Heegaard Splittings and Seiberg-Witten Monopoles

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math.GT/0409536

I. The Goal

Y**:** compact oriented 3-manifold.

$\mathfrak{s}$**:** a spin-c structure on Y.

$\mathrm{HF}^\bullet(Y, \mathfrak{s})$**:** $(\bullet = -, \infty, +, \wedge)$ 4 versions of Heegaard Floer homologies associated to $(Y, \mathfrak{s})$ defined by Ozsvath-Szabo.

$\overset{\bullet}{\mathrm{HM}}(Y, \mathfrak{s}; [\omega])$**:** $(\bullet = \wedge, -, \vee)$ Seiberg-Witten-Floer homologies defined by Kronheimer-Mrowka. $[\omega]$: the cohomology class of perturbation two form.

HM^{tot}**:** introduced by [L] (based on KM construction) in parallel to $\widehat{\mathrm{HF}}$.

R**:** coefficient ring, e.g. $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$.

In spite of their very different origins, HF and HM have identical formal properties:

- both are $R[U]$-modules, U being a degree -2 chain map,
- *fundamental exact sequences* relating the first 3 flavors:

$$\cdots \to \mathrm{HF}^{-} \to \mathrm{HF}^{\infty} \to \mathrm{HF}^{+} \to \cdots$$

$$\cdots \to \widehat{\mathrm{HM}} \to \overline{\mathrm{HM}} \to \check{\mathrm{HM}} \to \cdots$$

This talk will outline a long program to relate the two Floer homologies, and survey some partial results towards this goal. More precisely, we aim to prove:

Conjecture 1. *Let* $(Y, \mathfrak{s})$ *be as the above, and let*

$$[w] = 2\pi c_1(\mathfrak{s}). \tag{1}$$

Then there are isomorphisms of $R[U]$*-modules*

$$\begin{aligned}
\mathrm{HF}^{-}(Y, \mathfrak{s}) &\simeq \widehat{\mathrm{HM}}(Y, \mathfrak{s}; [w]),\\
\mathrm{HF}^{\infty}(Y, \mathfrak{s}) &\simeq \overline{\mathrm{HM}}(Y, \mathfrak{s}; [w]),\\
\mathrm{HF}^{+}(Y, \mathfrak{s}) &\simeq \check{\mathrm{HM}}(Y, \mathfrak{s}; [w]),\\
\widehat{\mathrm{HF}}(Y, \mathfrak{s}) &\simeq \mathrm{HM}^{tot}(Y, \mathfrak{s}; [w]),
\end{aligned}$$

which are natural with respect to the fundamental exact sequences of HM *and* HF.

Remarks. (1) This conjecture has been verified for all known computations of both sides. In addition, both HF and HM satisfy surgery exact sequences relating Y_0, Y, Y_1. If there is a map between two theories, natural with respect to the surgery exact sequences, then the conjecture holds.

The difficulty in proving the above conjecture lies precisely in *finding such a natural map*.

Will explain: the two theories are very different both in geometric contents and abstract frameworks.

(2) Variants of this conjecture: equivalence of 4-manifold invariants, twisted versions of Floer homologies, contact invariants, ...

The two theories are almost the same for practical purposes, but SW is more intimately related to the geometry (e.g. scalar curvature), while OS is more combinatorial (hence often more computable), and much easier to define. (500 pages vs. 100 pages; 10 years vs. 1 year; hard analysis vs. topological arguments)

II. Background

A Seiberg-Witten *configuration* on a 3- or 4-dimensional spin-c manifold $(X, \mathfrak{s})$ is a pair

A: a connection on $\det S$, $S =$ spinor bundle,
ψ: a section of S

Convention: we write
$$\begin{array}{l} (A, \psi) \text{ when } \dim X = 3, \\ (\hat{A}, \hat{\psi}) \text{ when } \dim X = 4. \end{array}$$

Simplest Seiberg-Witten invariant: for $(X, \mathfrak{s})$ closed spin-c 4-manifold, $\mathrm{Sw}_4(X, \mathfrak{s}) =$ signed count of

$$\left\{ \begin{array}{l} \text{solutions to SW equations:} \\[1ex] \quad \not{\partial}_{\hat{A}} \hat{\psi} = 0 \\ \quad \underset{\substack{\uparrow \\ \boxed{\text{self-dual part}}}}{\mathrm{SD}}(F_{\hat{A}}) - i\sigma(\hat{\psi}, \hat{\psi}) = i\, \underset{\substack{\uparrow \\ \boxed{\text{perturbation 2-form}}}}{\mathrm{SD}(\omega)} \end{array} \right\} \Big/ \text{(gauge equivalence)}$$

Sw_4 is an invariant of the differential structure of X: it is independent of metric and perturbation (under general assumptions).

SAMPLE APPLICATIONS OF SEIBERG-WITTEN (OS) THEORY.

- Short proof of Donaldson's theorem: SW, OS

Theorem. [Donaldson] *If a closed oriented 4-manifold X has negative definite intersection form Q_X, then the form is diagonalizable over $\mathbb{Z}$.*

Donaldson's theorem ⇝ existence of fake $\mathbb{R}^4$.

Proof: Seiberg-Witten theory ⇝

$$Q_X(\xi,\xi) + \mathrm{rank}(Q_X) \leq 0 \quad \text{for all characteristic } \xi$$

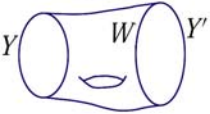
(i.e. $Q(\xi, v) \equiv Q(v, v) \mod 2 \forall v$). Combine with Elkies's result.

