SOME EXAMPLES IN THE NON-STABLE K-THEORY OF C^*-ALGEBRAS

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To My Parents

And

My Wife And My Son

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ABSTRACT

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We give a necessary and sufficient condition for lifting projections from the corona algebra of $I = C(X) \otimes \mathcal{K}$ to the multiplier algebra where X is $[0,1]$, $[0,\infty)$, $(-\infty,\infty)$, or $\mathbb{T}$. Also, we give criterions for homotopy equivalence, unitary equivalence, and Murray-von Neumann equivalence of two projections in the corona algebra. In addition, we show some examples of other lifting problems; lifting unitaries to unitaries, lifting unitaries to extremal partial isometries, lifting extremal partial isometries to extremal partial isometries.

1. Introduction

Our goal in this thesis is to study projections in the corona algebra of a non-simple stable rank one C^*-algebra. The work presented here originated from a lifting problem: Let A be a C^*-algebra and I be a closed ideal of A. Is every unitary in A/I liftable to a partial isometry in A ? It happens whenever I has an approximate identity of projections, in particular if I has real rank zero. We are concerned with the case where I has stable rank one. It is no surprise that this is impossible in general, but constructing explicit counter-example is still not so trivial. One way to do is to find a stably projectionless stable rank one algebra I, such that $K_0(I)$ is non-trivial. Then stabilize I and consider an extension of I by $C(\mathbb{T})$ corresponding to a unitary $\mathbf{u}$ in the corona algebra with non-trivial K_1-class. We can observe that $\mathbf{u}$ can't be lifted to a unitary (if so, $[\mathbf{u}]_{K_1} = 0$ which is a contradiction) and it can't be lifted to a partial isometry either, because there aren't any non-zero projections available to be the defect projections of a partial isometry. The cone and suspension of the C^*-algebra of compact operators on a Hilbert space, which are denoted by CK and SK respectively, are stably projectionless but their K_0-groups are trivial. But we can take I to be an algebra which has one of these as an ideal of codimension one.(i.e., I is an extension of one of these by one dimensional algebra.) To obtain a suitable extension, we need to have a projection $\mathbf{p}$ in the corona algebra which does not lift but its K_0-class does lift. Then we consider the C^*-algebra generated by $\mathbf{p}$ in the coronal algebra which is isomorphic to $\mathbb{C}$ and take I as the C^*-algebra generated by a lifting of $\mathbf{p}$ in the multiplier algebra of one of these. By showing the boundary map $\partial_0 : K_0(\mathbb{C}) \to K_1(E)$ where E is either CK or SK is trivial in the six-term exact sequence of K-theory, we can deduce that $K_0(I)$ is non-trivial. Thus this leads us to the question of lifting projections from the corona algebra of one of these examples and the question of lifting K_0-class of a projection in the coronal algebra.

This thesis is arranged as follows:

In chapter 2, we review some rudiments of continuous fields of Hilbert spaces and prove theorems and propositions which are crucial ingredients for the lifting problem of projection in the corona algebra and other equivalence relations as well.

In chapter 3, we define the notion of *essential codimension* and derive some facts using some techniques of Hilbert space decomposition. (In fact, the definition of essential codimension was given in [5] and some properties were provided without proofs in [2]. Here we give more precise definition and prove the properties carefully.)

In chapter 4, we give a necessary and sufficient condition for liftability of a projection in the corona algebra of $I = C(X) \otimes \mathcal{K}$ where X is $[0,1]$, $(-\infty, \infty)$, $[0, \infty)$, or $\mathbf{T}$.

In chapter 5, we give criterions for homotopy equivalence, unitary equivalence, and Murray-von Neumann equivalence of two projections $\mathbf{p}, \mathbf{q}$ in $\mathcal{C}(I) \otimes M_n$ for some n. In addition, we clarify the condition for the liftability of K_0-class of a projection; i.e., we clarify when it becomes trivial in K_0-group. Thus we give an example of a projection which is not liftable but its K_0-class is liftable. Also, we construct examples such that $[p]_0 = [q]_0$ in K_0 but $p \nsim q$, $p \sim q$ but $p \nsim_u q$, and $p \sim_u q$ but $p \nsim_h q$.

In chapter 6, we discuss some lifting problems which were our original interests through various affirmative and negative examples including a construction involving a projection we did in chapter 5.

2. Notations and Preliminaries

Throughout this thesis, H will denote a separable infinite dimensional Hilbert space, and $B(H)$ will denote the C^*-algebra of bounded operators on H. Furthermore, $K(H)$(shortly K without confusion) will be the C^*-algebra of compact operators on H. Given a C^*-algebra A, we denote by A^{**} the Banach space double dual of A which is also the enveloping von Neumann algebra of A. We call an element x in A^{**} a multiplier of A if xa, $ax \in A$ for all $a \in A$. The set of multipliers of A forms a C^*-algebra denoted by $M(A)$, which is called the multiplier algebra of A. From the definition A becomes an essential ideal of $M(A)$. Thus we can consider a C^*-algebra $\mathcal{C}(A) = M(A)/A$ which is called the corona algebra of A. In addition, we say A has stable rank one if the set of invertible elements of A(the unitization $\widetilde{A}$, if A is non-unital) is dense in A.

In this chapter we review definitions and some facts of continuous fields of Hilbert spaces. We also prove a technical result which will be used later.

Given a topological space T, by a continuous field $\mathcal{H}$ of Hilbert spaces over T we mean a family $(H_t)_{t\in T}$ of Hilbert spaces, together with a family Γ of vector fields on T (that is, elements of $\prod_{t\in T} H_t$) satisfying the following properties.

(i) Γ is a complex linear subspace of $\prod_{t\in T} H_t$

(ii) For every $t \in T$, the set of $x(t)$ for $x \in \Gamma$ is dense in H_t

(iii) For every $x \in \Gamma$ the function $t \to \|x(t)\|$ is continuous

(iv) If x is a vector field, and for each $t \in T$ and each $\epsilon > 0$ there is a an $x^{'} \in \Gamma$ such that $\|x - x^{'}\| < \epsilon$ on some neighborhood of t, then $x \in \Gamma$

The elements of Γ are called the *continuous* vector fields(or sections) of $\mathcal{H}$. If H_t is the same Hilbert space H for every t, and Γ consists of all continuous mappings of T into H, $\mathcal{H}$ is called a constant field. A field (isometrically) isomorphic to a constant field is said to be trivial. If $\mathcal{H}' = ((H_t')_{t\in T}, \Gamma')$ is a continuous filed of Hilbert spaces over T, H_t' is a closed subspace of H_t for each t, and $\Gamma' \subset \Gamma$, then $\mathcal{H}'$ is called a subfield of $\mathcal{H}$. Furthermore, we call $\mathcal{H}' = ((H_t')_{t\in T}, \Gamma')$ a complemented subfield of $\mathcal{H} = ((H_t)_{t\in T}, \Gamma)$ if there is a subfield $\mathcal{H}'' = ((H_t'')_{t\in T}, \Gamma'')$ such that $H_t' = H_t^{\perp}$ for every t. Also, we say that $\mathcal{H}$ is separable if Γ has a countable subset Λ such that $\{x(t) \mid x \in \Lambda\}$ is dense in H_t for each t.

Given a separable Hilbert space H, there is one to one correspondence between a complemented subfield $\mathcal{H}$ the constant filed over T defined by H and strongly continuous projection valued functions $p : T \mapsto B(H)$. Here H_t is the range of $p(t)$ (See P252-253 in [10].) Thus two continuous fields of Hilbert spaces defined by p and p' are isomorphic if and only if there is a double strongly continuous valued function u on T such that $uu^* = p'$ and $u^*u = p$.

Theorem 2.1. *Let T be paracompact space and $\mathcal{H}$ be a separable continuous field of Hilbert spaces over T. Then it is isomorphic to a complemented subfield of a trivial field, and thus is isomorphic to a continuous field defined by a strongly continuous projection valued function $p : T \mapsto B(H)$.*

Proof. If $\mathcal{H}_\infty$ is a trivial filed of infinite rank, then $\mathcal{H} \oplus \mathcal{H}_\infty \cong \mathcal{H}_\infty$ by Theorem 4 in [10]. ■

We are now going to see isomorphism between continuous fields of Hilbert spaces is sometimes automatically satisfied.

Given any continuous field $\mathcal{H} = ((H_t)_{t\in T}, \Gamma)$ and any closed subset A of T, we can get a continuous subfield $\mathcal{H}^0$ such that

$$H_t^0 = \begin{cases} H_t, t \notin A \\ 0,\ t \in A \end{cases}$$

Moreover, if $\mathcal{H}$ is separable and T is separable metric, then $\mathcal{H}^0$ is also separable. In fact, if f is a section such that $f(t) = 0$ for $t \in A$, for every $t \in T$, and $\epsilon > 0$, there is a continuous section g of $\mathcal{H}$ such that $\|f - g\| < \epsilon$ for some neighborhood O of t. If we choose ϕ such that $\phi|_A = 0$ and $\|\phi g - g\| < \epsilon$ on O, it follows that $\|f - \phi g\| < 2\epsilon$. Hence $\{\phi g \mid g \in \Gamma, \quad \phi : T \mapsto \mathbb{C} \text{ such that } \phi|_A = 0\}$ define continuous vector fields of $\mathcal{H}^0$.

The following Lemma 2.2 and Proposition 2.3 which was shown to the author by Larry Brown may be known by experts, but we shall give our proofs of these for the convenience of readers.

