)	because there are ∞-many critical points on a $\mathbb{Z}$-cover
long exact sequences relate $\widehat{\mathrm{HM}}$, $\overline{\mathrm{HM}}$, $\check{\mathrm{HM}}$, and HF^-, HF^∞, HF^+	this is a fundamental sequence of S^1-equivariant theory	this is the relative sequence associated with a filtration
both have a 4th version	HM^{tot} is modeled on homology of an S^1-bundle	$\widehat{\mathrm{HF}}$ models on homology of a fundamental domain

4 Filtration in Heegaard Floer theory

The key point in Ozsvath-Szabo's construction is a filtration argument which is less commonly seen in the Morse/Floer theory literature. We shall put their construction in an abstract formulation, since the same construction is needed again in §6.

4.1 1-cocycles, local coefficients, and infinite-cyclic coverings. Let M be a manifold, and Z be a 1-cocycle in M, or equivalently, a codimension 1 cycle in M. Suppose the cohomology class $[Z]$ is primitive; in particular, nontrivial. Such Z defines a local system, $\Gamma(Z)$, over M, by assigning each 1-chain in M its intersection number with Z. Let $\tilde{M}_Z$ be the infinite-cyclic covering associated to Z, namely, the covering obtained by cutting M open along Z, and gluing $\mathbb{Z}$ copies of such. The homology of M with local coefficients $\Gamma(Z)$ may alternatively be regarded as the homology of the covering $\tilde{M}_Z$. This is a graded $R[\mathbb{Z}] = R[t, t^{-1}]$ module.

In the Morse-Novikov context, given a closed Morse 1-form θ on M with cohomology class

$$[\theta] = \alpha[Z] \quad \text{for } \alpha \in \mathbb{R}, \tag{4.1}$$

one may define a Morse complex with local coefficients, $M_*(\theta, \Gamma(Z))$. This is a module of the completed ring $R[t^{-1}, t]]$ when $\alpha > 0$, of completion in the opposite direction, $R[t, t^{-1}]]$ when $\alpha < 0$. Here and below, t denotes a *negative* generator of the deck transformation, in the sense that t decreases the value of $\tilde{f}$, where $d\tilde{f}$ is the lift of θ to $\tilde{M}_Z$.

4.2 Filtration associated to semi-positive 1-cocycles. We say that Z is *semi-positive* with respect to θ if the cohomology class $[Z]$ satisfies (4.1), and $Z(\gamma) \geq 0$ if the path γ is a flow line of $-\check{\theta}$.

The positivity condition implies that Z gives a filtration on the Morse complex, and we may consider the associated subcomplexes (as R-modules), and quotient complexes, which are now $R[t]$-modules:

Let $\tilde{Z}$ be a lift of Z in the infinite-cyclic cover $\tilde{M}_Z$. It divides $\tilde{M}_Z$ into two halves. Let $\tilde{M}_Z^- \subset \tilde{M}_Z$ be the lower half (with respect to the flow of $-\check{\theta}$). The subcomplex $M_*^-(\theta, \Gamma(Z))$ may be understood as the Morse complex for $\tilde{M}_Z^-$. The quotient complex

$$M_*^+(\theta, \Gamma(Z)) := M_*(\theta, \Gamma(Z))/M_*^-(\theta, \Gamma(Z))$$

is then the complex of the pair $(\tilde{M}_Z, \tilde{M}_Z^-)$.

From the short exact sequence of R-modules

$$0 \to M_*^-(\theta, \Gamma(Z)) \to M_*(\theta, \Gamma(Z)) \to M_*^+(\theta, \Gamma(Z)) \to 0,$$

we have a long exact sequence of the pair $(\tilde{M}_Z, \tilde{M}_Z^-)$.

Notice that a different, though cohomologous choice of Z yields an equivalent local system, but a possibly different filtration. Thus, another semi-positive 1-cocycle Z' cohomologous to Z gives the same $H_*(M_*(\theta, \Gamma(Z)))$, but possibly different $H_*(M_*^-(\theta, \Gamma(Z)))$ and $H_*(M_*^+(\theta, \Gamma(Z)))$. Similarly, $H_*(M_*(\theta, \Gamma(Z)))$ is invariant under any exact perturbation to θ, but $H_*(M_*^-(\theta, \Gamma(Z)))$ and $H_*(M_*^+(\theta, \Gamma(Z)))$ in general are only invariant under *small* exact perturbations of θ.

Example Write $\theta = df$ for a circle-valued function f, and take Z to be the 1-cocycle θ, or the codimensional 1 cycle given by a level set of $-f$. Then Z is a semi-positive, and the associated filtration is the filtration by energy.

Example Let P be a Kähler manifold, and consider a symplectic version of Floer theory on P, in which $\mathcal{C}$ is a (relative) loop space Ω of P. A 1-cycle in Ω traces out a (relative) 2-cycle in P, and a flow line in Ω corresponds to a holomorphic curve in P. The intersection with a (complex) hypersurface in P thus defines a semi-positive 1-cocycle. See [24].

Example (Heegaard Floer homologies) The Heegaard Floer theory is a variant of the construction of the previous example. The hypersurface in this context is $\{z\} \times \mathrm{Sym}^{g-1} \Sigma$. The first three versions of Heegaard Floer homologies are respectively formal analogs of $H_*(M_*^-(\theta, \Gamma(Z)))$, $H_*(M_*(\theta, \Gamma(Z)))$, $H_*(M_*^+(\theta, \Gamma(Z)))$ above, and the fundamental exact sequence of Heegaard Floer homologies is modeled on the relative exact sequence of the pair $(\tilde{M}_Z, \tilde{M}_Z^-)$.

The fourth version, $\widehat{\mathrm{HF}}$, corresponds to the homology of a fundamental domain; more precisely, $H_*(\tilde{M}_Z^-, t\tilde{M}_Z^-)$.

Using topological arguments very special to this specific two-dimensional situation, Ozsvath-Szabo showed that these Floer homologies depend on z only through the spin-c structure.

5 Equivariant aspects of Seiberg-Witten-Floer theory

Here is a reformulation of the construction of Kronheimer-Mrowka. This formulation comes in particularly handy for our discussion on filtrated connected sum formulae, in §8 below.

5.1 The algebraic S^1-bundle. Let (C, ∂_C) be a complex of R-modules, and U be a degree -2 chain map on C. We form the following new complex

$$(S_U(C), \partial_S) = (C \otimes R[y],\ \partial_C \otimes \sigma + U \otimes y), \tag{5.1}$$

where $\deg(y) = 1$, $y^2 = 0$, and the homomorphism $\sigma : R[y] \to R[y]$ is defined by

$$\sigma(a + by) = a - by \quad \text{for } a, b \in R.$$

We call $S_U(C)$ an *algebraic S^1-bundle*, due to the following observation:

Lemma *If $C = C_*(B)$ is the chain complex of a manifold B, and $U_* : H_*(B) \to H_{*-2}(B)$ agrees with the cap product with a cohomology class $u \in H^2(B)$, then $H_*(S_U(C))$ is the homology of the S^1-bundle over B with Euler class u.*

To see this, view $S_U(C)$ as a double complex, and notice that the E^2 term of the associated spectral sequence agrees with the E^2 term of the Serre spectral sequence of the aforementioned S^1 bundle.

5.2 Jones's formulation of equivariant homologies. In retrospect, Ozsvath-Szabo's definition of the first three versions of Heegaard Floer homologies is reminiscent of Jones's formulation of S^1-equivariant homologies, which we sketch below.

Let T be a S^1 space. There are three versions of equivariant homologies (cohomologies) that fit into fundamental long exact sequences

$$\cdots \to H^n_{S^1}(T) \to \hat{H}^n_{S^1}(T) \to G^{n+2}_{S^1}(T) \to H^{n+1}_{S^1}(T) \to \cdots$$
$$\cdots \to G^{S^1}_n(T) \to \hat{H}^{S^1}_n(T) \to H^{S^1}_{n-2}(T) \to G^{S^1}_{n-1}(T) \to \cdots$$

where $\hat{H}$ denotes the Tate version (localized version), and G is the co-Borel version which is dual to the usual (Borel) equivariant (co)homology over $R[U]$.

Modeling on the Serre spectral sequence of the fibration

$$T \to T \times_{S^1} ES^1 \to BS^1, \tag{5.2}$$

J. D. S. Jones [14] wrote down the three versions of equivariant (co)homologies in the following alternative way.

Let S be the singular chain module or singular cochain algebra of T. The S^1 action on T,

$$g : S^1 \times T \to T,$$

equips S with a natural degree 1 chain map, J:

$$J(x) = \begin{cases} (-1)^{\deg(x)} g_* \delta(z \otimes x) & \text{when } S \text{ is the singular chain module,} \\ (-1)^{\deg(x)} g^* x / z & \text{when } S \text{ is the singular cochain module,} \end{cases}$$

where $\delta : C_*(S^1) \otimes C_*(T) \to C_*(S^1 \times T)$ is the Eilenberg-Zilber product, and z is the fundamental 1-cycle of S^1.

Writing

$$V^- := R[u],\ V^\infty := R[u, u^{-1}],\ V^+ := R[u, u^{-1}]/uR[u],$$

we define:

$$E^\bullet(S) := (S\otimes V^\bullet,\ \partial_S\otimes 1 + J\otimes u)\quad \text{for } \bullet = -,\infty,+. \tag{5.3}$$

(Jones's notation for E^-, E^∞, E^+ are $U^-, U^\wedge, U^+$ for homologies, and $V^-, V^\wedge, V^+$ for cohomologies respectively).

Lemma (Jones) *In the above notation,*

$$H_*(E^-(S)) = H^{-*}_{S^1}(T)\quad \textit{when } S \textit{ is the singular cochain algebra;}$$

$$H_*(E^+(S)) = H^{S^1}_*(T)\quad \textit{when } S \textit{ is the singular chain module.}$$

Furthermore, the long exact sequence induced by the short exact sequence

$$0\to uE^-(S)\to E^\infty(S)\to E^+(S)\to 0$$

is precisely the fundamental exact sequence for equivariant homologies/cohomologies, depending on whether S is the chain or cochain module.