To extend this result to 4-manifolds with boundary, need *relative Seiberg-Witten invariants*, which take values in Seiberg-Witten-Floer homologies (cohomologies)

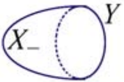
Atiyah: Gauge theoretic invariants form a TQFT

$$\left\{\begin{matrix}\text{category of 3-mfds}\\ \text{and cobordisms}\end{matrix}\right\} \xrightarrow[\mathcal{F}]{\text{gauge theory}} \left\{\begin{matrix}\text{category of}\\ \mathbb{F}\text{-vector spaces}\end{matrix}\right\}$$

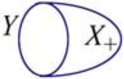
$$\mathcal{F}(Y) = \mathrm{FH}(Y) = \text{Floer homology of } Y$$



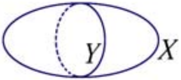
$$\mathcal{F}(W) = \mathrm{Hom}\big(\mathrm{FH}(Y), \mathrm{FH}(Y')\big)$$



$$\mathcal{F}(X_-) \in \mathrm{FH}_*(Y)$$



$$\mathcal{F}(X_+) \in \mathrm{FH}^*(Y)$$



$$\mathcal{F}(X) \in \mathbb{F}$$

FLOER HOMOLOGY HM:

Chain groups generated by translation invariant solutions on $\mathbb{R} \times Y$ (SW critical points);

Boundary map defined from 1-dim moduli space on $\mathbb{R} \times Y$ modulo translation. (SW flow lines)

A lot of hard analysis is required to make the picture work, but worthwhile: this is also important in computations of Seiberg-Witten invariants by *cut and paste*.

Theorem. [Froyshov OS, KM] *Let X be a compact negative-definite oriented 4-manifold bounding a homology sphere Y. Then*

$$Q_X(\xi,\xi) + \operatorname{rank}(Q_X) \leq 4d(Y) \quad \textit{for all characteristic } \xi,$$

where $d(Y)$ is an invariant of Y defined from Seiberg-Witten-Floer homologies.

- (Generalized) Thom conjectures: bounding minimal genus of embedded surfaces; bounding slice genera of knots. SW, OS
- Canonical class as a differential invariant for algebraic surfaces with $\kappa \geq 0$. SW
- Classifying Kahler surfaces with positive curvature. SW
- Classifying symplectic structures on ruled surfaces. SW
- Constraints on knots and 3-manifolds related by surgery.OS, SW

etc etc

Theorem. [Kronheimer-Mrowka] *Property P holds for all nontrivial knots $K \subset S^3$. I.e. $S_1^3(K)$ is not a homotopy sphere.* (SW)

Idea behind the proof: By Gabai, $S_0^3(K)$ admits a taut foliation. By Taubes-Kronheimer-Mrowka, this implies nontrivality of $\mathrm{HM}(S_0^3(K))$. Hence the nontriviality of $\mathrm{HM}(S_1^3(K))$ by the surgery exact sequence. By Witten's conjecture relating Seiberg-Witten theory and Donaldson theory, this implies nontriviality of instanton Floer homology of $S_1^3(K)$. HI is defined by counting representations of π_1; thus $\mathrm{HI} = 0$ for a homotopy sphere.

The actual proof indirectly follows the sketch above, because both Witten's conjecture and Floer homologies are hard.

In gauge theory, the degree of difficulty increases as one cuts open the manifolds:

closed manifold $\xrightarrow{cut}$ manifold with boundary $\xrightarrow{cut}$ manifold with corners

The route taken by Ozsvath-Szabo goes in the opposite direction: in OS theory, it is easier to define invariants for simple pieces, then using these as building blocks to define invariants for closed manifolds via pasting and TQFT. E.g. OS defined knot Floer homology; the analogous story in gauge theory is technically forbidding.

BASIC SETUP OF HEEGAARD FLOER THEORY.

Fix a Heegaard splitting of Y

$$Y = H_- \cup_\Sigma H_+,$$

H_+, H_-: two handlebodies;
Σ: Heegaard surface of genus g.

A Morse function $f : Y \to \mathbb{R}$ is *adapted to the Heegaard splitting* if $f^{-1}(0) = \Sigma$ is the Heegaard surface, and

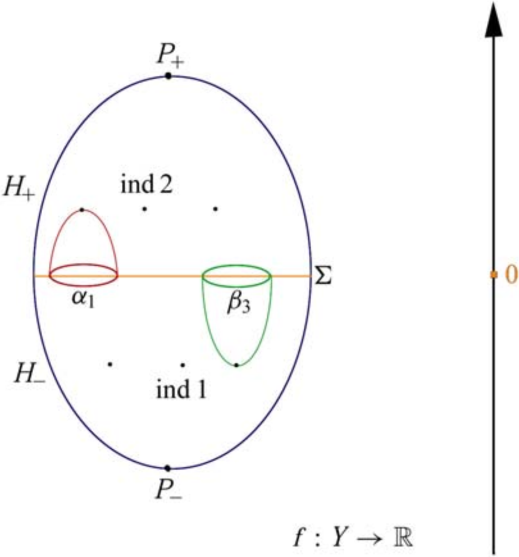
$$H_+ = f^{-1}\mathbb{R}_{\geq 0}, \quad H_- = f^{-1}\mathbb{R}_{\leq 0}$$

contain respectively one maximum p_+ and g index 2 critical points, and one minimum p_- and g index 1 critical points.

Let $\alpha_i,\ i = 1, \ldots, g$ denote the descending cycles on Σ from the g index 2 critical points. Let $\beta_i,\ i = 1, \ldots, g$ denote the ascending cycles on Σ from the g index 1 critical points.

$$\mathbb{T}_\alpha := \alpha_1 \times \cdots \times \alpha_g, \quad \mathbb{T}_\beta := \beta_1 \times \cdots \times \beta_g \subset \mathrm{Symp}^g\,\Sigma.$$

Suppose $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ intersects transversely.



Morse function adapted to Heegaard splitting

CF^∞ is generated by the “*Heegaard critical points*”, i.e. intersection points of $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$.

The boundary map is defined by counting “*Heegaard flow lines*”, i.e. holomorphic disks

$$\mu : \mathbb{R} \times [0,1] \to \mathrm{Symp}^g \Sigma, \quad \text{with } \mu(\cdot, 0) \in \mathbb{T}_\alpha, \mu(\cdot, 1) \in \mathbb{T}_\beta.$$

The more interesting versions of HF depend on the choice of a *base point*: Let $z \in \Sigma$ be a point avoiding α_i’s and β_j’s.

Let $\gamma_z \subset Y$ be the flow line of f from p_+ to p_- through z.