Lemma 2.2. *If X is a separable metric space such that $\dim X \leq 1$, if $\mathcal{H}$ is a continuous field of Hilbert spaces over X such that $H_x \neq 0$ for every $x \in X$, if f is a continuous vector field of $\mathcal{H}$, and if $\epsilon > 0$, then there is a continuous vector field g such that $\|g(x) - f(x)\| < \epsilon$ and $g(x) \neq 0$ for every $x \in X$.*

Proof. There is a countable open cover $\{U_n\}$ of X such that each $\mathcal{H}|_{U_n}$ has a non-vanishing section. By paracompactness of X there is a closed, locally finite refinement, $\{F_n\}$, for $\{U_n\}$. By construction, for each n we have

$$\mathcal{H}|_{F_n} = \mathcal{L}_n \oplus \mathcal{K}_n \tag{2.1}$$

where $\mathcal{L}_n$ is a trivial subfield of rank one. Choose a strictly increasing sequence $\{\epsilon_n\}$ of positive numbers such that $\epsilon_n < \epsilon$. We will recursively construct compatible sections g_n over F_n such that $g_n(x) \neq 0$ for every $x \in X$ and $\|g_n(x) - f(x)\| \leq \epsilon_n$. The local finiteness ensures that the resulting global section g is continuous on X.
To construct g_n, let $A = F_n \cap (\bigcup_{k=1}^{n-1} F_k)(A = \varnothing$ if $n = 1)$. We have a non-vanishing section g' on A such that $\|g'(x) - f(x)\| < \epsilon_{n-1}$ for every $x \in A$, and we wish to extend g' to F_n. Choose ϵ' and ϵ'' so that $\epsilon_{n-1} < \epsilon' < \epsilon'' < \epsilon_n$, and write $f = f^1 \oplus f^2$, $g' = g^1 \oplus g^2$ relative to the decomposition (2.1)
We first extend $g^2$to all of F_n so that $\|g_2(x) - f^2(x)\| \leq \epsilon'$ for every $x \in X$. To do this, let h be an arbitrary extension of g^2 to F_n, which exists by Proposition 7 in [10]. Then let $B = \{x \in F_n \mid \|h(x) - f^2\| \geq \epsilon'\}$, and note that $A \cap B = \varnothing$. Let

$\phi : F_n \mapsto [0,1]$ be a continuous function such that $\phi|_B = 1$ and $\phi|_A = 0$, and take $g^2 = \phi f^2 + (1-\phi)h$.

Next, we extend g^1 to a section k on all of F_n so that $\|k(x) - f^1(x)\|^2 + \|g^2(x) - f^2(x)\|^2 \le (\epsilon^{''})^2$. It will be convenient to identify sections of $\mathcal{L}_n$ with complex valued functions and define

$$\phi(s,z) = \begin{cases} z, & \text{if } |z| \le s \\ s\frac{z}{|z|}, & \text{if } |z| > s \end{cases}$$

for $s > 0$ and $z \in \mathbb{C}$. Thus ϕ is continuous on $(0,\infty) \times \mathbb{C}$. Now extend the function $g^1 - f^1$ to l on F_n and let $k(x) = f^1(x) + \phi(\sigma(x), l(x))$, where $\sigma(x) = ((\epsilon^{''})^2 - \|g^2(x) - f^2(x)\|^2)^{1/2}$.

Finally, we must modify k to obtain the non-vanishing property without changing $k|_A$. Let $C = x \in F_n \mid g^2(x) = 0$, $D = x \in C \mid k(x) = 0$, and $\delta = \frac{\epsilon_n - \epsilon^{''}}{2}$. Since $D \cap A = \varnothing$, there is an open neighborhood V of D such that $\overline{V} \cap A = \varnothing$. Let $G = \overline{V} \cap C$ and $E = ((\overline{V} - V) \cap C) \cup \{x \in G \mid \|k(x)\| = \delta\}$ By dimension theory, there is a non-vanishing continuous function r on G such that $r|_E = k|_E$. Then define $g^1|_C(x)$ by

$$g^1(x) = \begin{cases} \phi(\delta, r(x)) & \text{if } x \in V \cap C \text{ and } \|k(x)\| < \delta \\ k(x) & \text{otherwise on } C \end{cases}$$

Note that if $x_n \in V \cap C$, $\|k(x_n)\| < \delta$ and if $x_n \to x$ for some x not satisfying these conditions, then $r(x) = k(x)$ and $\|k(x)\| = \delta$ Thus

$$g^1(x_n) \to \phi(\delta, r(x)) = k(x)$$

This implies g^1 is continuous on C and $\|g^1 - k\| \le 2\delta$.

Now g^1 is defined on $C \cup A$, and we extend g^1 to F_n so that $\|g^1(x) - k(x)\| < 2\delta$ for every $x \in F_n$. This can be done as in the previous paragraph. Then $g_n = g^1 \oplus g^2$ satisfies all required properties. ■

Proposition 2.3. *If X is a separable metric space such that $\dim X \le 1$, and if $\mathcal{H}$ and $\mathcal{K}$ are separable continuous fields of Hilbert spaces over X such that $\dim H_x = \dim K_x$ for every $x \in X$, then $\mathcal{H} \cong \mathcal{K}$.*

Proof. For $n = 1, \cdots$, let $U_n = \{x \in X \mid \dim H_x \geq n\}$, an open set. Let $\mathcal{L}_n$ be a trivial line bundle over U_n, extended by zero as as to be a continuous field over X. Thus continuous section of $\mathcal{L}_n$ can be identified with continuous complex-valued functions on X which vanished on $X \setminus U_n$. We will show that $\mathcal{H} \equiv \bigoplus_1^\infty \mathcal{L}_n$. Since the same argument applies to $\mathcal{K}$, the result follows.

To do this we construct recursively a sequence $\{e_n\}$ such that e_n is a continuous section of $\mathcal{H}|_{U_n}$, such that

$$\|e_n(x)\| = 1 \quad \text{for every } x \in U_n, < e_n(x), e_m(x) >= 0 \quad \text{if } n < m \text{ and } x \in U_m.$$

We will impose additional conditions on the e_n's, but first we point out that any such e_n's give rise to complemented subfield $\mathcal{M}_n$, where $(\mathcal{M}_n)_x = \text{span}(e_n(x))$. Moreover if we write $\mathcal{H} = \mathcal{H}_n^{'} \oplus \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n$, then $\dim(H_n^{'})_x = \max(\dim H_x - n, 0)$. It is enough to consider the case $n = 1$. Note that for any continuous section f of $\mathcal{H}$, we have the function c_1 defined by

$$c_1(x) = \begin{cases} < f(x), e_1(x) > & , \quad x \in U_1 \\ 0 & , \quad x \notin U_1 \end{cases}$$

is continuous, since $| < f(x), e_1(x) > | \leq \|f(x)\|$ and f is vanishes on $X - U_1$. Thus $g = f - c_1 e_1$ may be regarded as a continuous section of $\mathcal{H}$ such that $\|g(x)\|^2 = \|f(x)\|^2 - \|c_1(x)\|^2$, this implies that $\mathcal{H}_1^{'} = \mathcal{M}_1^{\perp}$ is indeed a continuous subfield of $\mathcal{H}$. The dimension formula given above is obvious, and it follows that $\{x \mid \dim(\mathcal{H}_1^{'})_x \geq n\} = U_{n+1}$. Now we proceed by induction on n, working with $\mathcal{H}_1^{'}$ instead of $\mathcal{H}$.

Let $\{f_m\}$ be a sequence of continuous sections of $\mathcal{H}$ such that $\mathcal{H}_x = \overline{\text{span}(f_n(x))}$ for each x. Let $\{g_n\}$ be a sequence which includes each f_m infinitely many times. We choose the e_n's so that for each n and x, the projection of $g_n(x)$ on $(\mathcal{H}_n^{'})_x$ has norm at most $1/n$. If this is so, then $\mathcal{H} \cong \bigoplus \mathcal{M}_n$; and since $\mathcal{M}_n \cong \mathcal{L}_n$, the result follows.

Assume e_k has already been constructed for $k < n$. Let h be the $\mathcal{H}_{n-1}^{'}$ component of g_n. Apply Lemma 2.2 to $\mathcal{H}_{n-1}^{'}|_{U_n}$ and $h|_{U_n}$ with $\epsilon = 1/n$.(Recall that $U_n = \{x \mid$

$(H'_{n-1})_x \neq 0\}$.) Thus we obtain a non-vanishing section l on U_n, such that $\|h(x) - l(x)\| \leq 1/n$ for every x. If $e_n(x) = \dfrac{l(x)}{\|l(x)\|}$, then e_n satisfies all our requirements. ■

Corollary 2.4. *If X is a topological space such that* $\dim(X) \leq 1$ *and $\mathcal{H}$ is a continuous field of Hilbert spaces over X such that* $\dim(H_x) \geq n$ *for every $x \in X$, then $\mathcal{H}$ has a trivial subfield of rank n. Equivalently, if p is a strongly continuous projection valued function on X such that* $\operatorname{rank}(p(x)) \geq n$*, then there is a norm continuous projection valued function q such that $q \leq p$ and* $\operatorname{rank}(q(x)) = n$.

Proof. Suppose $\dim H_x \leq n$. Let $\mathcal{H}_0$ be the trivial field of rank n and $\mathcal{K} = ((K_x)_{x\in X}, \Gamma)$ be a continuous field of Hilbert spaces such that $\dim K_x = \dim H_x - n$: Let $m(x) = \dim H_x - n$. Since m is lower semi-continuous, for each k $O_k = \{x \mid m(x) \geq k\}$ is open so that we can construct a continuous field $\mathcal{H}_k$ of Hilbert spaces such that

$$\dim H_x^k = \begin{cases} 1 & \text{if } x \in O_k \\ 0 & \text{otherwise} \end{cases}$$

Now take $\mathcal{K}$ as $\bigoplus_k \mathcal{H}_k$.
If we let $\mathcal{H}' = \mathcal{H}_0 \oplus \mathcal{K}$, $\dim H'_x = \dim H_x$ from the above. Thus the conclusion follows from Proposition 2.3. ■

Corollary 2.5. *If X is a topological space such that* $\dim(T) \leq 1$ *and* $\dim(H_x) > 0$ *for every $x \in X$, then there is a vector field v such that $v(t) \neq 0$ for every t.*

One of remarkable results in [10] is the following triviality theorem of continuous fields of Hilbert spaces which we shall use later.