5.3 HM^{tot}**, and** $\widehat{\mathrm{HM}}, \overline{\mathrm{HM}}, \check{\mathrm{HM}}$ **as equivariant homologies.** We now combine the discussion in §5.1, 5.2 into a re-interpretation of Kronheimer-Mrowka's construction.

Recall that Kronheimer-Mrowka's Floer complexes $\widehat{\mathrm{CM}}$, $\check{\mathrm{CM}}$ are modeled on Morse theory of the (real) blow-up of an S^1-space along its fixed-point-set. Thus, they are analogs of the chain complex of the base space of a S^1-bundle. We may thus apply the algebraic S^1-bundle construction in §5.1 to these complexes. Let

$$\mathrm{SM}^- := S_U(\widehat{\mathrm{CM}});\quad \mathrm{SM}^+ := S_U(\check{\mathrm{CM}}).$$

These chain complexes come equipped with a degree 1 chain map, namely, multiplication by the nilpotent variable y. This is precisely the J-map above under the interpretation of S_U as the chain complex of an S^1-bundle. One may now apply the constructions E^-, E^∞, E^+ to either SM^+ or SM^- to obtain a sequence of equivariant homologies.

Lemma (a) *There is a degree 0 chain map $S_U(j) : \mathrm{SM}^+\to \mathrm{SM}^-$ commuting with the J-map, which induces isomorphism in homologies.*

(b) *$E^+(\mathrm{SM}^+)$ has the same homology as $\check{\mathrm{CM}}$; $E^-(\mathrm{SM}^-)$ has the same homology as $\widehat{\mathrm{CM}}$. $H_*(E^\infty(\mathrm{SM}^-)) = \widehat{\mathrm{HM}}\otimes_{R[U]} R[U, U^{-1}]$; similarly for the plus-check version.*

(c) *$E^+(\mathrm{SM}^-)$ has the same homology as $\check{\mathrm{CM}}$; $E^-(\mathrm{SM}^+)$ has the same homology as $\widehat{\mathrm{CM}}$.*

To see part (a), recall the maps

$$j : \check{\mathrm{CM}}_*\to \widehat{\mathrm{CM}}_*;\quad p : \widehat{\mathrm{CM}}_*\to \overline{\mathrm{CM}}_{*-1};\quad i : \overline{\mathrm{CM}}_*\to \check{\mathrm{CM}}_*$$

defined by Kronheimer-Mrowka, which induce the fundamental long exact sequence of Seiberg-Witten-Floer homologies. We define

$$S_U(j) : S_U(\check{\mathrm{CM}})_*\to S_U(\widehat{\mathrm{CM}})_*,$$
$$S_U(p) : S_U(\widehat{\mathrm{CM}})_*\to S_U(\overline{\mathrm{CM}})_{*-1},$$
$$S_U(i) : S_U(\overline{\mathrm{CM}})_*\to S_U(\check{\mathrm{CM}})_*$$

to be the natural generalization of the above maps j, p, i. For example, in terms of the notation of [22],

$$S_U(j) = \begin{pmatrix} 1 & 0\\ 0 & \bar\partial^s_u + \bar U^s_u y\end{pmatrix}.$$

A straightforward computation shows that these are indeed chain maps, and they induce a long exact sequence in homologies

$$\begin{array}{ccc} & H_*(S_U(\bar{\mathrm{CM}})) & \\ {\scriptstyle S_U(p)_*}\nearrow & & \searrow{\scriptstyle S_U(i)_*} \\ H_*(S_U(\hat{\mathrm{CM}})) & \xleftarrow[S_U(j)_*]{} & H_*(S_U(\check{\mathrm{CM}})) \end{array} \tag{5.4}$$

Furthermore, $S_U(j)$, $S_U(p)$, $S_U(i)$ are all J-preserving (i.e. commuting with J), since in all three cases, the J-map is simply multiplication by y.

On the other hand, it is easy to see from a spectral sequence calculation that

$$H_*(S_U(\bar{\mathrm{CM}})) = 0. \tag{5.5}$$

(5.4) and (5.5) together imply that $S_U(j)_*$ is an isomorphism.

Part (b) is also consequence of simple spectral sequence calculations.

Part (c) is in fact a special case of [14] Lemma 5.2, which states that:

If there is a J-preserving chain map from S_1 to S_2, inducing an isomorphism from $H_*(S_1)$ to $H_*(S_2)$, then the induced map from $H_*(E^\bullet(S_1))$ to $H_*(E^\bullet(S_2))$ is also an isomorphism, for $\bullet = -, \infty, +$.

This is a simple consequence of the observation that there is a spectral sequence to compute the homology of $E^\bullet(S)$ from the homology of S (analog of the Serre spectral sequence of the fibration (5.2)), which is natural with respect to J-preserving chain maps.

Applying the above statement to the J-preserving map $S_U(j)$ yields part (c).

Because of part (a) of the above lemma, we may now define

$$\mathrm{HM}^{tot} := H_*(S_U(\check{\mathrm{CM}})) = H_*(S_U(\hat{\mathrm{CM}})). \tag{5.6}$$

Parts (b) and (c) of the previous lemma, together with the next lemma, imply the following corollary:

Corollary *The fundamental exact sequence of the Seiberg-Witten-Floer homologies of Kronheimer-Mrowka agrees with the fundamental exact sequence of equivariant homologies:*

$$\begin{array}{ccccccc} \cdots H_*(uE^-(\mathrm{SM}^\pm)) & \longrightarrow & H_*(E^\infty(\mathrm{SM}^\pm)) & \longrightarrow & H_*(E^+(\mathrm{SM}^\pm)) & \longrightarrow & H_{*-1}(uE^-(\mathrm{SM}^\pm))\cdots \\ \| & & \| & & \| & & \| \\ \cdots \hat{\mathrm{HM}}_{*+1} & \longrightarrow & \bar{\mathrm{HM}}_* & \longrightarrow & \check{\mathrm{HM}}_* & \longrightarrow & \hat{\mathrm{HM}}_*\cdots \end{array} \tag{5.7}$$

Lemma (Localization Theorem) *We have*

$$H\hat{M}(Y, \mathfrak{s}; [w]) \otimes_{R[U]} R[U, U^{-1}] = H\bar{M}(Y, \mathfrak{s}; [w])$$

This is the analog of the familiar localization theorem in equivariant theory (see e.g. [1]). In this context, the proof relies on the nilpotence of the U action on CM^o, the submodule of $\hat{\mathrm{CM}}$ generated by irreducible critical points.

6 *HMT*: Seiberg-Witten-Floer theory with Taubes's perturbations

To bridge the gulf of differences between the two theories as pointed out above, our approach is to introduce an intermediate object: a third set of Floer homologies, $\text{HMT}^\bullet$, which also come in four flavors, $\bullet = -, \infty, +, \wedge$, and the first 3 versions fit into a fundamental long exact sequence. HMT should probably reads "Heegaard-Monopole-Taubes", meaning that it is a variant of the SWT Floer theory sketched in §2, whose definition also involves some Heegaard ingredients: the choice of a Heegaard splitting of Y, and a filtration associated to a 1-cycle $\underline{\gamma}_z$.

We shall show that the equivalence of HMT with either theory is easier to establish, and Conjecture 1.1 is thus broken into two:

Conjecture 6.1 *Let* $(Y, \mathfrak{s})$ *and* $[w] = 2\pi c_1(\mathfrak{s})$ *be as in Conjecture 1.1. Then there are isomorphisms of* $R[U]$*-modules:*

(**a**) $\overline{\text{HM}}(Y, \mathfrak{s}; [w]) = \text{HMT}^\infty(\underline{Y}, \underline{\mathfrak{s}})$; $\widehat{\text{HM}}(Y, \mathfrak{s}; [w]) = \text{HMT}^-(\underline{Y}, \underline{\mathfrak{s}}; \underline{\gamma}_z)$;

$\check{\text{HM}}(Y, \mathfrak{s}; [w]) = \text{HMT}^+(\underline{Y}, \underline{\mathfrak{s}}; \underline{\gamma}_z)$; $\text{HM}^{tot}(Y, \mathfrak{s}; [w]) = \widehat{\text{HMT}}(\underline{Y}, \underline{\mathfrak{s}}; \underline{\gamma}_z)$.

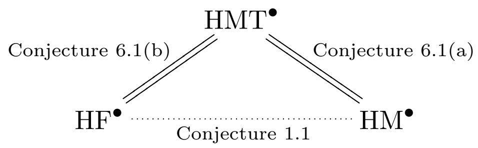
(**b**) $\text{HF}^\infty(Y, \mathfrak{s}) = \text{HMT}^\infty(\underline{Y}, \underline{\mathfrak{s}})$; $\text{HF}^\pm(Y, \mathfrak{s}) = \text{HMT}^\pm(\underline{Y}, \underline{\mathfrak{s}}; \underline{\gamma}_z)$;

$\widehat{\text{HF}}(Y, \mathfrak{s}) = \widehat{\text{HMT}}(\underline{Y}, \underline{\mathfrak{s}}; \underline{\gamma}_z)$.

Furthermore, these isomorphisms are all natural with respect to the fundamental sequences of $\text{HM}^\bullet$*,* $\text{HMT}^\bullet$*, and* $\text{HF}^\bullet$*.*

As explained before, the U-map in the Heegaard Floer theory acts by deck transformation, while in Seiberg-Witten-Floer theory, it is cap (cup) product with the generator of $H^*(BS^1)$. For the intermediate HMT theory, there are two natural module structures, one from deck transformation, and the other from equivariant theory. It turns out that these two module structures are identical, so U above denotes either action in this theory.