III. Motivation: Taubes's fundamental work.

In fact, it is not surprising that the Seiberg-Witten-Floer homologies should be related to curve-counting invariants. Since Taubes's seminal work, the relation between Seiberg-Witten theory and Gromov's theory of pseudo-holomorphic curves has been well-known.

Taubes's Theorem. *Let* (X, ϖ) *be a closed, oriented, symplectic 4-manifold, and* $\mathfrak{s}$ *be a spin-c structure on* X*. Then*

$$\mathrm{Sw}_4(X, \mathfrak{s}) = \mathrm{Gr}(X, \mathfrak{s}),$$

Gr is a variant of Gromov invariant that counts embedded, possibly disconnected, pseudo-holomorphic curves (with multiplicity) in the homology class determined by $\mathfrak{s}$.

Proof: 4×100 pages long.

In the case where X is an algebraic manifold, this is just a simple analog of the correspondence between line bundles and divisors.

Taubes's idea: choose a metric on X with respect to which the symplectic form ϖ is SD (hence harmonic). The metric, together withe ϖ, determines an almost complex structure on X.

The Clifford action by ϖ splits the spinor bundle into a direct sum of eigenspaces:

$$S = E \oplus E \otimes K^{-1}, \tag{2}$$

where K^{-1} is the anti-canonical bundle.

A connection $\hat{A}^E$ of E determines a spin-c connection A on S. Thus, we shall write Seiberg-Witten configuration as

$$(\hat{A}^E, (\hat{\alpha}, \hat{\beta})).$$

Consider perturbation to the Seiberg-Witten equations on X by the two-form

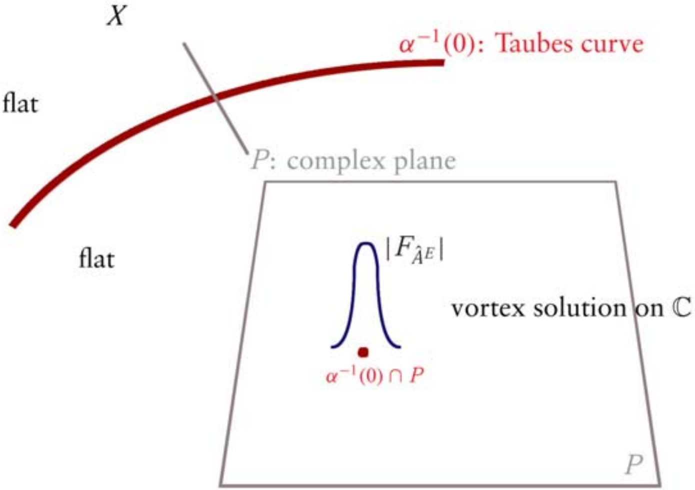
$$\omega = r\varpi + \text{insignificant terms}, \quad \text{where } r \gg 1 \text{ is a constant.}$$

As $r \to \infty$, the zero locus $\alpha^{-1}(0)$ approaches a holomorphic curve, which is also the support of the current, $\lim_{r\to\infty} F_{A^E}$. (The "*Taubes curve*").

Local description of large r solutions: Away from the Taubes curve, the solutions approximate the simple form

$$(\hat{A}^E, (\hat{\alpha}, \hat{\beta})) = (0, (\sqrt{r}, 0)), \quad \text{with respect to a trivialization of } E,$$

while near the Taubes curve, it approximates a family of vortex solutions parameterized by the Taubes curve.



Local behavior SWT solutions for large r

VARIANTS OF SEIBERG-WITTEN-TAUBES (SWT) THEORIES.

(i) SWT theory on closed 4-manifolds. Instead of a symplectic ϖ, one may take ϖ to be any self-dual harmonic 2-form. For generic metric, such ϖ vanishes along a set of circles in X. K^{-1} and the splitting (2) make sense away from these circles.

Taubes: extension of the convergence theorem (part I of the proof) to this setting:

Taubes curve $\sim$ pseudo-holomorphic subvariety with boundary at $\varpi^{-1}(0)$.

However, full equivalence is hard. Even the generalized Gromov invariant is undefined.

(ii) SWT theory on closed 3-manifolds: Morse-Novikov theory. The 3-d story is simpler.

$\varpi =$ harmonic 2-form on Y now.
$*\varpi =: \vartheta$ is a harmonic 1-form; *Morse* for generic metric.

$\varpi^{-1}(0) = \{g$ pairs of index 2 and index 1 critical points of $\vartheta\}$.

"*Taubes orbit*" (counterpart of Taubes curve)
$\mathcal{O} =$ union of finite-length flow lines of $-\check{\vartheta}$, $\partial\mathcal{O} = \varpi^{-1}(0)$.
($\check{\vartheta} =$ dual vector field of ϑ.)

"*constituent flow lines*"=the flow lines constituting a Taubes orbit.

I_3 =counting invariant of Taubes orbits. (Hutchings-Lee)

Conjecture. [HL1] $(Y, \mathfrak{s})$*: closed spin-c 3-manifold,* $b_1(Y) > 0$*. Then*

$$\mathrm{Sw}_3(Y, \mathfrak{s}) = I_3(Y, \mathfrak{s}).$$

I_3 is a special case of an invariant $I = \tau(\theta)\zeta(\theta)$ in general Morse-Novikov theory.

($\tau(\theta)$ =Reidemeister torsion of the Morse-Novikov complex,

$\zeta(\theta)$ =dynamical zeta function.)

This conjecture is indirectly proved, because of the Meng-Taubes theorem $\mathrm{Sw}_3 = \tau(Y)$, and the following

Theorem. [HL1, HL2] *For an oriented, closed manifold* M *(of any dimension),*

$$I(M) = \tau(M),$$

$\tau(M)$ *=combinatorially defined Reidemeister torsion of the manifold* M*.*

(iii) SWT Floer theory on closed 3-manifolds. The above picture may be generalized to Floer theory.

C_* generated by 3-d SWT solutions
$\sim$ Taubes orbits $\mathcal{O}$, $\partial\mathcal{O} = \varpi^{-1}(0)$
∂_* defined by counting 4-d SWT solutions on $\mathbb{R} \times Y$,
$\sim$ Taubes curves C, $\partial C = \mathbb{R} \times \varpi^{-1}(0)$.