Theorem 2.6. *If X is paracompact and of finite dimension, every separable continuous field $\mathcal{H} = ((H_x)_{x\in X}, \Gamma)$ of Hilbert spaces over X such that* $\dim(H_x) = \infty$ *for every x is trivial. Thus two continuous fields $\mathcal{H}$, $\mathcal{H}'$ of Hilbert spaces over X such that* $\dim H_x = \dim H'_x = \infty$ *are isomorphic.*

Proof. See Lemma 10.8.7 in [9]. ■

The following facts are useful and well-known. But we include proofs for the completion.

Proposition 2.7. *Given a closed interval X and a point a in X, there is a strongly continuous projection valued function p on X such that*

$$\operatorname{rank}(p(x)) = \begin{cases} 1 & \textit{if } x \neq a, \\ 0 & \textit{if } x = a \end{cases}$$

Proof. Since there is a separable continuous field $\mathcal{H}$ such that

$$\dim(H_x) = \begin{cases} 1 & \text{if } x \neq a, \\ 0 & \text{if } x = a \end{cases}$$

, it follows from Theorem 2.1. ■

Remark 2.8. In fact, we can prove above proposition using elementary arguments. Without loss of generality, we can assume $X = [0,1]$, $a = 0$. Since any separable Hilbert space is isomorphic to $L^2[0,1]$, it is enough to construct such a function in $L^2[0,1]$. We view a characteristic function corresponding to $[0,t]$ as a vector v_t. As t goes to 0, the functional defined by v_t goes to 0 weakly in $L^2[0,1]$ by Lebesgue Dominated Convergence Theorem. (In fact, the map $t \mapsto < v_t, \bullet >$ is continuous with respect to weak topology.) Now it is easy to check the map $t \mapsto$ rank one projection θ_{v_t, v_t} is strongly continuous projection valued function p such that $p(0) = 0$.

Corollary 2.9. *Given a closed interval X and a point a in X, there is a strongly continuous projection valued function p such that*

$$\operatorname{rank}(p(x)) = \begin{cases} \infty & \textit{if } x \neq a, \\ 0 & \textit{if } x = a \end{cases}$$

Proof. Let p_i be the function on X such that

$$\operatorname{rank}(p_i(x)) = \begin{cases} 1 & \text{if } x \neq a, \\ 0 & \text{if } x = a \end{cases}$$

for each i. Then the map $x \mapsto \bigoplus_i p_i(x) \in \bigoplus_i L^2[0,1]$ is the map in the required sense. ■

3. Essential codimension

Definition 3.1. (Brown, Douglas, and Fillmore) When p, q are projections in $B(H)$ such that $p-q \in K$, we define the **essential codimension** of p and q which will be denoted as $[p:q]$. Let V, W be isometries such that $VV^* = q, WW^* = p$ if p and q are infinite rank. Then

$$[p:q] = \begin{cases} \operatorname{Ind}(V^*W) & \text{if } p, q \text{ have infinite rank,} \\ \operatorname{rank}(p) - \operatorname{rank}(q) & \text{if } p, q \text{ have finite rank} \end{cases}$$

In this definition, when p, q have infinite rank, $[p:q]$ is independent of choice of V and W. In fact, if we have isometries V_1, V_2 such that $V_1V_1^* = q$ and $V_2V_2^* = q$, then it is easy to check that $U = V_2^*V_1$ is a unitary, $UV_1^*W = V_2^*W$. It follows that $\dim\ker V_1^*W = \dim\ker V_2^*W$ and $\dim\operatorname{coker} V_1^*W = \dim\operatorname{coker} V_2^*W$, and therefore $\operatorname{Ind}(V_1^*W) = \operatorname{Ind}(V_2^*W)$. The other case is proved similarly.

Proposition 3.2. $[\,:\,]$ *has the following properties.*

(1) If $p_2 \leq p_1$, then $[p_1 : p_2]$ is the usual codimension of p_2 in p_1 which is $\operatorname{rank}(p_1 - p_2)$*;*

(2) $[p_1 : p_2] = -[p_2 : p_1]$*;*

(3) $[p_1 : p_3] = [p_1 : p_2] + [p_2 : p_3]$*;*

(4) $[p_1 + p_1' : p_2 + p_2'] = [p_1 : p_2] + [p_1' : p_2']$*, when sensible.*

Proof. For (1), let V_i be the isometries corresponding to p_i for $i = 1, 2$. Then $V_2^*V_1$ is a co-isometry because $p_2p_1 = p_2$. Hence $\operatorname{Ind}(V_2^*V_1) = \dim\ker(V_2^*V_1) = \operatorname{rank}(1 - (V_2^*V_1)^*V_2^*V_1) = \operatorname{Tr}(V_1^*(p_1 - p_2)V_1) = \operatorname{Tr}(p_1 - p_2) = \operatorname{rank}(p_1 - p_2)$ since $p_1 - p_2 \in \mathcal{K}$.

(2) is evident from the definition.

For (3), if p_i's have finite rank, it is easy. If p_i's have infinite rank, we take the corresponding isometries such that $V_iV_i^* = p_i$. Then $V_3^*V_2V_2^*V_1 - V_3^*V_1 \in \mathcal{K}$, and therefore $\mathrm{Ind}(V_3^*V_1) = \mathrm{Ind}(V_3^*V_2V_2^*V_1) = \mathrm{Ind}(V_3^*V_2) + \mathrm{Ind}(V_2^*V_1)$.

Finally, note that $p_i + p_i^{'}$ is a projection if and only if p_i and $p_i^{'}$ have orthogonal ranges or $p_ip_i^{'} = 0$. If both p_i and $p_i^{'}$ have finite rank, then

$$\mathrm{rank}(p_i + p_i^{'}) = \mathrm{rank}(p_i) + \mathrm{rank}(p_i^{'}) \quad \text{for} \quad i = 1, 2$$

It follows that

$$\begin{aligned}[p_1 + p_1^{'} : p_2 + p_2^{'}] &= \mathrm{rank}(p_1 + p_1^{'}) - \mathrm{rank}(p_2 + p_2^{'}) \\ &= \mathrm{rank}(p_1) + \mathrm{rank}(p_1^{'}) - (\mathrm{rank}(p_2) + \mathrm{rank}(p_2^{'})) \\ &= \mathrm{rank}(p_1) - \mathrm{rank}(p_2) + \mathrm{rank}(p_1^{'}) - \mathrm{rank}(p_2^{'}) \\ &= [p_1 : p_2] + [p_1^{'} : p_2^{'}]\end{aligned}$$

If p_is have infinite rank and $p_i^{'}$s have finite rank, then $p_i + p_i^{'} - p_i \in K$ for $i = 1, 2$.

$$\begin{aligned}[p_1 + p_1^{'} : p_2 + p_2^{'}] &= [p_1 + p_1^{'} : p_1] + [p_1 : p_2] + [p_2 : p_2 + p_2^{'}] \\ &= \mathrm{rank}(p_1^{'}) + [p_1 : p_2] - \mathrm{rank}(p_2^{'}) \\ &= [p_1 : p_2] + [p_1^{'} : p_2^{'}]\end{aligned}$$

If both p_is and $p_i^{'}$s have infinite rank, note that $p_1^{'}p_2, p_2^{'}p_1 \in K$. If we let $U : H \to H \oplus H$ be a unitary element, then $V = U^*(V_1 \oplus V_1^{'})U$ and $W = U^*(V_2 \oplus V_2^{'})U$ be the isometries corresponding to $p_1 + p_1^{'}$ and $p_2 + p_2^{'}$ respectively where V_i and $V_i^{'}$ are isometries such that $V_iV_i^* = p_i$ and $V_i^{'}V_i^{'*} = p_i^{'}$ for $i = 1, 2$.

$$\begin{aligned}[p_1 + p_1^{'} : p_2 + p_2^{'}] &= \mathrm{Ind}(V^*W) \\ &= \mathrm{Ind}(U^*(V_1^*V_2 \oplus (V_1^{'})^*V_2^{'})U) \\ &= \mathrm{Ind}(V_1^*V_2 \oplus (V_1^{'})^*V_2^{'}) \\ &= \mathrm{Ind}(V_1^*V_2) + \mathrm{Ind}((V_1^{'})^*V_2^{'}) \\ &= [p_1 : p_2] + [p_1^{'} : p_2^{'}]\end{aligned}$$

■

Lemma 3.3. *Let p and q be projections in $B(H)$ such that $p - q \in K$. If there is a unitary $U \in 1+K$ such that $UpU^* = q$, then $[p : q] = 0$. In particular, if $\|p-q\| < 1$, then $[p : q] = 0$.*

Proof. If p and q are finite dimensional, $rank(p) = rank(UpU^*) = rank(q)$. Now assume p and q are infinite dimensional and let W be the isometry such that $WW^* = p$. If there is a unitary $U \in 1 + K$ such that $UpU^* = q$, then it is easily checked that $V = UW$ is an isometry such that $VV^* = q$. Therefore,

$$\begin{aligned}
[p : q] &= \mathrm{Ind}(V^*W) \\
&= \mathrm{Ind}(W^*UW) \\
&= \mathrm{Ind}(W^*W + \text{compact}) \\
&= \mathrm{Ind}(I) = 0.
\end{aligned}$$

Now if $\|p - q\| < 1$, we can take $a = (1 - q)(1 - p) + qp \in 1 + K$. Since $aa^* = a^*a = 1 - (p - q)^2 \in 1 + K$,

$$\|a^*a - 1\| = \|p - q\|^2 < 1, \quad \|aa^* - 1\| = \|p - q\|^2 < 1.$$

Moreover, it follows that

$$ap = qp = qa.$$

Hence a is invertible element and $U = a(a^*a)^{-\frac{1}{2}} \in 1 + K$ is a unitary such that $UpU^* = q$. ■

Since $p - q$ is a self adjoint compact operator, we can diagonalize $p - q$ by the Spectral theorem. From this point of view, we have a new characterization of $[p : q]$. Let $E_1(p - q)$ be the eigenspace $\ker(p - q - 1)$ and $E_{-1}(p - q)$ be the eigenspace $\ker(p - q + 1)$.