The following triangle best illustrates the relation among the three Floer theories.



6.2 The setup: $\underline{Y}$, $\underline{\gamma}_z$, and $\underline{\mathfrak{s}}$. We now describe the construction of $\text{HMT}^\bullet$.

Let $f : Y \to \mathbb{R}$ be a Morse function adapted to a Heegaard decomposition of Y, as in §1.3. As explained before, the obstruction to making f harmonic is the existence of the maximum and minimum of f, namely p_-, p_+. Attach a 1-handle to Y along the two points p_+, p_-, and call the resulting manifold $\underline{Y}$.

$$\underline{Y} = Y' \cup_{\partial B_{p_+} \sqcup \partial B_{p_-}} \left([-1, 1] \times S^2\right), \quad \text{where } Y' := Y \backslash (B_{p_+} \sqcup B_{p_-}).$$

One may now extend f to a *circle-valued* Morse function

$$\underline{f} : \underline{Y} \to S^1,$$

which has no extrema. It is then possible to choose a metric on $\underline{Y}$, making $\underline{f}$ harmonic [3].

We shall use the notation $\Sigma_H := \underline{f}^{-1}(0)$ to denote the Heegaard surface, to distinguish it from other genus g surfaces $\Sigma \subset \underline{Y}$. Given an interval $I \subset S^1$, $\underline{Y}_I \subset \underline{Y}$ will denote $\underline{f}^{-1}I$, and $\underline{Y}_{]a,b[} := \underline{Y}_{S^1\backslash(a,b)}$.

Let $z \in \Sigma_H$, $\gamma_z \subset Y$ be the base point and associated 1-chain defined in §1.3. Let $\underline{\gamma}_z \subset \underline{Y}$ be a 1-cycle through z completing γ_z: more precisely, we choose it such that:

- **(Γ1):** There are $\kappa \in S^1$, $\delta \in \mathbb{R}^+$, such that $\underline{Y}_{[\kappa-\delta,\kappa+\delta]}$ is contained in the added 1-handle, and $\underline{\gamma}_z \cap \underline{Y}_{]\kappa-\delta,\kappa+\delta[}$ is a gradient flow line of $\underline{f}$ through $z \in \Sigma_H$;
- **(Γ2):** $\underline{f}$ is monotone along $\underline{\gamma}_z$;
- **(Γ3):** $\underline{\gamma}_z$ avoids the ascending and descending manifolds from the critical points of $\underline{f}$ in $\underline{Y}\backslash\Sigma_H$.

The values κ, δ will be chosen so that Claim 6.4 (3) below holds. In addition, we also require that

$$[*d\underline{f}] = \alpha\, \mathrm{P.\,D.}[\underline{\gamma}_z] \in H^2(\underline{Y}) \quad \text{for some constant } \alpha > 0. \tag{6.1}$$

This may be achieved by fixing the metric, then choosing z so that the homology class $[\underline{\gamma}_z]$ meets the requirement. Alternatively, fixing the class $[\underline{\gamma}_z]$, a closer examination of Calabi's argument in [3] shows that a metric can be found so that the class $[*d\underline{f}]$ satisfies (6.1) [15].

Let S_z be the boundary 3-sphere of a tubular neighborhood of γ_z. It splits $\underline{Y}$ into a connected sum

$$\underline{Y} = Y \#_{S_z} S^1 \times S^2. \tag{6.2}$$

Given a spin-c structure $\mathfrak{s}$ on Y, let $\underline{\mathfrak{s}}$ be the spin-c structure on $\underline{Y}$ defined by

$$\underline{\mathfrak{s}} = \mathfrak{s} \#_{S_z} \mathfrak{s}_K, \tag{6.3}$$

where $\mathfrak{s}_K$ is the spin-c structure on $S^1 \times S^2$ corresponding to the standard nowhere-vanishing vector field on $S^1 \times S^2$, namely $\nabla\tau$, τ parameterizing S^1. Note that $\mathfrak{s}_K$ is *not* the trivial spin-c structure.

$$c_1(\mathfrak{s}_K) = 2\,\Omega_S,$$

where Ω_S is the positive generator of $H^2(S^1 \times S^2)$.

Remark We use here and below the following orientation convention for $S^1 \times S^2$ and its homologies: The parameterization τ orients S^1; S^2 is given the complex orientation, and $S^1 \times S^2$ given the product orientation. $H^1(S^1 \times S^2)$, $H^2(S^1 \times S^2)$ are oriented via their isomorphisms to $H^1(S^1)$, $H^2(S^2)$ respectively.

The Floer homologies HMT are constructed from the Seiberg-Witten-Floer theory on $(\underline{Y}, \underline{\mathfrak{s}})$, with perturbation of the form

$$\omega = r * d\underline{f} + w, \tag{6.4}$$

where $r \gg 0$ is a constant, and w is a closed 2-form with cohomology class $[w_Y]\#_{S_z}[0]$, where $[w_Y]$ satisfies (1.1). For the rest of this article, a "SWT" Floer theory on $\underline{Y}$, or a "SWT" solution will refer to this particular variant of SWT theory, unless otherwise specified.

6.3 Basic properties of HMT **theory. (1)** The decomposition (6.2) splits

$$H_1(\mathcal{C}) = H^1(\underline{Y}) = \mathbb{Z} \oplus H^1(Y).$$

In terms of this splitting,

$$\mathrm{SF} = c_1(\underline{\mathfrak{s}}) = 2 \oplus c_1(\mathfrak{s}),$$

$$\mathrm{PR} = 2\pi(2\pi c_1(\underline{\mathfrak{s}}) - [w + r * d\underline{f}]) = -C' \oplus 0, \quad \text{for some constant } C' > 0.$$

Notice the complete agreement of these formulae with the formulae for G, SF, and PR in Heegaard Floer theory given in §3.2.

(2) From the form of (SW_3), reducible critical points exist only when

$$c_1(\mathfrak{s}) - [\omega]/(2\pi) = 0.$$

With our choice of ω, the left hand side is $-r/(2\pi)[*d\underline{f}]$, which is never zero as $r > 0$. Thus, all critical points (of the CSD functional) are irreducibles (i.e. smooth points in $\mathcal{C}$).

This again agrees with Heegaard Floer theory formally.

(3) The above formula for PR indicates that it always lies in the positive ray along the negative generator of $H^1(S^1 \times S^2) \subset H^1(\underline{Y})$. Thus, the Morse-Novikov picture explained in §3.1 leads to the expectation that the Floer homology in this theory (HMT^∞) is an invariant, namely, independent of r, further exact perturbation of w, and depends on metric and $\underline{f}$ only through the cohomology classes $[d\underline{f}]$, $[*d\underline{f}]$. The compactness results for Seiberg-Witten moduli spaces proven in [22], [9] confirm this expectation.

6.4 Filtration by holonomy, and the definition of $\mathrm{HMT}^\bullet$**.** Because of the observation §6.3 (2), we may set

$$\mathrm{CMT}^\infty(\underline{Y}, \underline{\mathfrak{s}}) := \hat{\mathrm{CM}}(\underline{Y}, \underline{\mathfrak{s}}; [\omega]) = \check{\mathrm{CM}}(\underline{Y}, \underline{\mathfrak{s}}; [\omega]),$$

with ω given by (6.4). As we saw in §6.3 (1), this Floer complex has exactly the same formal properties as Ozsvath-Szabo's CF^∞. In particular, it is modeled on the Morse complex of the infinite-cyclic covering of $\mathcal{C}$ determined by the homomorphism

$$/[\underline{\gamma}_z] : H_1(\mathcal{C}) = H^1(\underline{Y}) \to \mathbb{Z}.$$

We denote this infinite-cyclic covering by $\hat{\mathcal{C}}_{\underline{\gamma}_z}$.

To complete the analogy, we shall introduce a filtration on CMT^∞ in parallel to the filtration by z in Heegaard Floer theory.

Notice that the holonomy of A^E along $\underline{\gamma}_z$ defines a circle-valued function

$$\mathrm{hol}_{\underline{\gamma}_z} : \mathcal{C} \to \mathbb{R}/2\pi\mathbb{Z}.$$

Let $Z_{\underline{\gamma}_z}$ be the codimension 1 cycle in $\mathcal{C}$ defined by

$$Z_{\underline{\gamma}_z} := \mathrm{hol}^{-1}_{\gamma_z}(c), \quad c \in (\mathbb{R}/2\pi\mathbb{Z})\backslash\{0\}. \tag{6.5}$$

Say, let $c = \pi$. As explained in §4.1, this 1-cocycle defines a local system on $\mathcal{C}$. In fact, the Floer complex with this local coefficient is precisely the Floer complex of the infinite-cyclic covering $\hat{\mathcal{C}}_{\underline{\gamma}_z}$, since by (6.1), $Z_{\underline{\gamma}_z}$ represents the same cohomology class as the above homomorphism $/[\underline{\gamma}_z]$.

Claim (1) $Z_{\underline{\gamma}_z}$ *is semi-positive.*

Thus, the construction in §4.2 may be carried over to this context to define the filtered versions of Floer complexes

$$\mathrm{CMT}^-(\underline{Y}, \underline{\mathfrak{s}}; \underline{\gamma}_z), \quad \mathrm{CMT}^+(\underline{Y}, \underline{\mathfrak{s}}; \underline{\gamma}_z), \quad \widehat{\mathrm{CMT}}(\underline{Y}, \underline{\mathfrak{s}}; \underline{\gamma}_z)$$

corresponding respectively to the Floer complexes of the lower half of $\hat{\mathcal{C}}_{\underline{\gamma}_z}$, of the pair, and of a fundamental domain.

It should now be clear that these filtered complexes carry two natural module structures over the polynomial ring of R: we denote by t the negative generator of deck transformation on $\hat{\mathcal{C}}_{\underline{\gamma}_z}$, and reserve U for the degree -2 chain map from S^1-equivariant theory (defined from higher dimensional moduli spaces of flows). These two module structures will be referred to as the $R[t]$- and $R[U]$ module structures respectively, before they are finally identified in §8.1.