Analog of Taubes's theorem:
SWT Floer homology = *"a symplectic version of Floer hom"*,
analogous to the contact homology of Eliashberg-Hofer-Givental.

SIMPLEST SPECIAL CASE: Y = mapping tori.
may choose ϑ such that $\varpi^{-1}(0) = \emptyset$.
Hutchings: some fundations of RHS, ("periodic Floer homlgy").

⇝ potentially interesting connection with contact homology, but hardly helps the understanding of SW Floer homologies, as its construction and computation is no simpler.

(iv) $\mathbb{R}$-valued Morse function & Heegaard splittings?

better to connect with $\mathbb{R}$-valued Morse theory than Morse-Novikov theory of closed 1-forms as in (ii) & (iii):

- The important case of $b_1(Y) = 0$ is excluded in (ii), (iii).
- $f : Y \to \mathbb{R} \Rightarrow$ Heegaard splitting $\Rightarrow$ reducing computation to 2-dimensions, i.e. Heegaard surface $\Rightarrow$ great simplifications!

However– Suppose $\vartheta = df$ for $f : Y \to \mathbb{R}$. ϑ harmonic $\Rightarrow$ f harmonic, but no non-constant harmonic function on a closed manifold.

This does not constitute a serious obstacle. Real-valued harmonic functions do exist on *non-compact* 3-manifolds. For instance, one may consider

1. **(3-d Euclidean SWT theory)** $Y\#\mathbb{R}^3$ with Euclidean metric at infinity. (Cf. [L0])
2. **(3-d cylindrical SWT theory)** Deleting two points p_+, p_- from Y, then choosing a complete metric on the resulting open manifold, so that it has two cylindrical ends $\mathbb{R}_\pm \times S^2$. There exist harmonic functions f on such cylindrical manifolds which are asymptotic to $\tau + C_\pm$on the cylindrical ends, where τ parametrizes $\mathbb{R}_\pm$.

   Consider spin-c structures such that $\deg E|_{S^2_{\pm\infty}} = 0$. (more generally, $\deg \geq 0$, or consider 3-manifolds with a number of positive and negative ends).

Taubes's pictures for both settings are similar; take #2.

f: Morse function adapted to a Heegaard splitting.

Constituent flow lines of Taubes orbits $\leftrightarrow$ intersection points of α_i and β_j in Σ;

SWT critical points $\sim$Taubes orbit
$\leftrightarrow$ intersection point of $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ in $\mathrm{Symp}^g\,\Sigma$. Note: no closed orbits in a Taubes orbit.

SWT flow lines $\sim$ Taubes curves C, $\partial C = \mathbb{R} \times \mathrm{Crit}(f)$
$\xleftrightarrow{\text{Atiyah-Floer}}$ holomorphic disks in $\mathrm{Symp}^g\,\Sigma$ with boundary along $\mathbb{T}_\alpha, \mathbb{T}_\beta$, i.e. Heegaard flow lines.

SWT Floer $\leftrightarrow$ Floer homology of "Lagrangian" intersections for
$(\mathrm{Symp}^g\,\Sigma; \mathbb{T}_\alpha, \mathbb{T}_\beta)$, ($\mathrm{HF}^\infty$ of Ozsvath-Szabo)

Independent of Seiberg-Witten and Atiyah-Floer, this Morse-theoretic picture defines certain topological invariants. However,

- These are decidedly different from the ordinary Seiberg-Witten invariant or Floer homologies of Y. i.e. $\check{HM}$. Not unexpected: these are supposedly gauge-theory invariants on *noncompact* manifolds.)
- They are not interesting invariants: only depend on homological information of Y.

E.g.

$$\begin{aligned} \#\left(\mathbb{T}_\alpha \cap \mathbb{T}_\beta\right) &= \det\left(\#(\alpha_i \cap \beta_j)\right) \\ &= \begin{cases} 0 & \text{as } b_1(Y) > 0; \\ |H_1(Y;\mathbb{Z})| & \text{as } b_1(Y) = 0, \end{cases} \end{aligned}$$

End of story until the advent of Ozsvath-Szabo's amazing idea—*filtration*.

IV. Abstract frameworks of two Floer theories.

BASIC INGREDIENTS OF A FLOER THEORY

Floer theory $\sim$ formal Morse theory over $\mathcal{C}$ (∞-dim'l space). (However, Floer homology does not compute $H_*(\mathcal{C})$. Instead, it is a "middle-dimensional homology": polarization of $T\mathcal{C}$ via the differential operator is important).

- possibly non-exact Morse 1-form over $\mathcal{C}$.
- (relative) index of the critical points defined $\mod N \in \mathbb{N}$. (∞-dim'l phenomenon–due to spectral flow)
- $\exists$ minimal abelian coverings $\tilde{\mathcal{C}} \to \mathcal{C}$ s.t. the Morse 1-form lifts to a differential of a $\mathbb{R}$-valued Morse function, and indices of lifts of the critical points are well-defined in $\mathbb{Z}$ *for all parameters of the theory*. (e.g perturbations, metric, spin-c structure) Let G be the covering group.

Let $$\mathrm{PR}: G \to \mathbb{R}; \quad \mathrm{SF}: G \to \mathbb{Z}$$

denote homomorphisms defined by the change in the values of the Morse function and the indices under deck transformation.