Proposition 3.4.

$$[p : q] = \dim E_1(p - q) - \dim E_{-1}(p - q)$$

Proof. Let M and N be the range of p and q respectively, and let $H_{11} = M \cap N$, $H_{10} = M \cap N^\perp$, $H_{01} = M^\perp \cap N$, $H_{00} = M^\perp \cap N^\perp$ and $H_0 = (H_{00} \oplus H_{10} \oplus H_{01} \oplus H_{11})^\perp$. It is possible to identify both $H_0 \cap M$ and $H_0 \cap M^\perp$ with $L^2(X)$ for some measure space X in such a way that the p on the H_0 which is denoted by p_0 and the q on the H_0 which is denoted by q_0 are given by

$$p_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad q_0 = \begin{pmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{pmatrix}$$

where ϕ is a measurable function on X such that $0 < \phi(x) < \dfrac{\pi}{2}$ for $x \in X$. Here p_0 and q_0 operate on $L^2(X) \oplus L^2(X)$ and the matrices are operator matrices whose entries are multiplication operators. (See p280 in [1].)

If we denote p on the H_{11} by p_{11} and p on the H_{10} by p_{10}, then $p = p_{11} + p_{10} + p_0$. Similarly, $q = q_{11} + q_{01} + q_0$. Now $p - q = p_{10} - q_{01} + p_0 - q_0 \in K$ implies that $p_0 - q_0 \in K(H_0)$ and $p_{10} - q_{01} \in K(H_{10} \oplus H_{01})$. Then

$$p_0 - q_0 = \begin{pmatrix} \sin^2\phi & -\cos\phi\sin\phi \\ -\cos\phi\sin\phi & -\sin^2\phi \end{pmatrix} \in K(H_0)$$

implies that X is a discrete space $\{x_n\}$ and $\phi(x_n) \to 0 \quad \text{as} \quad n \to \infty$. Therefore

$$U = \bigoplus_{n=1} \begin{pmatrix} \cos\phi(x_n) & \sin\phi(x_n) \\ -\sin\phi(x_n) & \cos\phi(x_n) \end{pmatrix} \in 1 + K(H_0)$$

From

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{pmatrix},$$

it follows that $U^* p_0 U = q_0$, and $[p_0 : q_0] = 0$ by the Lemma 3.3. On the other hand, $p_{10} - q_{01} \in K(H_{10} \oplus H_{01})$ means $\mathrm{rank}(p_{10})$ and $\mathrm{rank}(q_{01})$ are finite, and therefore

$$\begin{aligned} [p_{10} : q_{01}] &= \mathrm{rank}(p_{10}) - \mathrm{rank}(q_{01}) \\ &= \dim(E_1(p-q)) - \dim(E_{-1}(p-q)) \end{aligned}$$

Note that $[p:q] = [p_0 : q_0] + [p_{10} : q_{01}] + [p_{11} : q_{11}]$. Consequently,

$$\begin{aligned}[p:q] &= [p_{10} : q_{01}] \\ &= \operatorname{rank}(p_{10}) - \operatorname{rank}(q_{01}) \\ &= \dim(E_1(p-q)) - \dim(E_{-1}(p-q))\end{aligned}$$

■

Remark 3.5. (i) In fact, since $\dim(E_1(p-q)) = \#$ of 1 in the diagonalization of $p-q$ and $\dim(E_{-1}(p-q)) = \#$ of -1 in the diagonalization of $p-q$, $[p:q] =$ # of 1 in the diagonalization of $(p-q)$ - # of -1 in the diagonalization of $(p-q)$.

(ii) The other non-zero points in the spectrum of $p-q$ come from $p_0 - q_0$. They are $\sin\phi(x_n)$ and $-\sin\phi(x_n)$. Note that this part of the spectrum is symmetric about 0 (even considering multiplicity); i.e., $\dim\ker(p_0 - q_0 - \sin\phi) = \dim\ker(p_0 - q_0 + \sin\phi)$.

Corollary 3.6. *Suppose projections $p_t, q_t \in B(H)$ are defined for each t. Then $[p_t : q_t]$ is constant if the map $t \mapsto p_t - q_t$ is norm continuous.*

Proof. In fact, we are going to prove that there is a $\delta > 0$ such that whenever $A_1 = p_1 - q_1, A_2 = p_2 - q_2$ satisfy $\|A_1 - A_2\| < \delta$, $[p_1 : q_1] = [p_2 : q_2]$.
Since A_1 is compact, $\sigma(A_1)$ is discrete. So we can take a neighborhood $U = O_{-1}(\epsilon) \cup (-1+\epsilon, 1-\epsilon) \cup O_1(\epsilon)$ where $O_{\pm 1}(\epsilon)$ are open balls with radius ϵ centered at ± 1 respectively. By the semicontinuity of spectrum, if we take δ small enough, we know $\sigma(A_2) \subset U$. If we let $\gamma_{\pm 1}$ be the circles within $O_{\pm 1}$ with the same radius r centered at ± 1, by Riesz functional calculus, we have projections $r_i = \frac{1}{2\pi i}\int_{\gamma_1}(z - A_i)^{-1}dz$ and $s_i = \frac{1}{2\pi i}\int_{\gamma_{-1}}(z - A_i)^{-1}dz$ for $i = 1, 2$. Furthermore, if we take δ small enough, we also have $\|r_1 - r_2\| < 1, \|s_1 - s_2\| < 1$. Then, by Lemma 3.3, $\operatorname{rank}(r_1) = \operatorname{rank}(r_2)$ and $\operatorname{rank}(s_1) = \operatorname{rank}(s_2)$. Note that $\operatorname{rank}(s_i) = \sum_{-1<\lambda<-1+r} \dim\ker(A_i - \lambda 1) +$

$\dim\ker(A_i+1)$. Similarly, $rank(r_i) = \sum_{1-r<\lambda<1} \dim\ker(A_i - \lambda 1) + \dim\ker(A_i - 1)$. Then, by the Remark 3.5, we have

$$\begin{aligned} rank(r_1) - rank(s_1) &= \dim\ker(A_1 - 1) - \dim\ker(A_1 + 1) \\ &= [p_1 : q_1] \\ &= \mathrm{rank}(r_2) - \mathrm{rank}(s_2) \\ &= \dim\ker(A_2 - 1) - \dim\ker(A_2 + 1) \\ &= [p_2 : q_2] \end{aligned}$$

■

Now we want to prove the most important property of the essential codimension.

Theorem 3.7. *Let p and q be projections in $B(H)$ such that $p - q \in K$. There is a unitary $U \in 1 + K$ such that $UpU^* = q$ if and only if $[p : q] = 0$.*

Proof. "Only if "part was proved in Lemma 3.3.
For "if", suppose that $[p : q] = 0$. Using the decomposition of $p - q$ as is shown in Proposition 3.4, we know that p_0 and q_0 are unitarily equivalent, say the unitary U_0. In addition, from $[p : q] = \mathrm{rank}(p_{10} - \mathrm{rank}(q_{01}) = 0$, there is a partial isometry W in $B(H_{10} \oplus H_{01})$ such that $W^*W = p_{10}$ and $WW^* = q_{01}$. Note that $Wp_{10}W^* = q_{01}$ and W is a unitary in $1 + K(H_{10} \oplus H_{01})$. Now we have a unitary $U = U_0 + W + 1_{H_{11}} + 1_{H_{00}} \in 1 + K$ such that $UpU^* = q$. ■

In the next section, we need the following facts.

Proposition 3.8. *Let p, q be projections such that $p - q \in K$.*

(1) If q has finite rank, then $[p : q] \geq -\mathrm{rank}(q)$.

(2) If $1 - q$ has finite rank, then $[p : q] \leq \mathrm{rank}(1 - q)$

Proof. Straightforward. ■

Finally we close this section by showing another characterization of the essential codimension in terms of the Kasparov group $KK(\mathbb{C},\mathbb{C})$. $M(A)$ will denote the multiplier algebra of a C^*-algebra A. Recall that s Cuntz pair (ϕ,ψ) is a pair of representations $A\to M(B\otimes K)$ such that $\phi(a)-\psi(a)\in B\otimes K$ for every $a\in A$. They form a set denoted by $\mathbb{E}_h(A,B)$. A homotopy of Cuntz pairs consists of a Cuntz pair $(\Phi,\Psi):A\to M(B[0,1]\otimes K)$, and the quotient of $\mathbb{E}_h(A,B)$ by this homotopy equivalence becomes the Kasparov group $KK(A,B)$(See [12] for more details).

Theorem 3.9. *There is a one to one correspondence between the elements of* $KK(\mathbb{C},\mathbb{C})$ *and* (p,q)*'s where* p,q *are projections in* $B(H)$ *such that* $p-q\in\mathcal{K}$*. The map* $(p,q)\mapsto[p:q]$ *induces (the standard) isomorphism from* $KK(\mathbb{C},\mathbb{C})$ *onto* $\mathbb{Z}$*.*

Proof. Note that a Cuntz pair in $\mathbb{E}_h(\mathbb{C},\mathbb{C})$ is a pair of representations $\mathbb{C}\to B(H)$ and this corresponds to the pair $(p,q)=(\phi(1),\psi(1))$.

If (ϕ_0,ψ_0) and (ϕ_1,ψ_1) are homotopic, there is a path (Φ_t,Ψ_t) in $\mathbb{E}_h(\mathbb{C},\mathbb{C}), t\in[0,1]$ such that $t\to\Phi_t(1)-\Psi_t(1)$ is norm continuous. Thus Corollary3.6 implies $[p_0:q_0]=[p_1:q_1]$.

Then the map $(p,q)\mapsto[p:q]$ is group homomorphism by Proposition 3.2-(4). Suppose $[p:q]=0$. Then there is a unitary $U\in 1+K$ such that $UpU^*=q$ by the Theorem 3.7. Note that, since $U\in 1+K$, U can be written as e^{iT} where T is a self-adjoint compact operator. Using $U_t=e^{itT}$ for $t\in[0,1]$, we have a path of unitaries connecting 1 and U. Therefore we can find a homotopy between (ϕ,ψ) and (ϕ,ϕ) which is a zero element in $KK(\mathbb{C},\mathbb{C})$.