Claim (2) *For $r \gg 1$, the corresponding homologies* $\mathrm{HMT}^\bullet$ *are independent of the choice of level c.*

The verification of these two claims requires a modification of Taubes work in Part I of [25].

Let $(\hat{A}^E, (\hat{\alpha}, \hat{\beta}))$ be a large r solution of the SWT equations on $\mathbb{R} \times \underline{Y}$. Recall that according to Taubes's description of the local behavior of SWT solutions, $\hat{A}^E$ or A^E approximates 0 away from the corresponding Taubes curve or Taubes orbit.

According to (Γ3), $\underline{\gamma}_z$ avoids the Taubes orbits; thus the holonomy map $\mathrm{hol}_{\underline{\gamma}_z}$ always takes values near 0. This justifies our choice of the level c. Furthermore, note that two homologous codimension 1 cycles Z, Z' defines the same filtered Floer complexes if they together bound a region containing no critical points. The above observation on $\mathrm{hol}_{\underline{\gamma}_z}$ then implies that different level sets of $\mathrm{hol}_{\underline{\gamma}_z}$ define the same filtration as long as the level c stays away from 0.

To see Claim (1), we need the following interpretation of the holonomy filtration in terms of intersection numbers.

Let $\tilde{\mathrm{hol}}_{\underline{\gamma}_z} : \tilde{\mathcal{C}} \to \mathbb{R}$ be a lift of $\mathrm{hol}_{\underline{\gamma}_z}$, and $(\hat{A}^E, (\hat{\alpha}, \hat{\beta}))$ be a SWT flow line between the SWT critical points $c_- := (A^E_-, (\alpha_-, \beta_-))$ and $c_+ := (A^E_+, (\alpha_+, \beta_+))$. Then

$$\tilde{\mathrm{hol}}_{\underline{\gamma}_z}(c_+) - \tilde{\mathrm{hol}}_{\underline{\gamma}_z}(c_-) = \int_{\mathbb{R}\times\underline{\gamma}_z} F_{\hat{A}^E} \sim \# \Big(\text{Taubes curve} \cap (\mathbb{R} \times \underline{\gamma}_z)\Big).$$

According to the next Claim and the condition (Γ1), the cylinder $\mathbb{R} \times \underline{\gamma}_z$ intersects the Taubes curve in a region where it is pseudo-holomorphic. Thus, the intersection number is non-negative. This implies the semi-positivity of $Z_{\underline{\gamma}_z}$.

Claim (3) *Let the spin-c structure be given by (6.3). Then there exist κ, δ such that all Taubes orbits in $\underline{Y}$ are contained in $\underline{Y}_{]\kappa-\delta,\kappa+\delta[}$. Moreover, with respect to appropriately chosen metric, all Taubes curves in $\mathbb{R} \times \underline{Y}$ are contained in $\mathbb{R} \times \underline{Y}_{]\kappa-\delta,\kappa+\delta[}$.*

An "appropriately chosen metric" can be one of the following: (i) when there is a holomorphic sphere of class $* \times [S^2]$ in

$$\mathbb{R} \times \underline{Y}_{(\kappa-\delta,\kappa+\delta)} = \mathbb{R} \times (\kappa - \delta, \kappa + \delta) \times S^2$$

with respect to the associated almost complex structure; (ii) when $\underline{Y}_{(\kappa-\delta,\kappa+\delta)}$ is included in a long neck $[-L, L] \times S^2 \subset \underline{Y}$; or, according to the Atiyah-Floer picture explained in §7.2, 9.2 below, (iii) when $\underline{Y}$ includes a long neck along the Heegaard surface, $[-L, L] \times \Sigma \supset \Sigma_H$. Notice that case (i) can always be arranged, since the sphere $\underline{f}^{-1}(\kappa)$ is a symplectic curve, and hence there exists almost complex structures with respect to which it is pseudo-holomorphic.

To see that the Taubes orbits in $\underline{Y}$ avoid $\underline{Y}_{(\kappa-\delta,\kappa+\delta)}$, notice that the choice of $\underline{\mathfrak{s}}$ imposes the homological constraint that the intersection number of the Taubes orbit with any level surface $S_\tau := \underline{f}^{-1}(\tau)$, $\tau \in (\kappa - \delta, \kappa + \delta)$ is 0. Moreover, as the constituent flow lines are oriented by the gradient flow of $\underline{f}$, they all intersect S_τ positively. Thus, the Taubes orbit does not intersect any S_τ.

The assertion on the Taubes curves can be seen as follows: In case (iii), by reducing to the case of Taubes orbits; in case (ii), by reducing to the case of a product complex structure on $\mathbb{R} \times \underline{Y}_{(\kappa-\delta,\kappa+\delta)}$; in case (i), by combining the the homological constraint from the spin-c structure (which says that the intersection number of the Taubes curve with any closed surface in $\mathbb{R} \times \underline{Y}_{(\kappa-\delta,\kappa+\delta)}$ is zero), with the observation that in this case, there is a 2-parameter family of pseudo-holomorphic spheres covering $\mathbb{R} \times \underline{Y}_{(\kappa-\delta,\kappa+\delta)}$.

Remark (a) Notice that the definition of $\underline{\mathfrak{s}}$, the splitting $H^1(\underline{Y}) = H^1(Y) \oplus \mathbb{Z}$, and the filtration all depends on the class $[\underline{\gamma}_z]$. This is similar to the dependence of spin-c structure and filtration on z in Heegaard Floer theory.

(b) It might appear that, by adding a 1-handle, we are forced from the simpler (in the sense of §2.3) $\mathbb{R}$-valued Morse theory back to the more complicated Morse-Novikov situation in §2.2. Claim (3) above shows that, when the spin-c structure is chosen as (6.3), this is not the case, and the simple picture in §2.3 is retained. Notice that these spin-c structures only span the subspace $H^2(Y) \subset H^2(\underline{Y})$ in the space of all spin-c structures. For general spin-c structures on $\underline{Y}$, one would indeed need the more complicated picture in §2.2.

6.5 A fundamental example: $(\underline{Y}, \underline{\mathfrak{s}}) = (S^1 \times S^2, \mathfrak{s}_K)$. The following is the simplest example of HMT$^\bullet$, which also plays an essential role in the proof of Conjecture 6.1 (a).

Let $Y = S^3$, and $\mathfrak{s}$ be the unique spin-c structure on S^3. Then $\underline{Y} = S^1 \times S^2$, and $\underline{\mathfrak{s}} = \mathfrak{s}_K$. Endow $\underline{Y}$ with the product metric, and notice that with this choice of $\underline{\mathfrak{s}}$, the line bundle E is trivial.

There is an obvious S^1-valued harmonic function on $\underline{Y}$, namely

$$\underline{f} = \tau, \quad \text{where } \tau \text{ parameterizes } S^1.$$

The perturbation two form is now $\omega = r * d\tau$, and there is an obvious solution to the 3-dimensional Seiberg-Witten equations with this perturbation:

$$(A^E, (\alpha, \beta)) = (0, (\sqrt{r}, 0)) \quad \text{with respect to a trivialization of } E.$$

It is also not hard to see that this is the unique solution.

Recalling that $H_1(\mathcal{C}) = H^1(\underline{Y}) = \mathbb{Z}$, and SF $= 2$, while PR $= -C' < 0$, this unique solution generates CMT$^\infty$ as a $R[t, t^{-1}]$-module.

Choose $\underline{\gamma}_z = S^1 \times \{z\}$ for a point $z \in S^2$. This is a gradient flow line of $\underline{f}$. The map $\mathrm{hol}_{\underline{f}}$ sends the unique solution above to 0.

Thus, we have

$$\begin{aligned}\mathrm{HMT}^- &= uR[u],\\ \mathrm{HMT}^\infty &= R[u,u^{-1}],\\ \mathrm{HMT}^+ &= R[u,u^{-1}]/(uR[u]),\\ \widehat{\mathrm{HMT}} &= R,\end{aligned}$$

where $\deg u = -2$, and the deck transformation acts by multiplication by u.

Notice that this agrees both with $\mathrm{HF}^\bullet(S^3)$ and $\mathrm{HM}^\bullet(S^3)$, confirming Conjecture 6.1 in this case.

Next, we demonstrate that the U-map from equivariant theory agrees with the deck transformation.

Recall the following geometric interpretation of the U-map (see e.g. [6] Lemma 7.6 for a Yang-Mills analog): Let p, q be two SWT critical points with $\mathrm{Ind}(p) - \mathrm{Ind}(q) = 2$, then

$$\langle q, Up\rangle = \deg \mathrm{hol}_\nu,$$

where hol_ν is the following holonomy map

$$\mathrm{hol}_\nu : \mathcal{M}(p,q) \to S^1,$$

and $\mathcal{M}(p,q)$ is the 1-dimensional (reduced) moduli space of SWT flow lines between p and q, $\mathrm{hol}_\nu(c)$ is the holonomy of $\hat{A}^E$ along the path $\mathbb{R}\times\nu$, with respect to a chosen framing of E over $\nu \in \underline{Y}$, and $c = (\hat{A}^E, (\hat{\alpha}, \hat{\beta}))$.

In our situation, let p be the unique critical point described above, and $q = up$. The unreduced moduli space of flows between p and q is $\mathbb{R}\times S^1$, consisting of pull-backs of vortex solutions of vortex number 1 on $\mathbb{R}\times S^1$ to $\mathbb{R}\times S^1\times S^2$. The degree of hol_ν can be computed from the integral of $F^{\hat{A}^E}$ over the cylinder $\mathbb{R}\times S^1\times \mathrm{pt}$, namely the vortex number, 1.