- subcovers $\hat{\mathcal{C}}$ of $\tilde{\mathcal{C}}$ on which the Morse 1-form lifts to be exact
$\leftrightarrow$ subgroups N of $\ker \mathrm{PR}$
$\leftrightarrow$Floer complexes with different local coefficients. (finitely generated $R[G/N]$ modules, coefficients: $R[\ker \mathrm{SF}|_N]$, graded mod $\gcd(\mathrm{SF}|_N)$)

  Unless otherwise specified, a "Floer complex" means the version corresonding to minimal $\hat{\mathcal{C}}$. It has ordinary R-coefficients when:

$$\mathrm{PR}|_{\ker \mathrm{SF}} = \emptyset. \tag{3}$$

- Morse-Novikov picture $\rightsquigarrow$ Floer homology invariant under variation of metric or perturbations, as long as

$$\text{PR remains on a positive ray from 0.} \tag{4}$$

Comparing the formal frameworks of two Floer theories:

	Seiberg-Witten	Heegaard
$\mathcal{C}$	$\{(A,\psi)\}/\mathcal{G}$: quotient of the spaces of configurations by gauge group action	$\Omega(\mathrm{Sym}^g(\Sigma);\mathbb{T}_\alpha,\mathbb{T}_\beta)$: space of paths $\gamma:[0,1]\to\mathrm{Sym}^g(\Sigma)$, with $\gamma(0)\in\mathbb{T}_\alpha$, $\gamma(1)\in\mathbb{T}_\beta$
	$\mathcal{C}$ singular at fixed points (reducibles)	$\mathcal{C}$ smooth
G	$H^1(\mathcal{C};\mathbb{Z})=H^1(Y;\mathbb{Z})$	$\pi_2(\mathbf{x},\mathbf{x})=\mathbb{Z}\oplus H^1(Y;\mathbb{Z})$ (*)
PR	$2\pi(2\pi c_1(\mathfrak{s})-[w])$ (†)	$-C\oplus 0$ (‡) C is a positive constant
SF	$c_1(\mathfrak{s})$	$2\oplus c_1(\mathfrak{s})$

(*) Projection to the $\mathbb{Z}$ factor is given by the intersection number of the base point z with the 2-chain in Σ associated to the 1-cycle in $\Omega(\mathrm{Sym}^g(\Sigma);\mathbb{T}_\alpha,\mathbb{T}_\beta)$.

(†) Here and below, we regard a cohomology class in $H^2(Y)$ as a homomorphism $H^1(Y)\to\mathbb{R}$ via Poincaré duality.

Note: **(1)** G of the Heegaard Floer theory contains an extra $\mathbb{Z}$- factor than that of SW theory. To put both theories on the same footing, regard CF^{∞} as an *infinitely generated* $R[H^1(Y;\mathbb{Z})]$ module. I.e. instead of $\mathcal{C}$, model the Heegaard Floer theory on the Morse theory on the infinite-cyclic covering $\hat{\mathcal{C}}_z$ of $\mathcal{C}$, that corresponds to the projection of G to $\mathbb{Z}$.

(2) Choosing $[w] = 2\pi c_1(\mathfrak{s})$ (Condition (1) in Conjecture 1)
$\rightsquigarrow \mathrm{PR}_{SW} = 0$. $\rightsquigarrow$
$\mathrm{PR}_{SW} = \mathrm{PR}_{OS}\,|_{H^1(Y;\mathbb{Z})}$;
$\mathrm{SF}_{SW} = \mathrm{SF}_{OS}\,|_{H^1(Y;\mathbb{Z})}$.
Condition (3) holds in both theories $\rightsquigarrow$
both Floer complexes are of R-coefficients.

Continuing the table of comparison, the two theories are now formally parallel. However, these parallel aspects come from entirely different origins.

Both Floer theories require refinements of the basic Floer theory framework outlined above, but the refinement for each theory is based on different principles: the Heegaard Floer theory relies on a *filtration* of the complex associated to the infinite-cyclic covering, while the Seiberg-Witten theory is an S^1-*equivariant* theory.

Formal analogies	Seiberg-Witten	Heegaard
both are $R[u]$-modules	because this is an equivariant theory: U generates $H^*(BS^1)$	because this models on Morse theory on a $\mathbb{Z}$-cover: U generates deck transf'n
both complexes are ∞-generated R-modules	#(crit pts) finite, including reducible critical points (fixed points)	because there are ∞-many critical points on a $\mathbb{Z}$-cover
long exact sequences relate $\mathrm{H\hat{M}}$, $\mathrm{H\bar{M}}$, $\mathrm{H\check{M}}$, and HF^-, HF^∞, HF^+	[L]: this is formally a fundamental sequence of S^1-equivariant theory	this is the relative sequence associated to a filtration
both have a 4th version	[L]: HM^{tot} models on homology of S^1-bundle	$\widehat{\mathrm{HF}}$ models on homology of a fundamental domain

V. Filtration from semi-positive 1-cocycles

OS CONSTRUCTION IN ABSTRACT FORMULATION:

Z: 1-cocycle in M (manifold), or a codimension 1 cycle in M, with primitive cohomology class $[Z]$.

$Z \rightsquigarrow \Gamma(Z)$, local system over M, (assigning each 1-chain in M its intersection number with Z).

$\tilde{M}_Z$: infinite-cyclic covering associated to Z.
$H_*(M; \Gamma(Z)) = H_*(\tilde{M}_Z)$ is a $R[\mathbb{Z}] = R[t, t^{-1}]$ module.

θ: closed Morse 1-form on M with cohomology class

$$[\theta] = \alpha[Z] \quad \text{for } \alpha \in \mathbb{R}, \tag{5}$$

$\rightsquigarrow M_*(\theta, \Gamma(Z))$, Morse complex with local coefficients. module over $R[t^{-1}, t]]$ when $\alpha > 0$, over $R[t, t^{-1}]]$ when $\alpha < 0$. t: *negative* generator of the deck transformation.

Definition. Z is *semi-positive* with respect to θ if $[Z]$ satisfies (5), and $\Gamma(Z)(\gamma) \geq 0$ for flow lines γ of $-\check{\theta}$.

Z semipositive ⇝ filtration on the Morse complex M_*
⇝ M_*^-, M_*^+ sub- and quotient complexes:

$\tilde{Z}$: a lift of Z in $\tilde{M}_Z$ dividing $\tilde{M}_Z$ into two halves.

$\tilde{M}_Z^- \subset \tilde{M}_Z$: lower half (with respect to the flow of $-\check{\theta}$).

$M_*^-(\theta, \Gamma(Z))$: Morse complex for $\tilde{M}_Z^-$.