The onto part is obvious. ■

Remark 3.10. Let E be a Hilbert B-module. If $\pi,\sigma:A\to\mathcal{L}(E)$ are representations, we say that π and σ are properly asymptotically unitarily equivalent and write $\pi\approxeq\sigma$ if there is a continuous path of unitaries $u:[0,\infty)\to\mathcal{U}(K(E)+\mathbb{C}1_E)$, $u=(u_t)_{t\in[0,\infty)}$ such that

- $\lim_{t\to\infty}\|u_t\pi(a)u_t^*-\sigma(a)\|=0$ for all $a\in A$
- $u_t\pi(a)u_t^*-\sigma(a)\in K(E)$, for all $t\in[0,\infty)$, and $a\in A$

The use of the word 'proper' reflects the crucial fact that all unitaries are of the form 'identity + compact'. A result of Dadarlat and Eilers shows that if $\phi, \psi : A \to M(B \otimes K)$ is a Cuntz pair, then $[\phi, \psi]$ vanishes in $KK(A, B)$ if and only if $\phi \oplus \gamma \approxeq \psi \oplus \gamma$ for some representation $\gamma : A \to M(B \otimes K)$. As a corollary which is originally due to Voiculescu, they have shown if $\phi, \psi : A \to B(H)$ is a Cuntz pair of admissible representations(faithful, non-degenerate, and its image does not contain any non-trivial compacts), then $[\phi, \psi]$ vanishes in $KK(A, \mathbb{C})$ if and only if $\phi \approxeq \psi$. Note that we don't need the extra term γ. Theorem 3.7 deals with the case where also $A = \mathbb{C}$. But contrast to asymptotic version, we was able to find the unitary equivalence of a pair.

4. Lifting projections

Let X be $[0,1], [0,\infty)$, $(-\infty,\infty)$ or $\mathbb{T}=[0,1]/\{0,1\}$ and let $I=C(X)\otimes K$. Then $M(I)$ is given by $C_b(X,B(H))$ which is the set of bounded functions from X to $B(H)$ where $B(H)$ is given the double-strong topology. Let $\mathcal{C}(I)=M(I)/I$ be the corona algebra of I and also let $\pi : M(I)\to\mathcal{C}(I)$ be the natural quotient map. Then an element $\mathbf{f}$ of the corona algebra can be represented as follows: Consider a finite partition of X or $X\setminus\{0,1\}$ where $X=\mathbb{T}$ given by partition points $x_1<x_2<\cdots<x_n$ all of which are in the interior of X and divide X into $n+1$ (closed)subintervals $X_0,X_1,\cdots,X_n$. We can take $f_i\in C_b(X_i,B(H))$ such that $f_i(x_i)-f_{i-1}(x_i)\in K$ for $i=1,2,\cdots,n$ and $f_0(x_0)-f_n(x_0)\in K$ where $x_0=0=1$ if X is $\mathbb{T}$. The coset in $\mathcal{C}(I)$ represented by $(f_0,\cdots,f_n)$ consists of function f in $M(I)$ such that $f-f_i\in C(X_i,K)$ for every i and $f-f_i$ vanishes (in norm) at any infinite end point of X_i. Then $(f_0,\cdots,f_n)$ and $(g_0,\cdots,g_n)$ define the same element of $\mathcal{C}(A)$ if and only if $f_i-g_i\in C_b(X_i,K)$ for $i=0,\cdots,n$ and $f_0-g_0\in C_0((-\infty,x_1],K)$ $f_n-g_n\in C_0([x_n,\infty),K)$ if applicable.

We proceed to show that a projection in $\mathcal{C}(I)$ is locally liftable to a projection in $M(I)$.

Lemma 4.1. *(Calkin) Let T be a self-adjoint element of $B(H)$ such that $T-T^2\in K$ with $\|T\|\leq 1$. Then $\sigma(T)\subset[0,a)\cup(b,1]$ for some $a<b$ and for any such a, b, if we let $E_1(T)$ be the spectral projection of T corresponding to $(\frac{a+b}{2},1]$, then $E_1(T)-T\in K$.*

Proof. This is well-known(See [8].)

Theorem 4.2. *If $\mathbf{f}$ is a projection in $\mathcal{C}(I)$, we can find an $(f_0,\cdots,f_n)$ as above such that each f_i is projection valued.*

Proof. Let f be the element of $M(I)$ such that $\pi(f) = \mathbf{f}$. Without loss of generality, we can assume f is self-adjoint and $0 \leq f \leq 1$.

(i) Suppose X does not contain any infinite point. Choose a point $t \in X$ and let $T = f(t)$. Note that $T - T^2 \in K$. Since $r : t \to f(t) - f(t)^2$ is norm continuous. if we pick a point z in $\left(0, \frac{1}{4}\right)$ such that $z \notin \sigma(r(T))$, then $\sigma(f(s))$ omits $r^{-1}(J)$ for s sufficiently close to t where J is an interval containing z. In other words, there is a $\delta > 0$ and $b > a > 0$ such that if $|t - s| < \delta$, then $\sigma(f(s)) \subset [0, a) \cup (b, 1]$.
If we let $f_t(s) = E_1(f(s))$ for s in $(t - \delta, t + \delta)$, then it is continuous projection valued such that $f_t(s) - f(s) \in K$ by the Lemma 4.1.
Since X is compact, we can find $n + 1$ points $t_0, \cdots, t_n$ and open interval of t_i denoted by O_i such that $O_i \cap O_{i-1} \neq \varnothing$ and $X = \bigcup_i O_i$. Now let $f_i = f_{t_i}$ as above. Take the point $x_i \in O_{i-1} \cap O_i$ for $i = 1, \cdots, n$. Then $f_i(x_i) - f_{i-1}(x_i) = f_i(x_i) - f(x_i) + f(x_i) - f_{i-1}(x_i) \in K$ and $f_0(x_0) - f_n(x_0) \in K$ if applicable. Let $X_i = [x_i, x_{i+1}]$ for $i = 1, \cdots, n-1$, $X_0 = [0, x_1]$, and $X_n = [x_n, 1]$. Since each f_i is also defined on X_i, $(f_0, \cdots, f_n)$ is what we want.

(ii) let X be $[0, \infty)$. Since $f^2(t) - f(t) \to 0$ as t goes to ∞, for given δ in $(0, 1/2)$, there is $M > 0$ such that whenever $t \geq M$ then $\|f^2(t) - f(t)\| < \delta - \delta^2$. It follows that $\sigma(f(t)) \subset [0, \delta) \cup (1 - \delta, 1]$ for $t \geq M$. Then again $E_1(f(t))$ is a continuous projection valued function for $t \geq M$ such that $f(t) - E_1(f(t))$ vanishes as t goes to ∞. . By applying the argument in (i) to $[0, M]$, we get a closed subintervals X_i for $i = 0, \cdots, n-1$ of $[0, M]$ and $f_i \in C_b(X_i, B(H))$. Now if we let $X_n = [M, \infty]$ and $f_n(t) = E_1(f(t))$, we are done.

(iii) The case $X = (-\infty, \infty)$ is similar to (ii).

■

Given a representation $(f_0, \cdots, f_n)$ of $\mathbf{f}$ in $\mathcal{C}(I)$, we can associate integers $k_i = [f_i(x_i) : f_{i-1}(x_i)]$ for $i = 1, \cdots, n$ and $k_0 = [f_0(x_0) : f_n(x_0)]$ if applicable.

Proposition 4.3. *If all k_i are equal to 0, then* $\mathbf{f}$ *is liftable to a projection in* $M(I)$.

Proof. First we consider the case $X = [0,1], [0,\infty] or (-\infty,\infty)$. We recursively construct perturbations $f_i^{'}, \cdots f_n^{'}$ of $f_1, \cdots, f_n$ so that $f_i^{'}(x_i) = f_{i-1}^{'}(x_i)$ ($f_0^{'} = f_0$) and $f_n^{'}$ agrees with f_n in the some neighborhood of ∞ if the right end point of X is ∞.

Observe that, if $k_i = [f_i(x_i) : f_{i-1}(x_i)] = 0$, by Theorem 3.7, we can find a unitary $U \in 1+K$ which is path connected to 1 such that $Uf_i(x_i)U^* = f_{i-1}(x_i)$. If we denote the (norm continuous) path of unitaries by $U(t)$ such that $U(x_i) = U$, $U(x_{i+1}) = 1$ and let $f_i^{'}(t) = U(t)f_i(t)U(t)^*$ for each $t \in X_i$, then $f_i^{'} - f_i \in C_b(X_i, K)$ and $f_i^{'}$ agree at x_i with f_{i-1}. Furthermore, by Corollary 3.6 and Lemma 3.3

$$\begin{aligned}[f_{i+1}(x_{i+1}) : f_i^{'}(x_{i+1})] &= [f_{i+1}(x_{i+1}) : f_i(x_{i+1})] + [f_i(x_{i+1}) : f_i^{'}(x_{i+1})] \\ &= [f_{i+1}(x_{i+1}) : f_i(x_{i+1})] + [f_i(x_i) : f_i^{'}(x_i)] \\ &= [f_{i+1}(x_{i+1}) : f_i(x_{i+1})]\end{aligned}$$

Thus all k_i are equal to 0, we can perturb f_i's for $i = 1, \cdots, n-1$ inductively so that they will still be projection valued and will agree at $x_1, \cdots, x_{n-1}$. If the right end point of X is ∞, let V be the unitary such that $Vf_n(x_n)V^* = f_{n-1}^{'}(x_n)$, we apply the same process to perturb f_n with the path of unitaries defined by

$$U(t) = \begin{cases} V & \text{if } t = x_n \\ 1 & \text{if } t \geq x_n + M \text{ for some M large enough} \end{cases}$$

If $X = \mathbb{T}$, we can recursively construct $f_1^{'}, f_2^{'}, \cdots, f_n^{'}, f_0^{'}$ of $f_1, f_2, \cdots, f_n, f_0$ using the same argument as above since the perturbed map $f_i^{'}$ agrees with f_i at the end point of the interval X_i. ∎

If $\mathbf{f}$ is liftable to a projection g in $M(I)$, we can use the same partition of X so that $(g_0, \cdots, g_n)$ and $(f_0, \cdots, f_n)$ define the same element $\mathbf{f}$ where g_i is the restriction of g on X_i. Then, for each i, $[g_i(x) : f_i(x)]$ is defined for all x. From Corollary 3.6 this function must be constant on X_i since $g_i - f_i$ is norm continuous. So we can let $l_i = [g_i(x) : f_i(x)]$.