7 Heuristic explanation of the equivalence

Here is this author's best attempt (for now) at a conceptual explanation of the somewhat mysterious relation (as highlighted in §3.2) between the two theories. The actual proof of equivalence will not adhere to the heuristic picture sketched below, as it is hard to make rigorous. However the picture does provide a useful guideline.

7.1 From HMT **to** HM**.** It has been gauge theorists' dream to understand Floer homologies as the homologies of certain generalized spaces (pro-spectra?) [5]. Just as the Floer homology is modeled on the Morse homology on $\tilde{\mathcal{C}}$, this "space" is modeled on the set of points in $\tilde{\mathcal{C}}$ contained in finite-energy flow lines. This idea is difficult to realize, see however [2, 21] for some recent progress in this direction, in the Seiberg-Witten context.

Let's assume for the moment the existence of such objects: suppose HM and HMT are S^1-equivariant homologies of the "S^1-spaces" S_M, S_{MT} respectively. Due to this author's ignorance of homotopy theory, we shall regard them as ordinary topological spaces. The topological meaning of Conjecture 6.1 (a) then hinges on the special properties of the "space" $S_{MT}(S^1\times S^2, \mathfrak{s}_K)$, and the behavior of these spaces under connected sums of 3-manifolds. In fact, $S_{MT}(S^1\times S^2, \mathfrak{s}_K)$ provides the mechanism that transforms the equivariant Seiberg-Witten theory into the non-equivariant Heegaard Floer theory of filtered $\mathbb{Z}$-covers.

From the computation in §6.5, one expects $S_{MT}(S^1 \times S^2, \mathfrak{s}_K)$ to be, very roughly, of the (S^1-equivariant) homotopy type of the infinite dimensional Hopf sphere, "$S^{\infty}_{-\infty}$", i.e. S^1-equivariant version of the pro-spectrum $\mathbb{C}P^{\infty}_{-\infty}$ (cf. e.g. Example 6.2 in [5]). Moreover, the filtration by deck transformation on $\tilde{\mathcal{C}}$ induces a filtration on S_{MT}: letting $t^n\tilde{\mathcal{C}}^-$ denote the half of $\tilde{\mathcal{C}}$ below the hypersurface $t^n\tilde{Z}$,

$$S_{MT} = \lim_{n\to-\infty} t^n S^-_{MT},$$

where $t^n S^-_{MT}$ is defined from the flows in $t^n\tilde{\mathcal{C}}^-$. On the other hand, let $S^+_{MT} = S_{MT}/S^-_{MT}$.

Both $S^{\pm}_{MT}$ are homotopic to $S^\infty = ES^1$, and $t^n S^-_{MT}$ is homotopic to $t^{n+1}S^-_{MT}$ with a free S^1-cell e_S attached,

$$e_S := S^{2*+1}\backslash S^{2*-1} = \left\{ v \mid v \in \mathbb{C}^{*+1}\backslash\mathbb{C}^*, |v| = 1 \right\},$$

where $*$ is taken to the infinity in the limit. In addition,

- Due to spectral flow, $t^{n-1}S^-_{MT} \simeq \Sigma^{\mathbb{C}} t^n S^-_{MT}$, and $S_{MT} = \Sigma^{\mathbb{C}} S_{MT}$.
- The Euler class of the S^1 action is represented by a codimension 2 cycle $\mathcal{E}$, such that $\mathcal{E} \cap t^n S^-_{MT} = t^{n+1}S^-_{MT}$.

On the other hand, one expects the "S^1-spaces" of a connected sum of 3-manifolds to be a product

$$S_M(Y_1 \# Y_2) \simeq S_M(Y_1) \times S_M(Y_2); \quad \text{in particular,}$$
$$S_{MT}(\underline{Y}, \underline{\mathfrak{s}}) \simeq S_M(Y, \mathfrak{s}; [w]) \times S_{MT}(S^1 \times S^2, \mathfrak{s}_K).$$

Notice that the S^1 action on $S_{MT}(S^1 \times S^2, \mathfrak{s}_K)$ is free, implying that the S^1 action on the product $S_{MT}(\underline{Y}, \underline{\mathfrak{s}})$ is always free, even though the S^1 action on $S_M(Y, \mathfrak{s}; [w])$ has fixed points. Thus, the quotient is a fiber product

$$S_{MT}(\underline{Y}, \underline{\mathfrak{s}})/S^1 \simeq S_M(Y, \mathfrak{s}; [w]) \times_{S^1} S_{MT}(S^1 \times S^2, \mathfrak{s}_K),$$

which is smooth.

This explains why, in Seiberg-Witten-Taubes theory, all critical points are irreducibles, and via Conjecture 6.1 (b), why the Heegaard Floer theory is a *non-equivariant* theory of a *smooth* $\mathcal{C}$.

Furthermore, the filtration on $S_{MT}(S^1 \times S^2, \mathfrak{s}_K)$ transfers to a filtration on $S_{MT}(\underline{Y}, \underline{\mathfrak{s}})$, and the diagonal S^1 action on the product is determined by the free S^1 action on $S_{MT}(S^1 \times S^2, \mathfrak{s}_K)$. Thus, the agreement of deck transformation and U action in HMT theory merely reflects the relation between Euler class and the filtration via deck transformation on $S_{MT}(S^1 \times S^2, \mathfrak{s}_K)$, as described above. This in terms translates the equivariant aspects of Seiberg-Witten theory into the deck transformation in Heegaard Floer theory.

The hat versions of HF and HMT are both equivariant homologies of flows in a fundamental domain. According to the above picture, They would compute the homology of

$$S_M(Y, \mathfrak{s}; [w]) \times_{S^1} e_S \simeq S_M(Y, \mathfrak{s}; [w]),$$

namely the (non-equivariant) homology of the total S^1-space.

The + versions of HF and HMT are both equivariant homologies of flows in the upper half of a infinite cyclic cover. According to the above picture, they would compute the homology of

$$S_M(Y, \mathfrak{s}; [w]) \times_{S^1} ES^1,$$

namely the equivariant homology of $S_M(Y, \mathfrak{s}; [w])$.

7.2 From HMT **to** HF**.** Because of Claim 6.4 (3), this is predicted by Taubes's picture and a Seiberg-Witten analog of the Atiyah-Floer conjecture, similar to the 3-dimensional cylindrical SWT theory sketched in §2.3.

Let

$$\underline{Y}(L) = \underline{Y} \backslash \Sigma_H \cup [-L/2, L/2] \times \Sigma$$

denote $\underline{Y}$ endowed with a metric such that it contains a cylinder of length L about the Heegaard surface Σ_H, and let

$$Y(L)^\circ := \underline{Y}(L) \Big\backslash \left(-\frac{L}{2+\epsilon}, \frac{L}{2+\epsilon}\right) \times \Sigma,$$

namely, a connected sum of the handlebodies $H_\pm$ along $p_\pm$. Partition $\mathbb{R} \times \underline{Y}(L)$ into the union of $\mathbb{R} \times [-L/2, L/2] \times \Sigma$ and $\mathbb{R} \times Y(L)^\circ$. We call the former the *"inside piece"*, the latter the *"outside piece"*.

The idea is that as $L \to \infty$, most of the "energy" is expected to reside on the inside piece. Thus, the Taubes curve in the outside piece would then approach a path of Taubes orbits

$$\bigcup_{s \in \mathbb{R}} \{s\} \times \mathcal{O}_s, \quad \text{where } \mathcal{O}_s \text{ is a Taubes orbit on } Y^\circ, \text{ and}$$

$$Y^\circ = Y(L)^\circ \cup \mathbb{R}_+ \times \Sigma \cup \mathbb{R}_- \times \Sigma$$

completes $Y(L)$ into a 3-manifold with two cylindrical ends.

Let $\mathfrak{s}^\circ$ denote the spin-c structure on Y° induced by $\underline{\mathfrak{s}}$.

A constituent flow line of a Taubes orbit in Y° is specified by its asymptotics at $\tau \to \infty$ or $-\infty$; in other words, a point on the corresponding descending cycle on the limiting surface Σ_∞ at infinity, or ascending cycle on the limiting surface $\Sigma_{-\infty}$. With slight abuse of notation, let $\mathbb{T}_\alpha$ be the product of descending cycles in $\mathrm{Sym}^g \Sigma_+$, and $\mathbb{T}_\beta$ be the product of descending cycles in $\mathrm{Sym}^g \Sigma_-$. Then a Taubes orbit in Y° is specified by a point in $\mathbb{T}_\alpha \times \mathbb{T}_\beta$. Namely, there is a diffeomorphism

$$\mathcal{O}_{Y^\circ} : \mathbb{T}_\alpha \times \mathbb{T}_\beta \to \text{the space of Taubes orbits of spin-c structure } \mathfrak{s}^\circ \text{ in } Y^\circ.$$

Next, we describe the more interesting inside piece.

Note that the metric on $[-L/2, L/2] \times \Sigma \subset \underline{Y}(L)$ is a product metric, and $\underline{f}$ approximates a linear function $\tau + C_L$ as $L \to \infty$. Thus, in the limit, the projection

$$\text{inside piece} \to \mathbb{R} \times [-L/2, L/2]$$

is complex-linear. This implies that the Taubes curves in the limit is a union of a multi-section of the Σ-bundle $\mathbb{R} \times [-L/2, L/2] \times \Sigma$, and a number of fibers. An index computation shows that fibers do not appear as irreducible components of a Taubes curve in a reduced moduli space of dimension ≤ 0. On the other hand, we know that the multi-section is a g-branched cover of $\mathbb{R} \times [-L/2, L/2]$, as in the $s \to \pm\infty$ limit it is asymptotic to a

$$\text{Taubes orbit in } \underline{Y}(L) \cap [-L/2, L/2] \times \Sigma,$$

which consists of g constituent flow lines. Furthermore, since this needs to match with the Taubes curve in the outside piece, the multi-section restricts at the two boundary components, $\mathbb{R} \times \{-L/2, L/2\} \times \Sigma$ of the inside piece, to the union of g paths on the g descending/ascending cycles.