$M_*^+(\theta, \Gamma(Z)) := \frac{M_*(\theta,\Gamma(Z))}{M_*^-(\theta,\Gamma(Z))}$: Morse complex of the pair $(\tilde{M}_Z, \tilde{M}_Z^-)$

⇝ $$0 \to M_*^-(\theta, \Gamma(Z)) \to M_*(\theta, \Gamma(Z)) \to M_*^+(\theta, \Gamma(Z)) \to 0$$

(short exact sequence of $R[t]$-modules)

⇝ long exact sequence of the pair $(\tilde{M}_Z, \tilde{M}_Z^-)$.

(relative exact sequence of a filtration)

Note: a different, cohomologous Z' yields an equivalent local system, but a possibly different filtration. Thus, $H_*(M_*(\theta, \Gamma(Z))) = H_*(M_*(\theta, \Gamma(Z')))$ but the filtrated versions in general are invariant only under *small* exact perturbations of θ or cohomologous perturbation of Z.

Example. $\theta = df$ for a Morse $f : M \to S^1$,
$Z =$ the 1-cocycle θ, or the codim 1 cycle $f^{-1}(c)$.
Z is a semi-positive; associated filtration=filtration by energy.

Example. (Seidel) $P =$ Kähler manifold. Consider a symplectic version of Floer theory on P:
$\mathcal{C} =$ (relative) loop space Ω of P.
flow line in $\Omega \leftrightarrow$ holomorphic disk/sphere in P.
semi-positive 1-cocycle: intersection with a (complex) hypersurface in P.

Example. (Heegaard Floer homologies) variant of the previous example, with Hypersurface$= \{z\} \times \mathrm{Symp}^{g-1}\,\Sigma$.

$\mathrm{HF}^-, \mathrm{HF}^\infty, \mathrm{HF}^+ \leftrightarrow H(M_*^-), H(M_*), H(M_*^+)$,

fundamental exact sequence of Heegaard Floer homologies
$\leftrightarrow$ relative exact sequence of the pair $(\tilde{M}_Z, \tilde{M}_Z^-)$.

$\widehat{\mathrm{HF}} \leftrightarrow H_*(\tilde{M}_Z^-, t\tilde{M}_Z^-)$ (homology of a fundamental domain)

Using topological arguments very special to this specific two-dimensional situation, OS showed that these Floer homologies depend on z only through the spin-c structure.

VI. Bridging the two Floer theories: HMT

PROPOSAL: construct an intermediate Floer theory, $\mathrm{HMT}^{\bullet}$, (Heegaard-Monopole-Taubes) exhibiting characteristics of both theories:

- $\bullet = -, \infty, +, \wedge$; the first 3 versions fit into a fundamental long exact sequence.
- This is a variant of SWT Floer theory, with Heegaard ingredients: the choice of a Heegaard splitting of Y, and a filtration associated to a 1-cycle $\underline{\gamma}_z$.
- there are two natural $R[U]$-module structures for HMT, one from deck transformation, and the other from equivariant theory, which turn out to be identical. (Recall: U-map for HF acts by deck transformation; for HM, it is cap (cup) product with the generator of $H^*(BS^1)$).

Equivalence of HMT with either theory is easier to establish, and Conjecture 1 is broken into two:

Conjecture. *Let* $(Y, \mathfrak{s})$ *and* $[w] = 2\pi c_1(\mathfrak{s})$ *be as in Conjecture 1. Then there are isomorphisms of* $R[U]$*-modules:*

$$\begin{tikzcd}
& \mathrm{HMT}^\bullet(\underline{Y}, \underline{\mathfrak{s}}; \underline{\gamma}_z) \arrow[dl, equal, "\textit{Conjecture (b)}"'] \arrow[dr, equal, "\textit{Conjecture (a)}"] & \\
\mathrm{HF}^\bullet(Y, \mathfrak{s}) \arrow[rr, dotted, no head, "\textit{Conjecture 1}"'] & & \mathrm{HM}^\bullet(Y, \mathfrak{s}; [w])
\end{tikzcd}$$

Furthermore, these isomorphisms are all natural with respect to the fundamental sequences of $\mathrm{HM}^\bullet$, $\mathrm{HMT}^\bullet$, *and* $\mathrm{HF}^\bullet$.

BASIC SETUP FOR HMT.

$f : Y \to \mathbb{R}$: Morse function adapted to Heegaard splitting.

Attach a 1-handle to Y along the two points p_+, p_- (max, min of f) $\rightsquigarrow \underline{Y}$.

$\underline{f} : \underline{Y} \to S^1$: extension of f, which has no extrema.

Calabi: $\exists$ metric on $\underline{Y}$, making $\underline{f}$ harmonic.

$z \in \Sigma$, $\gamma_z \subset Y$: base point and associated 1-chain as in HF.
$\underline{\gamma}_z \subset \underline{Y}$: 1-cycle through z completing γ_z; choosing metric and z such that

$$[*d\,\underline{f}] = \alpha \, \mathrm{P.\,D.}[\underline{\gamma}_z] \in H^2(\underline{Y}) \quad \text{for some constant } \alpha > 0. \tag{6}$$

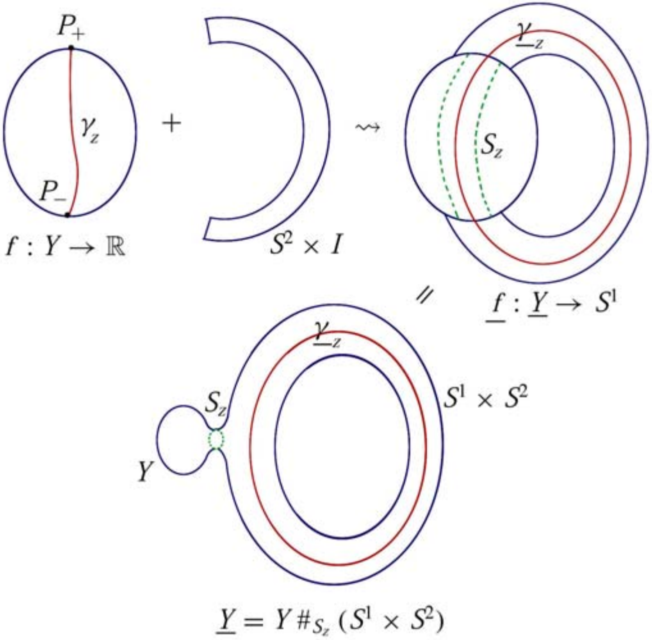
More precisely, $\underline{\gamma}_z$ satisfies:

(Γ1) There are $\kappa \in S^1$, $\delta \in \mathbb{R}^+$, such that $\underline{Y}_{[\kappa-\delta,\kappa+\delta]}$ is contained in the added 1-handle, and $\underline{\gamma}_z \cap \underline{Y}_{]\kappa-\delta,\kappa+\delta[}$ is a gradient flow line of $\underline{f}$ through $z \in \Sigma_H$;

(Γ2) $\underline{f}$ is monotone along $\underline{\gamma}_z$;

(Γ3) $\underline{\gamma}_z$ avoids the ascending and descending manifolds from the critical points of $\underline{f}$ in $\underline{Y} \backslash \Sigma_H$.

$\Sigma_H := \underline{f}^{-1}(0)$: Heegaard surface; $\underline{Y}_I := \underline{f}^{-1} I \subset \underline{Y}$; $\underline{Y}_{]a,b[} := \underline{Y}_{S^1 \backslash (a,b)}$.


P_+
γ_z
P_-
$f : Y \to \mathbb{R}$
+
$S^2 \times I$
⇝
$\underline{\gamma}_z$
S_z
$\underline{f} : \underline{Y} \to S^1$
=
$\underline{\gamma}_z$
S_z
$S^1 \times S^2$
Y
$\underline{Y} = Y \#_{S_z} (S^1 \times S^2)$

S_z: boundary 3-sphere of a tubular neighborhood of γ_z.
$\rightsquigarrow$ z-dependent decomposition

$$\underline{Y} = Y \#_{S_z} S^1 \times S^2. \tag{7}$$

$\mathfrak{s}$: spin-c structure on Y $\rightsquigarrow$ $\underline{\mathfrak{s}}$: spin-c structure on $\underline{Y}$,

$$\underline{\mathfrak{s}} := \mathfrak{s} \#_{S_z} \mathfrak{s}_K, \tag{8}$$

where $\mathfrak{s}_K$ is the spin-c structure on $S^1 \times S^2$ corresponding to the standard nowhere-vanishing vector field on $S^1 \times S^2$, i.e. $\nabla \tau$, τ parametrizing S^1.

Note: $\mathfrak{s}_K$ is *not* the trivial spin-c structure. $c_1(\mathfrak{s}_K) = 2\,\Omega_S$, where Ω_S is the positive generator of $H^2(S^1 \times S^2)$.

$\mathrm{HMT}^{\infty}(\underline{Y}, \underline{\mathfrak{s}})$: HM of $(\underline{Y}, \underline{\mathfrak{s}})$, with perturbation

$$\omega = r * d\underline{f} + w, \tag{9}$$

where $r \gg 0$ is a constant, and w is a closed 2-form with $[w] = [w_Y]\#_{S_z}[0]$, $[w_Y]$ satisfying (1). $\check{\mathrm{HM}} = \hat{\mathrm{HM}}$ in this case, see (2) below.

BASIC PROPERTIES:

(1) The decomposition (7) splits

$$H_1(\mathcal{C}) = H^1(\underline{Y}) = \mathbb{Z} \oplus H^1(Y).$$

In terms of this splitting,

$$\begin{aligned} \mathrm{SF} &= c_1(\underline{\mathfrak{s}}) = 2 \oplus c_1(\mathfrak{s}), \\ \mathrm{PR} &= 2\pi(2\pi c_1(\underline{\mathfrak{s}}) - [w + r * df]) = -C' \oplus 0, \quad C' > 0. \end{aligned}$$

Notice the complete agreement with the formulae G, SF, and PR in Heegaard Floer theory. In particular, it models on the Morse complex of $\hat{\mathcal{C}}_{\underline{\gamma}_z}$, the infinite-cyclic covering of $\mathcal{C}$ determined by the homomorphism

$$/[\underline{\gamma}_z] : H_1(\mathcal{C}) = H^1(\underline{Y}) \to \mathbb{Z}.$$

(2) From the form of 3-d SW eqns, reducible critical points exist only when

$$c_1(\mathfrak{s}) - [\omega]/(2\pi) = 0.$$

With our choice of ω, the left hand side is $-r/(2\pi)[*d\underline{f}]$, which is never zero as $r > 0$. Thus, all critical points (of the CSD functional) are irreducibles (i.e. smooth points in $\mathcal{C}$). again agreeing with Heegaard Floer theory formally.

(3) Formula above $\rightsquigarrow$ PR always lies in the positive ray along the negative generator of $H^1(S^1 \times S^2) \subset H^1(\underline{Y})$.

$\rightsquigarrow$ HMT$^\infty$ is an invariant, (independent of r, further exact perturbation of w, and depends on metric and $\underline{f}$ only through the cohomology classes $[d\underline{f}]$, $[*d\underline{f}]$).

We now introduce a filtration on CMT^∞ in parallel to the filtration by z in Heegaard Floer theory.

$\mathrm{hol}_{\underline{\gamma}_z} : \mathcal{C} \to \mathbb{R}/2\pi\mathbb{Z}$: holonomy of A^E along $\underline{\gamma}_z$.

$Z_{\underline{\gamma}_z} := \mathrm{hol}_{\gamma_z}^{-1}(c),\ c \in (\mathbb{R}/2\pi\mathbb{Z})\setminus\{0\}$: codimension 1 cycle in $\mathcal{C}$.

By (6), Floer complex with local coefficient $\Gamma(Z_{\underline{\gamma}_z}) =$ Floer complex of $\hat{\mathcal{C}}_{\underline{\gamma}_z}$.

Claim. *$Z_{\underline{\gamma}_z}$ is semi-positive.* (by positive intersection of Taubes curve with $\mathbb{R} \times \underline{\gamma}_z$)

$\rightsquigarrow$ $\mathrm{CMT}^-, \mathrm{CMT}^+, \widehat{\mathrm{CMT}}$: Floer complexes of the lower half of $\hat{\mathcal{C}}_{\underline{\gamma}_z}$, of the pair, and of a fundamental domain.