Since $g_i(x_i) = g_{i-1}(x_i)$, we have $[g_i(x_i) : f_i(x_i)] + [f_i(x_i) : f_{i-1}(x_i)] = [g_{i-1}(x_i) : f_{i-1}(x_i)]$ by Proposition 3.2-(3). In other words,

$$l_i - l_{i-1} = -k_i \quad \text{for} \quad i > 0 \quad \text{and} \quad l_0 - l_n = -k_0 \quad \text{if applicable.} \tag{4.1}$$

Moreover, if we apply Proposition 3.8 and Lemma 3.3 to projections $g_i(x)$ and $f_i(x)$, then we have the following restrictions on l_i.

(i) If for some x on X_i, $f_i(x)$ has finite rank, then

$$l_i \geq -rank(f_i(x)) \tag{4.2}$$

(ii) If for some x on X_i, $1 - f_i(x)$ has finite rank, then

$$l_i \leq rank(1 - f_i(x)) \tag{4.3}$$

(iii) If either end point of X_i is infinite, then

$$l_i = 0 \tag{4.4}$$

However, these necessary conditions are also sufficient.

Theorem 4.4. *A projection* $\mathbf{f}$ *in* $\mathcal{C}(I)$ *represented by* $(f_0, \cdots, f_n)$ *is liftable to a projection in* $M(I)$ *if and only if there exist* $l_0, \cdots, l_n$ *satisfying above conditions (4.1), (4.2), (4.3), (4.4).*

Proof. Given l_i's satisfying (4.1), (4.2), (4.3), (4.4), we will show there exist $g_0, \cdots, g_n$ such that $[g_i(x_i) : g_{i-1}(x_i)] = 0$ for $i > o$ and $[g_0(x_0) : g_n(x_0)] = 0$ if applicable.

First observe that if we have g_i's such that $l_i = [g_i(x_i) : f_i(x_i)]$, we have $[g_i(x_i) : g_{i-1}(x_i)] = 0$ by (4.1). Thus it is enough to show that there exist $g_0, \cdots, g_n$ such that $[g_i(x_i) : f_i(x_i)] = l_i$.

$l_i = 0$: Take $g_i = f_i$.

$l_i > 0$: By Corollary 2.4 the continuous field determined by $1 - f_i$ has a rank l_i trivial subfield which is given by projection valued function $q \leq 1 - f_i$. So we take $g_i = f_i + q$.

$l_i < 0$: Similarly, the continuous field determined by f_i has a rank $-l_i$ trivial subfield which is given by projection valued function $q^{'} \leq f_i$. So we take $g_i = f_i - q^{'}$.

Then the conclusion follows from Proposition 4.3. ■

5. Applications

Applying the representation to matrix algebra over I, we can represent an element $\mathbf{f}$ in $M_n(\mathcal{C}(I))$ as $(f_1, f_2, \cdots, f_n)$ where f_i's are in $M_n(M(I)) = M(M_n(I))$ and each of them is projection valued: $f_i(x)$ is in $M_n(B(H)) \simeq B(H^n)) \simeq B(H)$. In addition, since $f_i(x_i) - f_{i-1}(x_i) \in M_n(K) \simeq K$, we can also associate the integers $[f_i(x_i) : f_{i-1}(x_i)]$ for $i = 1, \cdots, n+1$ modulo $n+1$.

Lemma 5.1. *If a projection $\mathbf{f} \in \mathcal{C}(I)$ where $I = C(-\infty,\infty)\otimes K$ or $I = C(\mathbb{T})\otimes K$ has two different local liftings $(f_1, \cdots f_n)$ and $(g_1, \cdots, g_n)$, then $\sum_{i=1}[f_i(x_i) : f_{i-1}(x_i)] = \sum_{i=1}[g_i(x_i) : g_{i-1}(x_i)]$ or $\sum_{i=1}[f_i(x_i) : f_{i-1}(x_i)] + [f_0(x_0) : f_n(x_0)] = \sum_{i=1}[g_i(x_i) : g_{i-1}(x_i)] + [g_0(x_0) : g_n(x_0)]$, respectively.*

Proof. Note that

$$[g_i(x_i) : f_i(x_i)] + [f_i(x_i) : f_{i-1}(x_i)] + [f_{i-1}(x_i) : g_{i-1}(x_i)] = [g_i(x_i) : g_{i-1}(x_i)]$$

where $i > 0$ in the first case, and indices are taken modulo $n+1$ in the circle case. Equivalently,

$$[f_i(x_i) : f_{i-1}(x_i)] - [g_i(x_i) : g_{i-1}(x_i)] = [f_i(x_i) : g_i(x_i)] - [f_{i-1}(x_i) : g_{i-1}(x_i)] \quad (5.1)$$

Hence we have

$$\sum_{i=1}^{n}[f_i(x_i) : f_{i-1}(x_i)] - \sum_{i=1}^{n}[g_i(x_i) : g_{i-1}(x_i)] = [f_n(x_n) : g_n(x_n)] - [f_0(x_0) : g_0(x_0)] \quad (5.2)$$

for the first case. Since $f_n - g_n \in C_0(x_n, \infty) \otimes K$ and $f_0 - g_0 \in C_0(-\infty, x_0) \otimes K$, $[f_n(x_n) : g_n(x_n)] = [f_0(x_0) : g_0(x_0)] = 0$ by Theorem 3.7, we have

$$\sum [f_i(x_i) : f_{i-1}(x_i)] = \sum [g_i(x_i) : g_{i-1}(x_i)]$$

In the circle case, since we have $f_n - g_n \in C_0[x_n, x_0] \otimes K$, then from the equation5.1 for $i = 1, \cdots, n+1$ modulo $n+1$, it follows that

$$\sum [f_i(x_i) : f_{i-1}(x_i)] - \sum [g_i(x_i) : g_{i-1}(x_i)] + [f_0(x_0) : f_n(x_0)] - [g_0(x_0) : g_n(x_0)] = 0$$

■

Given two projections $\mathbf{p}, \mathbf{q}$ in $\mathcal{C}(I)$ let's assume there is a partial isometry $\pi(u)$ in $\mathcal{C}(I)$ such that $\pi(u)^*\pi(u) = \mathbf{p}$ and $\pi(u)\pi(u)^* = \mathbf{q}$ for some u in $M(I)$. If we take $(p_0, \cdots, p_n)$ and $(q_0, \cdots, q_n)$ as local liftings of $\mathbf{p}$ and $\mathbf{q}$ respectively, using the same partition, we can have a representation of u as $(u_0, \cdots, u_n)$. In fact, $u_i = q_i u|_{X_i} p_i$. Hence we have

$$q_i - u_i {u_i}^*, \quad p_i - {u_i}^* u_i \in C(X_i) \otimes K$$

For any $x \in X_i$, we can view $u_i(x)$ as a Fredholm operator from $p_i(x)H$ to $q_i(x)H$, and therefore we can define Fredholm index for $u_i(x)$ for each point x in X_i.

Lemma 5.2. *$Ind(u_i(x))$ is constant on X_i*

Proof. Consider either end point of X_i is not infinite. Fix a point x_0 and observe that $ker(u_i(x_0)) = E_1(p_i(x_0) - u_i(x_0)^* u_i(x_0))$. Similarly, $ker(u_i(x_0)^*) = E_1(p_i(x_0) - u_i(x_0)u_i(x_0)^*)$.
Note that 1 is an isolated point in the spectrum of $p_i(x_0) - u_i(x_0)^* u_i(x_0)$, and therefore we can consider a neighborhood $\{1\} \cup [0, 1-\epsilon)$ for some $\epsilon > 0$. Since $p_i - u_i^* u_i, q_i - u_i u_i^*$ is norm-continuous, there is a δ such that if $|x_0 - x| < \delta$ then $\sigma(p_i(x) - u_i(x)^* u_i(x)) \subset O_1(\epsilon) \cup [0, 1-\epsilon)$ of $\sigma(p_i - u_i^* u_i)$ where $O_1(\epsilon)$ is the open ball centered at 1 with radius ϵ. By the Riesz functional calculus (see the proof of Corollary 3.6),

$$\dim(E_1(p_i(x_0) - u_i(x_0)^* u_i(x_0))) = \sum_{0 \le \lambda_j < \epsilon} \dim(E_{1-\lambda_j}(p_i(x) - u_i(x)^* u_i(x))).$$

Similarly,

$$\dim(E_1(q_i(x_0) - u_i(x_0) u_i(x_0)^*)) = \sum_{0 \le \nu_j < \epsilon} \dim(E_{1-\nu_j}(q_i(x) - u_i(x) u_i(x)^*)).$$

Since $\dim E_{1-\lambda_i}(p_i(x) - u_i(x)^* u(x)) = \dim E_{\lambda_i}(u_i(x)^* u_i(x)) = \dim E_{\lambda_i}(u_i(x) u_i(x)^*)$ $= E_{1-\lambda_i}(q_i(x) - u_i(x)^* u(x))$ for $\lambda_j > 0$,

$$\sum_{0<\lambda_j<\epsilon} \dim(E_{1-\lambda_j}(p_i(x) - u_i(x)^* u_i(x))) = \sum_{0<\nu_j<\epsilon} \dim(E_{1-\nu_j}(q_i(x) - u_i(x) u_i(x)^*)).$$

Thus, we have

$$\begin{aligned} \operatorname{Ind}(u_i(x_0)) &= \dim(E_1(p_i(x_0) - u_i(x_0)^* u_i(x_0))) - \dim(E_1(q_i(x_0) - u_i(x_0) u_i(x_0)^*)) \\ &= \dim(E_1(p_i(x) - u_i(x)^* u_i(x))) - \dim(E_1(q_i(x) - u_i(x) u_i(x)^*)) \\ &= \operatorname{Ind}(u_i(x)) \quad \text{for} \quad |x - x_0| < \delta \end{aligned}$$