Thus, the Taubes curve of a large r SWT solution used for the definition of the boundary map in the HMT Floer theory defines a holomorphic map:

$$\mu_L : \mathbb{R} \times [-L/2, L/2] \to \mathrm{Sym}^g \Sigma \quad \text{with } \mu_L(\cdot, -L/2) \in \mathbb{T}_\alpha,\ \mu_L(\cdot, L/2) \in \mathbb{T}_\beta.$$

Composing with the conformal map resc_L^{-1}, where

$$\mathrm{resc}_L : \mathbb{R} \times [-L/2, L/2] \to \mathbb{R} \times [0,1] : (s, \tau) \mapsto (s/L, \tau/L + 1/2),$$

we obtain a holomorphic disk

$$\mu : \mathbb{R} \times [0,1] \to \mathrm{Sym}^g \Sigma, \quad \text{with } \mu(\cdot, 0) \in \mathbb{T}_\alpha,\ \mu(\cdot, 1) \in \mathbb{T}_\beta,$$

namely, a Heegaard flow line.

Conversely, a Heegaard flow line uniquely determines a limiting Taubes curve. Thus, *if* an analog of Taubes's theorem (Theorem 2.1) holds in this context, this means $\mathrm{CMT}^\infty = \mathrm{CF}^\infty$.

The matching of filtration between the two Floer theories will not require the full strength of Taubes's theorem, but only Part I of his proof. We shall postpone its discussion to §9.2.

Remark It follows that all irreducible components of a Taubes curve in this context have multiplicity 1: All irreducible components with boundary has multiplicity one, since we saw that the multiplicities at the boundary are all 1. A closed irreducible component can only be a copy of Σ, which is of genus $g > 1$ according to our assumption. However, according to Taubes, the multiplicity can be larger than 1 only if $g \leq 1$.

8 Towards a real proof, part (a)

This section and the next outline our plan for proving the two halves of the conjecture, Conjecture 6.1 (a) and Conjecture 6.1 (b) respectively.

Conjecture 6.1 (a) is now "almost a theorem", in the sense that details are still being written down [17], but there shouldn't be additional difficulties.

Though the picture in §7.1 is far from rigorous, it is nevertheless possible to prove a connected sum formula for Floer *homologies* consistent with the fiber product picture, via a cobordism proof. In the context of instanton Floer homologies, this is proven by Fukaya [10], see also the exposition in [6].

Because the Floer complexes $\mathrm{CM}^\bullet$ of Kronheimer-Mrowka are built from real blown-ups of S^1-spaces along fixed-point-sets, on which S^1 acts *freely*, the connected sum formula in the context of Kronheimer-Mrowka theory takes a cleaner form than the pre- blown-up version (as in [10, 6]). The author learned of the following formulation from Mrowka and Ozsvath. We only state the hat version, since it alone suffices for our purpose.

Theorem 8.1 (Connected sum formula) *Let Y_1, Y_2 be closed, oriented 3-manifolds and $\mathfrak{s}_1$, $\mathfrak{s}_2$ be spin-c structures on Y_1, Y_2 respectively. For $i = 1, 2$, let $[w_i] \in H^2(Y_i; \mathbb{R})$, and $U_i : \hat{\mathrm{CM}}_*(Y_i, \mathfrak{s}_i; [w_i]) \to \hat{\mathrm{CM}}_{*-2}(Y_i, \mathfrak{s}_i; [w_i])$ be the U-map defined in* [22]*. Then there is an isomorphism of $R[U]$-modules:*

$$\hat{\mathrm{HM}}_*(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2; [w_1] \# [w_2]) = H_* \Big(S_{U_1+U_2}(\hat{\mathrm{CM}}(Y_1, \mathfrak{s}_1; [w_1]) \otimes_R \hat{\mathrm{CM}}(Y_2, \mathfrak{s}_2; [w_2])) \Big).$$

The $R[U]$-module structure of the right hand side of the above isomorphism is given by the interpretation of the complex

$$S_{U_1+U_2}(\hat{\mathrm{CM}}(Y_1, \mathfrak{s}_1; [w_1]) \otimes_R \hat{\mathrm{CM}}(Y_2, \mathfrak{s}_2; [w_2]))$$

as an "algebraic fiber product".

Indeed, suppose $C(B_1)$, $C(B_2)$ are chain complexes of the base manifolds of two S^1-bundles $E_i \to B_i$ for $i = 1, 2$, and $U_i : C_*(B_i) \to C_{*-2}(B_i)$ are chain maps such that they induces maps on homologies that agree with cap products with the respective Euler class. Then $C(B_1) \otimes C(B_2)$ is the chain complex of the base manifold of $E_1 \times_{S^1} E_2$, and according to Lemma 5.1, $H_*(S_{U_1+U_2}(C(B_1) \otimes C(B_2)))$ computes the homology of the fiber product $E_1 \times_{S^1} E_2$.

The chain maps

$$U_1, -U_2 \, : S_{U_1+U_2}(C(B_1) \otimes C(B_2)) \to S_{U_1+U_2}(C(B_1) \otimes C(B_2))$$

are chain homotopic, and induce a degree -2 map on homologies agreeing with cap product with the Euler class of the S^1-bundle

$$E_1 \times E_2 \to E_1 \times_{S^1} E_2.$$

The proof of Conjecture 6.1 (a) requires a filtered version of the above connected sum theorem. Let

$$U_Y : \hat{\mathrm{CM}}_*(Y, \mathfrak{s}; [w]) \to \hat{\mathrm{CM}}_{*-2}(Y, \mathfrak{s}; [w]),$$
$$U_{S^1\times S^2} : \mathrm{CMT}^{\bullet}_*(S^1 \times S^2, \mathfrak{s}_K; S^1 \times \mathrm{pt}) \to \mathrm{CMT}^{\bullet}_{*-2}(S^1 \times S^2, \mathfrak{s}_K; S^1 \times \mathrm{pt})$$

be respective the U-maps for Y and $S^1 \times S^2$. For $\bullet = -, \infty, +, \wedge$, denote by

$$S^{\bullet}_{\otimes}(Y, \mathfrak{s}) := S_{U_Y + U_{S^1\times S^2}} \left(\hat{\mathrm{CM}}(Y, \mathfrak{s}; [w]) \otimes_R \mathrm{CMT}^{\bullet}(S^1 \times S^2, \mathfrak{s}_K; S^1 \times \mathrm{pt}) \right).$$

The first three of these complexes are endowed with both $R[U]$- and $R[t]$- module structures, the former via its definition as an algebraic fiber product; the latter via the $R[t]$-module structure of $\mathrm{CMT}^{\bullet}(S^1 \times S^2, \mathfrak{s}_K; S^1 \times \mathrm{pt})$. Furthermore, the short exact sequence

$$0 \to \mathrm{CMT}^-(S^1 \times S^2) \to \mathrm{CMT}^{\infty}(S^1 \times S^2) \to \mathrm{CMT}^+(S^1 \times S^2) \to 0$$

induces the short exact sequence

$$0 \to S^-_{\otimes}(Y, \mathfrak{s}) \to S^{\infty}_{\otimes}(Y, \mathfrak{s}) \to S^+_{\otimes}(Y, \mathfrak{s}) \to 0,$$

and hence a long exact sequence relating the homologies of $S^{\bullet}_{\otimes}(Y, \mathfrak{s})$.

Theorem 8.2 (Filtered connected sum formula) *Let $\bullet$ be any of the four: $-, \infty, +$ or $\wedge$, and $r \gg 1$. Then there is an isomorphism*

$$\mathrm{HMT}^{\bullet}_*(\underline{Y}, \underline{\mathfrak{s}}; \underline{\gamma}_z) = H_* \left(S^{\bullet}_{\otimes}(Y, \mathfrak{s}) \right).$$

both as $R[U]$- and as $R[t]$-modules.

Furthermore, these isomorphisms are natural in the sense that the following diagram is commutative:

$$\begin{array}{ccccccc}
\cdots \mathrm{HMT}^-_*(\underline{Y}, \underline{\mathfrak{s}}; \underline{\gamma}_z) & \longrightarrow & \mathrm{HMT}^{\infty}_*(\underline{Y}, \underline{\mathfrak{s}}) & \longrightarrow & \mathrm{HMT}^+_*(\underline{Y}, \underline{\mathfrak{s}}; \underline{\gamma}_z) & \longrightarrow & \mathrm{HMT}^-_{*-1}(\underline{Y}, \underline{\mathfrak{s}}; \underline{\gamma}_z) \cdots \\
\| & & \| & & \| & & \| \\
\cdots H_*(S^-_{\otimes}(Y, \mathfrak{s})) & \longrightarrow & H_*(S^{\infty}_{\otimes}(Y, \mathfrak{s})) & \longrightarrow & H_*(S^+_{\otimes}(Y, \mathfrak{s})) & \longrightarrow & H_{*-1}(S^-_{\otimes}(Y, \mathfrak{s})) \cdots
\end{array}$$

Sketch of proof. To establish the isomorphism between the unfiltered versions, i.e. $\bullet = \infty$, let $(Y_1, \mathfrak{s}_1) = (Y, \mathfrak{s})$ and $Y_2 = (S^1 \times S^2, \mathfrak{s}_K)$, and $[w_1] = [w]$, $[w_2] = [r *_3 d\tau]$ in Theorem 8.1. The idea of the typical proof of connected sum formulae is to consider the pairs (V, γ_V), (W, γ_W), where V, W are respectively the cobordisms giving the connected sum and its reverse operation. $\gamma_V \subset V$ and $\gamma_W \subset W$ are paths between the Y_1-end and the Y_2-end. In Morse-theoretic picture, there is a monotonically decreasing Morse function on V with exactly one critical point, of index 1, and γ_V is the descending manifold from this critical point. (W, γ_W) is obtained by reversing this Morse function.