FUNDAMENTAL EXAMPLE. $(\underline{Y}, \underline{\mathfrak{s}}) = (S^1 \times S^2, \mathfrak{s}_K)$:

$$\begin{aligned} \mathrm{HMT}^- &= uR[u], \\ \mathrm{HMT}^\infty &= R[u, u^{-1}], \\ \mathrm{HMT}^+ &= R[u, u^{-1}]/(uR[u]), \\ \widehat{\mathrm{HMT}} &= R, \end{aligned}$$

where $\deg u = -2$; deck transformation and (equivariant) U-map both act by multiplication by u.

Agree both with $\mathrm{HF}^\bullet(S^3, \mathfrak{s}_0)$ and $\mathrm{HM}^\bullet(S^3, \mathfrak{s}_0)$.

CONJECTURE (A): “almost a theorem”.

Proof: *filtrated connected sum theorem* plus computation for $(\underline{Y}, \underline{\mathfrak{s}}) = (S^1 \times S^2, \mathfrak{s}_K)$.

Heuristics behind of proof: Floer homology should be the S^1-equivariant homology of a generalized “S^1-space” (pro-spectrum?).

Connected sum of 3-manifolds ⇝ product of S^1-spaces, with diagonal S^1 action.

S^1-space for the fundamental example: $S^{\infty}_{-\infty}$: universal S^1-bundle over $\mathbb{C}P^{\infty}_{-\infty}$.

$\Rightarrow$ mysterious relation between SW and Heegaard Floer theories.

- Free S^1 action on $S^{\infty}_{-\infty}$ ⇝ Free S^1 action on product ⇝ $\mathcal{C}$ smooth in HMT and HF theory.
- filtration by deck transformation ↔ natural filtration of $\mathbb{C}P^{\infty}_{-\infty}$ by attaching even dim cells.
  “Positive-half”$\sim S^{\infty}_{-\infty}/S^{0}_{-\infty} \sim ES^1$.
  $\rightsquigarrow \mathrm{HMT}^{+} = H_{S^1}(S_Y \times ES^1)$
  equivariant homology of S_Y.
  $\widehat{\mathrm{HMT}} = H_{S^1}(S_Y \times \text{free } S^1\text{-cell}) = H(S_Y)$
  homology of total space
- filtration $+$ periodicity of $H_*(\mathbb{C}P^{\infty}_{-\infty})$ ⇝ mechanism that relates filtration in OS theory and equivariant theory of SW.

Thus, adding 1-handle (to form $\underline{Y}$) is really *essential*. (Also used: for existence harmonic Morse function; to get identical G, SF, PR with Heegaard theory)

CONJECTURE (B): Variant of the Taubes picture for cylindrical SWF Floer theory. Partial results.

Ingredients:

(1) Modification of part I of Taubes's proof.

(2) Structure of 3-d cylindrical SWT moduli spaces [L1]

(3) Simpler variant of Atiyah-Floer (stretching & gluing arguments) missing!

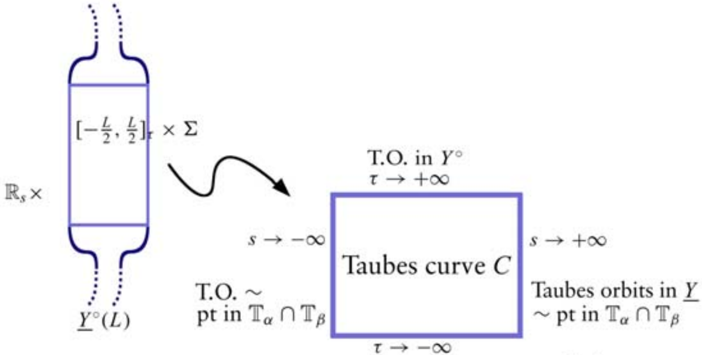
Heuristics behind:

$\underline{Y}(L)$: Stretch $\underline{Y}$ along the Heegaard surface Σ_H, so that it contains a long neck $[-L/2, L/2] \times \Sigma$.

$$Y(L)^\circ := \underline{Y}(L) \Big\backslash \left(-\tfrac{L}{2+\epsilon}, \tfrac{L}{2+\epsilon}\right) \times \Sigma.$$ (connected sum of $H_\pm$ along $p_\pm$)

$$\xrightarrow{L\to\infty} Y^\circ$$ mfds with cylindrical ends

As $L \to \infty$, energy on $\mathbb{R}_s \times \underline{Y}$ concentrates on
"middle square": $[-\Lambda/2, \Lambda/2]_s \times [-L/2, L/2]_\tau \times \Sigma$.



$[-\frac{L}{2}, \frac{L}{2}]_t \times \Sigma$

$\mathbb{R}_s \times$

$\underline{Y}^\circ(L)$

T.O. in Y°

$\tau \to +\infty$

$s \to -\infty$

$s \to +\infty$

Taubes curve C

T.O. $\sim$ pt in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$

Taubes orbits in $\underline{Y}$ $\sim$ pt in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$

$\tau \to -\infty$

Taubes orbits in $Y^\circ \sim$ pts in $\mathbb{T}_\alpha \times \mathbb{T}_\beta \xrightarrow{\partial_-, \partial_+} \mathbb{T}_\alpha, \mathbb{T}_\beta$

$$C \cap (\mathbb{R}_s \times [-\frac{L}{2}, \frac{L}{2}] \times \Sigma) \xrightarrow{\text{proj}} \mathbb{R}_s \times [-\frac{L}{2}, \frac{L}{2}]$$

$$\xrightarrow{\text{rescale}} \mathbb{R} \times [0, 1] : \text{holomorphic}$$

$\rightsquigarrow$ (g) multi-section of $\mathbb{R} \times [0, 1] \times \Sigma \to \mathbb{R} \times [0, 1]$

(trivial Σ-bundle)

$\rightsquigarrow$ $\mu : \mathbb{R} \times [0, 1] \to \mathrm{Sym}^g(\Sigma)$ & B.C. (matching outside)

i.e. Heegaard flow line