Finally, the claim follows since each X_i is connected. ■

We will denote the index of u_i on X_i by t_i. If $u_i(x_i) = v_i |u_i(x_i)|$ is a polar decomposition of $u_i(x_i)$ in $B(H)$ and $u_{i-1}(x_i) = v_{i-1} |u_{i-1}(x_i)|$ is a polar decomposition of $u_{i-1}(x_i)$, then

$$t_i = [p_i(x_i) : v_i^* v_i] - [q_i(x_i) : v_i v_i^*]$$

$$t_{i-1} = [p_{i-1}(x_i) : v_{i-1}^* v_{i-1}] - [q_{i-1}(x_i) : v_{i-1} v_{i-1}^*]$$

Since $p_i(x_i) - p_{i-1}(x_i), q_i(x_i) - q_{i-1}(x_i) \in K$, we have $u_i(x_i) - u_{i-1}(x_i) \in K$. Thus we can deduce that $v_i - v_{i-1} \in K$ and $v_i^* v_i - v_{i-1}^* v_{i-1}, v_i v_i^* - v_{i-1} v_{i-1}^* \in K$.
Note that

$$\begin{aligned} [p_i(x_i) : p_{i-1}(x_i)] &= [p_i(x_i) : v_i^* v_i] + [v_i^* v_i : v_{i-1}^* v_{i-1}] + [v_{i-1}^* v_{i-1} : p_{i-1}(x_i)] \quad i > 0 \\ [q_i(x_i) : q_{i-1}(x_i)] &= [q_i(x_i) : v_i v_i^*] + [v_i v_i^* : v_{i-1} v_{i-1}^*] + [v_{i-1} v_{i-1}^* : q_{i-1}(x_i)] \quad i > 0 \end{aligned}$$

Thus we have

$$[p_i(x_i) : p_{i-1}(x_i)] - [q_i(x_i) : q_{i-1}(x_i)] = t_i - t_{i-1} + [v_i^* v_i : v_{i-1}^* v_{i-1}] - [v_i v_i^* : v_{i-1} v_{i-1}^*] \tag{5.3}$$

Let W, V be isometries such that $WW^* = v_i^* v_i, VV^* = v_{i-1}^* v_{i-1}$. Then $V' = v_{i-1}V, W' = v_i W$ are isometries such that $V'V'^* = v_{i-1}v_{i-1}{}^*, W'W'^* = v_i v_i^*$.

$$\begin{aligned}[v_i v_i^* : v_{i-1}v_{i-1}^*] &= \mathrm{Ind}(V'^* W') \\ &= \mathrm{Ind}(V^* v_{i-1}^* v_i W) \\ &= \mathrm{Ind}(V^* v_i{}^* v_i W) \quad (v_{i-1} - v_i \in K) \\ &= \mathrm{Ind}(V^* W W^* W) \\ &= \mathrm{Ind}(V^* W) \\ &= [v^* v_i : v_{i-1}{}^* v_{i-1}]\end{aligned}$$

Thus, if we let $k_i = [p_i(x_i) : p_{i-1}(x_i)]$, $l_i = [q_i(x_i) : q_{i-1}(x_i)]$, then we have

$$t_i - t_{i-1} = k_i - l_i \quad i \geq 1 \tag{5.4}$$

Lemma 5.3. *Suppose there is a partial isometry* $\pi(u)$ *in* $\mathcal{C}(I)$ *where* $I = C(-\infty, \infty) \otimes K$ *or* $I = C(\mathbb{T}) \otimes K$ *such that* $\pi(u)^*\pi(u) = \mathbf{p}$ *and* $\pi(u)\pi(u)^* = \mathbf{q}$ *for some* u *in* $M(I)$. *If* $(p_0, \cdots, p_n)$ *and* $(q_0, \cdots, q_n)$ *are local liftings of* $\mathbf{p}$ *and* $\mathbf{q}$ *respectively, then* $\sum_{i>0}[p_i(x_i) : p_{i-1}(x_i)] = \sum_{i>0}[q_i(x_i) : q_{i-1}(x_i)]$ *or* $\sum_{i>0}[p_i(x_i) : p_{i-1}(x_i)] + [p_0(x_0) : p_n(x_0)] = \sum_{i>0}[q_i(x_i) : q_{i-1}(x_i)] + [q_0(x_0) : q_n(x_0)]$ *respectively.*

Proof. From the equation (5.4), we have $\sum_{i=1}^n k_i - \sum_{i=1}^n l_i = t_n - t_0$ for the first case. Since $q_0 - u_0 u_0{}^*, p_0 - u_0{}^* u_0 \in C_0(-\infty, x_1] \otimes K$, and $q_n - u_n u_n{}^*, p_n - u_n{}^* u_n \in C_0[x_n, \infty) \otimes K$ we have $t_n = t_0 = 0$. In the circle case, it follows by adding the equations (5.4) for $i = 1, \cdots, n+1$ modulo $n+1$. ■

Lemma 5.4. *Let* $(p_0, \cdots, p_n)$ *and* $(q_0, \cdots, q_n)$ *be local liftings of* $\mathbf{p}$ *and* $\mathbf{q}$ *such that* $\mathrm{rank}(1 - q_i(x)) = \mathrm{rank}(q_i(x)) = \infty$ *for each* x *in* X_i.
If $\sum_{i=1}^n [p_i(x_i) : p_{i-1}(x_i)] = \sum_{i=1}^n [q_i(x_i) : q_{i-1}(x_i)]$ *or* $\sum_{i=1}^n [p_i(x_i) : p_{i-1}(x_i)] + [p_0(x_0) : p_n(x_0)] = \sum_{i=1}^n [q_i(x_i) : q_{i-1}(x_i)] + [q_0'(x_0) : q_n'(x_0)]$, *then we can find a perturbation* $(q_0', \cdots, q_n')$ *of* $\mathbf{q}$ *such that* $[p_i(x_i) : p_{i-1}(x_i)] = [q_i'(x_i) : q_{i-1}'(x_i)]$ *for each* $i > 0$ *and* $[p_0(x_0) : p_n(x_0)] = [q_0'(x_0) : q_n'(x_0)]$ *in the circle case.*

Proof. Let $[p_i(x_i) : p_{i-1}(x_i)] = k_i, [q_i(x_i) : q_{i-1}(x_i)] = l_i$ for $i > 0$ and $[p_0(x_0) : p_n(x_0)] = k_0, [q_0(x_0) : q_n(x_0)] = l_0$. If $d_i = k_i - l_i$, note that

$$\sum [p_i(x_i) : p_{i-1}(x_i)] = \sum [q_i(x_i) : q_{i-1}(x_i)] \quad \text{if and only if} \quad \sum d_i = 0$$

First we consider the case $\sum_{i=1}^n k_i = \sum_{i=1}^n l_i$. Let $q_0^{'} = q_0$. Suppose that we have constructed $q_0^{'}, \cdots, q_i^{'}$ such that $[p_j(x_j) : p_{j-1}(x_j)] = [q_j^{'}(x_j) : q_{j-1}^{'}(x_j)]$ for $j = 1, \cdots, i$ and $[q_{i+1}(x_{i+1}) : q_i^{'}(x_{i+1})] = l_{i+1} - \sum_{k=1}^i d_k$.
If $d_{i+1} + \sum_{k=1}^i d_k > 0$, let q be a projection valued (norm continuous) function such that $q \le 1 - q_{i+1}$ and $\text{rank}(q(x)) = d_{i+1} + \sum_{k=1}^i d_k$ which is possible since rank $(1 - q_{i+1})(x) \ge d_{i+1} + \sum_{k=1}^i d_k > 0$.
Take $q_{i+1}^{'} = q + q_{i+1}$. Then

$$\begin{aligned}
[q_{i+1}^{'}(x_{i+1}) : q_i^{'}(x_{i+1})] &= [q_{i+1}(x_{i+1}) : q_i^{'}(x_{i+1})] + [q(x_{i+1}) : 0] \\
&= l_{i+1} - \sum_{k=1}^{i} d_k + d_{i+1} + \sum_{k=1}^{i} d_k \\
&= l_{i+1} + k_{i+1} - l_{i+1} \\
&= k_{i+1}
\end{aligned}$$

$$\begin{aligned}
[q_{i+2}(x_{i+2}) : q_{i+1}^{'}(x_{i+2})] &= [q_{i+2}(x_{i+2}) : q_{i+1}(x_{i+2})] + [0 : q(x_{i+2})] \\
&= l_{i+2} - (d_{i+1} + \sum_{k=1}^{i} d_k) \\
&= l_{i+2} - \sum_{k=1}^{i+1} d_k
\end{aligned}$$

If $d_{j+1} + \sum_{k=1}^j d_k < 0$, let q be a projection valued (norm continuous) function such that $q \le q_{i+1}$ and $\text{rank}(q(x)) = -(d_{i+1} + \sum_{k=1}^i d_k)$ which is possible since rank $(q_{i+1})(x) \ge -(d_{i+1} + \sum_{k=1}^i d_k) > 0$.
Take $q_{i+1}^{'} = q_{i+1} - q$.
Note that

$$[q_{i+1}^{'}(x_{i+1}) : q_i^{'}(x_{i+1})] + [q(x_{i+1}) : 0] = [q_{i+1}^{'}(x_{i+1}) : q_i(x_{i+1})]$$

Thus

$$\begin{aligned}[q'_{i+1}(x_{i+1}) : q'_i(x_{i+1})] &= l_{i+1} - \sum_{k=1}^{i} d_k + (d_{i+1} + \sum_{k=1}^{i} d_k) \\ &= l_{i+1} + d_{i+1} \\ &= k_{i+1}\end{aligned}$$

Also,

$$\begin{aligned}[q_{i+2}(x_{i+2}) : q'_{i+1}(x_{i+2})] &= [q_{i+2}(x_{i+2}) : q_{i+1}(x_{i+2})] + [q(x_{i+2}) : 0] \\ &= l_{i+2} - (d_{i+1} + \sum_{k=1}^{i} d_k) \\ &= l_{i+2} - \sum_{k=1}^{i+1} d_k\end{aligned}$$