The pair (V, γ_V) defines a chain map V_* from the Floer complex of the connected sum to the algebraic fiber product. Conversely, (W, γ_W) defines a chain map W_* from the latter to the former. Next, observe that the compositions of cobordisms $W \cup_{Y_1 \coprod Y_2} V$ and $V \cup_{Y_1 \# Y_2} W$ are related to the product cobordisms, respectively by a 1-surgery along the 1-cycle $\gamma_V \cup \gamma_W$, and collapsing the 2-sphere formed by the 2-dimensional descending and ascending manifolds from the two critical points in $V \cup_{Y_1 \# Y_2} W$. One may then show that the compositions $W_* \circ V_*$ and $V_* \circ W_*$ are chain homotopic to the identity, by proving some surgery formulae.

To prove the filtered versions, one needs in addition to show that V_*, W_* preserve the filtration. The proof is a variant of the proof of the semi-positivity of $Z_{\underline{\gamma}_z}$.

Remark The cobordism proof sketched above works when one of $\mathrm{CM}(Y_1)$, $\mathrm{CM}(Y_2)$ involves only irreducibles. In general, W_* is not well-defined, due to compactness problems. Instead, the plan of Mrowka et al. is to use naturality of V_* with respect to the surgery exact sequences.

As we showed in §6.5 that the $R[t]$- and $R[U]$-module structures of $\mathrm{CMT}^\bullet(S^1 \times S^2, \mathfrak{s}_K; S^1 \times \mathrm{pt})$ agree, this implies that the $R[t]$- and $R[U]$-module structures of $S^\bullet_\otimes(Y, \mathfrak{s})$ also agree, and hence, as an immediate corollary:

Corollary 8.3 *The U-action on $\mathrm{HMT}^\bullet_*(\underline{Y}, \underline{\mathfrak{s}}; \underline{\gamma}_z)$ agrees with deck transformation.*

This corollary may also be obtained via a geometric argument generalizing the computation for the $S^1 \times S^2$ case in §6.5.

As another immediate corollary, we have

Corollary 8.4 *Conjecture 6.1 (a) holds.*

To see this, recall that with the standard choice of $\underline{f}$ and $\underline{\gamma}_z$ for $S^1 \times S^2$ described in §6.5,

$$\mathrm{CMT}^\bullet(S^1 \times S^2, \mathfrak{s}_K; S^1 \times \mathrm{pt}) = V^\bullet,$$

where $V^\bullet$ is as in (5.3), and $U_{S^1 \times S^2}$ is simply multiplication by u.

Combining this with the filtered connected sum formulae, the simple fact that

$$E^\bullet S_U(C) = S_{U+u}(C \otimes_R V^\bullet) \quad \text{for a complex of } R[U]\text{-modules } C,$$

and Corollary 5.3, Conjecture 6.1 (a) follows immediately.

9 Towards a real proof, part (b)

Progress towards the harder half of the conjecture, Conjecture 6.1 (b), is still in a very preliminary stage. We shall however present some partial results and ideas in this direction.

9.1 Structure of moduli spaces in 3-dimensional Seiberg-Witten-Taubes theory. Let Y°, $\mathfrak{s}^\circ$ be as in §7.2. A metric on Y° is called *cylindrical* if it restricts to a fixed product metric on the attached cylinders. Corresponding to each cylindrical metric, there is a harmonic function

$$f^\circ : Y^\circ \to \mathbb{R}, \quad \text{with the asymptotic condition (2.2) as } \tau \to \pm\infty,$$

Furthermore, when L is large, the restrictions $f^\circ|_{Y(L)^\circ}$, $\underline{f}|_{Y(L)^\circ}$ approximate each other.

Let $\mathcal{M}_{Swt_3}(Y^\circ, \mathfrak{s}^\circ; r, \eta)$ denote the moduli spaces of the 3 dimensional Seiberg-Witten equations on the cylindrical manifold $(Y^\circ, \mathfrak{s}^\circ)$, with perturbation

$$\omega = r * df^\circ + \eta.$$

This is a special case of the 3-d cylindrical SWT theory sketched in §2.3. Solutions to this perturbed Seiberg-Witten equations approach "pull-backs" of vortex solutions on $E \to \Sigma$ exponentially as $\tau \to \pm\infty$ (see [18, 16] for a definition). Thus, there are well-defined *end-point maps* from this moduli space to the moduli space of vortices on $E \to \Sigma$:

$$\partial_\pm : \mathcal{M}_{Swt_3}(Y^\circ, \mathfrak{s}^\circ; r, \eta) \to \operatorname{Sym}^g \Sigma.$$

(Recall that $c_1(\mathfrak{s}^\circ)[\Sigma] = 2$. This implies, via (2.1), that $\deg E = g$).

Proposition *For generic pair of cylindrical metric and exact 2-form η of $O(1)$, the moduli space $\mathcal{M}_{Swt_3}(Y^\circ, \mathfrak{s}^\circ; r, \eta)$ is a compact, orientable manifold of dimension $2g$. Furthermore, in the limit $r \to \infty$,*

$$\partial_- \times \partial_+ : \mathcal{M}_{Swt_3}(Y^\circ, \mathfrak{s}^\circ; r, \eta) \to \operatorname{Sym}^g \Sigma \times \operatorname{Sym}^g \Sigma$$

is a smooth, proper submersion of degree 1 to $\mathbb{T}_\alpha \times \mathbb{T}_\beta \subset \operatorname{Sym}^g \Sigma \times \operatorname{Sym}^g \Sigma$.

See [18] for a more precise statement. An analogous theorem on the structures of moduli spaces for manifolds with Euclidean ends may be found in [16].

Sketch of proof. Consider any general 3-manifolds with cylindrical ends H, and let $\mathcal{M}_{Swt_3}(H)$ be the moduli space of 3-d SWT solutions. A simplified version of the arguments in [16] establishes the smoothness, compactness, and orientability properties of these moduli spaces. An adaptation of Taubes's convergence theorem in part I of [25] describes the image of the limiting end-point maps in terms of products of descending/ascending cycles.

The last statement of the Proposition on the degree of the limiting $\partial_- \times \partial_+$ follows from gluing theorems for SWT moduli spaces on cylindrical manifolds, and known computation of the Seiberg-Witten invariants of closed 3-manifolds.

Details will appear in [18].

Remark Taubes's picture (cf. §7.2) leads one to expect the map $\partial_- \times \partial_+$ to be a diffeomorphism, but the above weaker statement will hopefully suffice for our purpose. To prove the the stronger statement predicted by Taubes's picture, one would need a counterpart of part III of [25] (Gr $\to$ Sw) in this situation. This is not an easy task; see however [16] for some progress.

9.2 Comparing CMT **and** CF**.** This subsection will be of even sketchier character, since details for this part are not existent at this moment.

Conjecture *For all sufficiently large L, there is an isomorphism of $R[U]$ modules*

$$\mathrm{CMT}^\bullet(\underline{Y}(L), \underline{\mathfrak{s}}; \underline{\gamma}_z) = \mathrm{CF}^\bullet(Y, \mathfrak{s}).$$

To prove this conjecture, notice first that the identification of the chain groups follows readily from the description of $\mathcal{M}_{Swt_3}(Y^\circ)$ in the previous Proposition, a simple gluing theorem for moduli of 3-dimensional SWT solutions mentioned already in §9.1 above, and the computation of SF and PR of both sides explained before. Thus, it remains to identify the spaces of flow lines on both sides.

(i) From Heegaard flow lines to SWT flow lines: What this part requires is a gluing theorem of the following form. According to [19], the Heegaard Floer homologies are invariant under isotopies of the α- and β-cycles. We may thus choose $\mathbb{T}_\alpha$, $\mathbb{T}_\beta$ to be as in Proposition 9.1, namely, as products of descending/ascending cycles of the two surfaces at infinity of Y°.

Given a Heegaard flow line

$$\mu : \mathbb{R} \times [0,1] \to \mathrm{Sym}^g \Sigma, \quad \text{with } \mu(\cdot, 0) \in \mathbb{T}_\alpha,\ \mu(\cdot, 1) \in \mathbb{T}_\beta,$$

the goal of the gluing theorem is to give a gauge-equivalence class of SWT solution on $\mathbb{R} \times \underline{Y}(L)$, for large L.

Composing μ with the rescaling map resc_L introduced in §7.2, we obtain a family of vortex solutions on Σ parameterized by (s, τ):

$$\left\{(a_{(s,\tau)}, \nu_{(s,\tau)}) \,|\, (s,\tau) \in \mathbb{R} \times [-L/2, L/2]\right\}.$$

This defines a Seiberg-Witten configuration on the inside piece, $\mathbb{R} \times [-L/2, L/2] \times \Sigma$, such that

$$(\hat{A}^E, (\hat{\alpha}, \hat{\beta}))|_{\{(\tau,s)\}\times\Sigma} = (a_{(\tau,s)}, (\nu_{(\tau,s)}, 0)).$$

When L is large, $\partial_s(\hat{A}^E, (\hat{\alpha}, \hat{\beta}))$ and $\partial_\tau(\hat{A}^E, (\hat{\alpha}, \hat{\beta}))$ for such configurations are small, and thus this gives an approximate SWT solution on the inside piece.

On the other hand, the restriction of μ to the boundary $\{0, 1\} \times \mathbb{R}$ defines a map $\partial\mu : \mathbb{R} \to \mathbb{T}_\alpha \times \mathbb{T}_\beta$, which lifts, under $\partial_- \times \partial_+$, to paths in $\mathcal{M}_{Swt_3}(Y^\circ, \mathfrak{s}^\circ)$:

$$c_s^j \in \mathcal{M}_{Swt_3}(Y^\circ, \mathfrak{s}^\circ), \quad s \in \mathbb{R},\ j = 1, \ldots, k.$$

This in turn defines k Seiberg-Witten configurations on $\mathbb{R} \times Y^\circ$, so that

$$(\hat{A}^E, (\hat{\alpha}, \hat{\beta}))|_{\{s\}\times Y^\circ} = c^j_{s/L} \quad \text{for some } j.$$

These are again approximate SWT solutions, because $\partial_s(\hat{A}^E, (\hat{\alpha}, \hat{\beta}))$ is small due to the rescaling by L.

Noting that the above approximate solutions over the inside piece and the outside piece are close to each other on the overlap, a splicing construction then yields k approximate SWT solutions over $\mathbb{R} \times \underline{Y}(L)$. A first task is then to perform an error estimate on these approximate solutions showing:

$$L \cdot \text{error} \to 0 \quad \text{as } L \to \infty.$$

To perturb the approximate solutions to true solutions, one needs the deformation operator at these approximate solutions, D_j, to be invertible, and that

$$L^{-1} \cdot D_j^{-1} \quad \text{is uniformly bounded.}$$

This should follow from the fact that similar statements hold both for the outside piece and the inside piece, as long as μ is nondegenerate: both the outside piece and the inside piece have a good Fredholm theory via a separation of variables argument (see [25] part III and [16] for some example of this argument in the Seiberg-Witten context). For the inside piece, this argument reduces the Fredholm theory to the Fredholm theory of a Cauchy-Riemann operator on $\mathbb{R} \times [-L/2, L/2]$ with (finite dimensional) totally real boundary conditions. For the outside piece, the uniform boundedness follows from an eigenvalue estimate on the compact manifold $Y(L)^\circ$ with APS boundary conditions. (See e.g. [4]).

If the orientation works out, the last statement of Proposition 9.1 on the degree of the end-point map would imply that the signed count of the k SWT solutions corresponding to μ equals 1.

(ii) From SWT flow lines to Heegaard flow lines: This part requires a convergence theorem. Very roughly, consider a sequence of SWT solutions c_i on $\mathbb{R} \times \underline{Y}(L_i)$ with $L_i \to \infty$.

Rescale the restrictions to the inside piece by L_i^{-1} in the τ and s directions to get a sequence of configurations on $\mathbb{R} \times [0,1] \times \Sigma$. These configurations satisfy equations that break up into two parts: the first being L_i^{-1} times the Cauchy-Riemann equation, and the essential part of the second being the vortex equation on $E \to \Sigma$. One thus expects that some adiabatic analysis akin to that in [7] to show that this gives, in the $L \to \infty$ limit, a pseudo-holomorphic disk in the moduli space of vortices, namely $\mathrm{Sym}^g \Sigma$.

On the other hand, rescale the restrictions to the outside piece $\mathbb{R} \times Y(L)^\circ$ in the s direction by L_i^{-1}. The ∂_s term in the SWT equations is replaced by $L_i^{-1}\partial_s$ in the rescaled equations. Thus, one expects the $L_i \to \infty$ limit to yield a path of 3-dimensional SWT solutions, namely, a path in $\mathcal{M}_{Swt_3}(Y^\circ, \mathfrak{s}^\circ)$, and hence a path in $\mathbb{T}_\alpha \times \mathbb{T}_\beta$ via the end-point map $\partial_- \times \partial_+$ described in Proposition 9.1.

The above $L \to \infty$ limits for the outside and the inside piece have to match, as they both come from SWT solutions that agree over the overlaps $\mathbb{R} \times (-L/2, -L/(2+\epsilon)) \times \Sigma$ and $\mathbb{R} \times (L/(2+\epsilon), L/2) \times \Sigma$. Thus, the holomorphic map

$$\mu : \mathbb{R} \times [0,1] \to \mathrm{Sym}^g \Sigma$$

arising from limits in the inside region must satisfy the boundary condition

$$\mu(\cdot, 0) \in \mathbb{T}_\alpha, \quad \mu(\cdot, 1) \in \mathbb{T}_\beta.$$

This is precisely a Heegaard flow line.

Remark Notice that rescaling is also crucial for the desired compactness results to be possible. While a variant of the usual compactness results for Seiberg-Witten moduli spaces may very likely yield a "local compactness theorem" for the SWT solutions in our context, "global compactness" for gauge-theoretic moduli spaces over infinite cylinders $\mathbb{R} \times M$ requires additional decay estimates for the solutions in the ends $s \to \pm\infty$. This is typically achieved by an eigenvalue estimate for the relevant deformation operator over M: more precisely, the minimum of the absolute values of the eigenvalues should be bounded away from zero. This will not hold when M itself contains an infinite cylinder. Instead, our approach involves M with a long, but finite cylinder of length L. As $L \to \infty$, the minimal absolute value of the eigenvalues goes to 0. The effect of rescaling is that, instead of the minimal absolute value, only a uniform lower bound on

$$L \cdot (\text{ minimal absolute value of the eigenvalues})$$

is required for compactness.

(iii) Matching the filtrations: We explained in §6.4 that the holonomy filtration in HMT theory is given by the intersection number of the Taubes curve with $\mathbb{R}\times\underline{\gamma}_z$. Our task now is to compare this intersection number with the intersection number giving the filtration in Heegaard Floer theory.

For this purpose, we construct curves that approximate the large L Taubes curve in the inside and outside pieces respectively.

Regarding a Heegaard flow line,

$$\mu : \mathbb{R} \times [0,1] \to \mathrm{Sym}^g \Sigma, \quad \text{with } \mu(\cdot,0) \in \mathbb{T}_\alpha,\ \mu(\cdot,1) \in \mathbb{T}_\beta,$$

as a multi-section of the Σ-bundle $\mathbb{R} \times [0,1] \times \Sigma$, its graph defines, via composing with the holomorphic map

$$\mathbb{R} \times [-L/2, L/2] \times \Sigma \to \mathbb{R} \times [0,1] \times \Sigma : \ (s,\tau,w) \mapsto (s/L, \tau/L + 1/2, w),$$

a curve in the inside piece $C_\mu \subset \mathbb{R} \times [-L/2, L/2] \times \Sigma$.

The restriction $\partial\mu$, on the other hand, defines a surface $B_{\partial\mu}$ with boundary at the zero locus of $\mathrm{SD}(*_3 d\underline{f})$ in the outside piece $\mathbb{R} \times Y(L)^\circ$:

$$B_{\partial\mu} \cap (\{s\} \times Y(L)^\circ) = (\mathcal{O}_{Y^\circ}(\partial\mu(s/L))) \cap Y(L)^\circ,$$

where $\mathcal{O}_{Y^\circ}$ is the diffeomorphism introduced in §7.2.

If the gluing construction in (i) works, C_μ and $B_{\partial\mu}$ approximates the Taubes curve on the inside and outside piece respectively. However, via (Γ3), the cylinder $\mathbb{R} \times \underline{\gamma}_z$ does not intersect $B_{\partial\mu}$ in the outside piece. Thus, the intersection number of the Taubes curve with $\mathbb{R} \times \underline{\gamma}_z$ is given by

$$\#\left((\mathbb{R} \times \gamma_z) \cap C_\mu\right) = \#\left(\{z\} \times \mathrm{Sym}^{g-1}\Sigma \cap \mathrm{image}(\mu)\right),$$

which also gives the filtration in Heegaard Floer theory.

9.3 Final remarks. Two main technical components of this program are respectively analogs of the Atiyah-Floer conjecture, and Taubes's work. The following comments compare the variants needed in our context and the "original" versions.

(1) In the Seiberg-Witten context, the Atiyah-Floer conjecture should relate the Seiberg-Witten-Floer homology with the Floer homology of Lagrangian intersections for $(P; L_1, L_2)$, where P is the moduli space of the dimension-reduction of the Seiberg-Witten equations over the Heegaard surface, and L_1, L_2 are respectively images of the end-point maps from the moduli spaces of 3-dimensional Seiberg-Witten solutions over the two handlebodies H_+, H_-.

An important difference, though, is that we work with a different spin-c structure: the restriction of $\underline{\mathfrak{s}}$ on $\underline{Y}$ minus the 1-handle does not extend across the two 3-balls to give a spin-c structure on Y. In particular, in our program, $c_1(\underline{\mathfrak{s}})[\Sigma_H] = 2$, while $c_1(\mathfrak{s})[\Sigma_H] = 0$ in the Atiyah-Floer conjecture. We also work with a Taubes type perturbation which does not extend across the two balls.

These differences make our program substantially easier than the Atiyah-Floer conjecture. As we saw, there are no irreducibles in our program, and the "Lagrangians" and other moduli spaces in our story are all smooth.

In an earlier work [20], Ozsvath-Szabo defined a "theta-invariant" corresponding to the Euler characteristic of Seiberg-Witten-Floer homology, modeling more directly with the Atiyah-Floer conjecture. It remains interesting to understand the corresponding Floer homology, which is yet to be defined.

(2) Though Taubes's picture has been the guiding principle for this work, our current plan does not involve running the whole gamut of Taubes's fundamental work. As outline above, we shall only need variants of Part I of [25] (Sw $\to$ Gr), which is the more accessible part when $\varpi^{-1}(0) \neq \emptyset$, as pointed out in the beginning of §2.2. Part II of [25] defines Gr, which in our context is replaced by the work of Ozsvath-Szabo. Part III of [25], Gr $\to$ Sw, is in our plan replaced by the weaker result Proposition 9.1 above. Part IV of [25] compares the Kuranishi structures of Gr and Sw theories, which is complicated mainly due to curves with multiplicity > 1. As we saw that in our context, all irreducible components of the Taubes curves have multiplicity 1, most of the discussion in this part can therefore be avoided.

Acknowledgments. The author's debt to Taubes should be apparent from the frequent appearance of his name in this article. We are grateful to Kronheimer and Mrowka, for inspiring remarks, and for generously answering many questions on the details of their book in preparation. We also benefited from helpful conversations with Bauer, Manolescu, Ozsvath, and Szabo.

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