By induction, we can get $q'_0, \cdots, q'_{n-1}$ such that $[p_j(x_j) : p_{j-1}(x_j)] = [q'_j(x_j) : q'_{j-1}(x_j)]$ for $j = 1, \cdots, n-1$ as we want. Since we also have $[q_n(x_n) : q'_{n-1}(x_n)] = l_n - \sum_{k=1}^{n-1} d_k = l_n + d_n = k_n$ from $\sum_{k=1}^{n-1} d_k + d_n = 0$, we take $q'_n = q_n$.
If $\sum_{i=0}^{n} k_i = \sum_{i=0}^{n} l_i$, we also perturb q_n and get q'_n such that $[q'_n(x_n) : q'_{n-1}(x_n)] = k_n$ and $[q_0(x_0) : q'_n] = l_0 - \sum_{k=1}^{n} d_k = l_0 + d_0 = k_0$. ■

Lemma 5.5. *Let* $(q_0, \cdots, q_n)$ *be local lifting of* $\mathbf{q}$ *such that* $\mathrm{rank}(1 - q_i(x)) = \mathrm{rank}(q_i(x)) = \infty$ *for each* x *in* X_i*. We can find a perturbation* $(q'_0, \cdots, q'_n)$ *of* $\mathbf{q}$ *such that* $[q'_i(x_i) : q'_{i-1}(x_i)] = 0$ *for* $i \geq 1$, $[q'_n(x_n) : q'_{n-1}(x_n)] = \sum_{i=1}^{n-1}[q_i(x_i) : q_{i-1}(x_i)]$ *and* $[q'_0(x_0) : q'_n(x_0)] = \sum_{i=1}^{n}[q_i(x_i) : q_{i-1}(x_i)]$ *in the circle case.*

Proof. Keeping the same notation, the proof is similar to Lemma 5.4: Suppose that we have constructed $q'_0, \cdots, q'_i$ such that $[q'_j(x_j) : q'_{j-1}(x_j)] = 0$ for $j = 1, \cdots, i$ and $[q_{i+1}(x_i) : q'_i(x_i)] = l_{i+1} + \sum_{j=1}^{i} l_j$. If $l_{i+1} + \sum_{j=1}^{i} l_j \geq 0$, we make $q'_{i+1} \leq q_{i+1}$. Otherwise, we make $q'_{i+1} > q_{i+1}$. ■

Next is an analogous result to Lemma 5.3 that is more symmetrical.

Lemma 5.6. *Let* $(p_0, \cdots, p_n)$ *and* $(q_0, \cdots, q_n)$ *be local liftings of* $\mathbf{p}$ *and* $\mathbf{q}$ *such that* $\operatorname{rank}(p_i(x)) = \operatorname{rank}(q_i(x)) = \infty$ *for each* x *in* X_i.
If $\sum_{i=1}^{n}[p_i(x_i) : p_{i-1}(x_i)] = \sum_{i=1}^{n}[q_i(x_i) : q_{i-1}(x_i)]$ *or* $\sum_{i=1}^{n+1}[p_i(x_i) : p_{i-1}(x_i)] = \sum_{i=1}^{n+1}[q_i(x_i) : q_{i-1}(x_i)]$ *modulo* $n+1$, *then we can find perturbations* $(q_0^{'}, \cdots, q_n^{'})$ *of* $\mathbf{q}$ *and* $(p_0^{'}, \cdots, p_n^{'})$ *of* $\mathbf{p}$ *such that* $[p_i^{'}(x_i) : p_{i-1}^{'}(x_i)] = [q_i^{'}(x_i) : q_{i-1}^{'}(x_i)]$ *for each* $i \geq 1$.

Proof. The proof proceeds as Lemma 5.4 with one exception: If $d_{i+1} + \sum_{k=1}^{i} d_k \geq 0$, we make $p_{i+1}^{'} \leq p_i$ rather than making $q_{i+1}^{'} \geq q_i$. ∎

Recall that $K_i(\mathcal{C}(I)) = K_{i+1}(I)\,(\mathrm{mod}\,2)$. Thus

$$K_1(\mathcal{C}(I)) = \begin{cases} \mathbb{Z} & \text{, if } X = [0,1] \text{ or } \mathbb{T} \\ 0 & \text{, if } X \text{ contains } \infty \text{ or } -\infty \end{cases} \qquad K_0(\mathcal{C}(I)) = \begin{cases} \mathbb{Z} & \text{, if } X = (-\infty,\infty) \text{ or } \mathbb{T} \\ 0 & \text{, otherwise} \end{cases}$$

Now in the case $I = C(X) \otimes K$ where $X = (-\infty, \infty)$ or $\mathbb{T}$ we can define a map $\chi : K_0(\mathcal{C}(I)) \to \mathbb{Z}$ as follows; Let $\alpha = [\mathbf{p}] - [\mathbf{q}]$ be an element of $K_0(\mathcal{C}(I))$ and $(p_0, \cdots, p_n)$ and $(q_0, \cdots, q_n)$ be local liftings of $\mathbf{p}$ and $\mathbf{q}$ respectively. Then $\chi(\alpha) = \sum_{i=1}^{n}[p_i(x_i) : p_{i-1}(x_i)] - \sum_{i=1}^{n}[q_i(x_i) : q_{i-1}(x_i)]$ and $\sum_{i=1}^{n+1}[p_i(x_i) : p_{i-1}(x_i)] - \sum_{i=1}^{n+1}[q_i(x_i) : q_{i-1}(x_i)]$ modulo $n+1$ in the circle case.

Theorem 5.7. *The map*

$$\chi : K_0(\mathcal{C}(I)) \to \mathbb{Z}$$

is an isomorphism.

Proof. χ is well-defined by Lemma 5.1, Lemma 5.3.
For injectivity of χ, let $\alpha = [\mathbf{p}] - [\mathbf{q}]$ be an element of $K_0(\mathcal{C}(I))$ such that $\chi(\alpha) = 0$. If $(p_0, \cdots, p_n)$ and $(q_0, \cdots, q_n)$ are local liftings of $\mathbf{p}$ and $\mathbf{q}$ respectively, then $\mathbf{p}$ and $\mathbf{q}$ are Murray-von Neumann equivalent if and only if there is $\mathbf{u}$ in $\mathcal{C}(I)$ such that $\mathbf{u}^*\mathbf{u} = \mathbf{p}$ and $\mathbf{u}\mathbf{u}^* = \mathbf{q}$. To construct $\mathbf{u}$, it suffices to find a representation $(u_0, \cdots, u_n)$ such that

$$u_i{}^* u_i = p_i, \quad u_i u_i{}^* = q_i, \quad u_i(x_i) - u_{i-1}(x_i) \in K$$

By replacing $\mathbf{p}$ with $\mathbf{p}\oplus 1_N \oplus 0_N$ and $\mathbf{q}$ with $\mathbf{p}\oplus 1_N \oplus 0_N$ for some N, we may assume that $\operatorname{rank} p_i(x) = \operatorname{rank}(1-p_i(x)) = \operatorname{rank} q_i(x) = \operatorname{rank}(1-q_i(x)) = \infty$. Moreover, if we let $k_i = [p_i(x_i) : p_{i-1}(x_i)]$, $l_i = [q_i(x_i) : q_{i-1}(x_i)]$ for $i/geq1$ and $k_0 = [p_0(x_0) : p_n(x_0)]$, $l_0 = [q_0(x_0) : q_n(x_0)]$ if applicable, by Lemma 5.4 we may also assume that $k_i = l_i$ for each i. Since $\operatorname{rank}(p_i(x)) = \operatorname{rank}(q_i(x)) = \infty$ and $\dim(X_i) < \infty$, there is a (double) strongly continuous function u_i on each X_i such that ${u_i}^* u_i = p_i, u_i {u_i}^* = q_i$ by Theorem 2.6. But we do not claim $(u_0, \cdots, u_n) \in \mathcal{C}(I)$ because we have not yet achieved $u_i(x_i) - u_{i-1}(x_i) \in K$ for $i \geq 1$ and $u_0(x_0) - u_n(x_0) \in K$ if necessary.

On each X_i, we are going to change u_i on half-open interval including x_i such that $u_i(x_i) - u_{i-1}(x_i) \in K$ and ${u_i}^* u_i = p_i, u_i {u_i}^* = q_i$.

Since $\operatorname{Ind}(u_{i-1}(x_i)) = 0$,

$$\operatorname{Ind}(q_i(x_i) u_{i-1}(x_i) p_i(x_i)) = -l_i + \operatorname{Ind}(u_{i-1}(x_i)) + k_i = 0$$

and $q_i(x_i)u_{i-1}(x_i)p_i(x_i) - u_{i-1}(x_i) \in K$. Thus there is a $v_i \in B(H)$ which is a unitary from $p_i(x_i)H$ to $q_i(x_i)H$ such that $v_i - q_i(x_i)u_{i-1}(x_i)p_i(x_i) \in K$. Now $u_i(x_i)^* v_i$ is a unitary from $p_i(x_i)H$ to $p_i(x_i)H$. Combining triviality of the continuous field of Hilbert spaces determined by p_i and path connectedness of unitary group of B(H), we can form a path $\{v(t) : t \in [x_i, x]\}$ of unitaries such that $v(x_i) = u_i(x_i)^* v_i$ and $v(x) = I_{p_i(x)H}$ for some $x \in X_i$. Let's take $w_i = u_i v$, then

$$w_i(x_i) - u_{i-1}(x_i) = v_i - u_{i-1}(x_i) \in K$$

$$ {w_i}^* w_i = v^* {u_i}^* u_i v = v^* p_i v = p_i$$

$$ w_i {w_i}^* = u_i v v^* {u_i}^* = u_i p_i {u_i}^* = q_i$$

Thus, we define

$$u_i^{'} = \begin{cases} w_i, & \text{on} \quad [x_i, x] \\ u_i, & \text{on} \quad [x, x_{i+1}] \end{cases}$$

Now $(u_0, u_1^{'}, \cdots, u_n^{'})$ for $X = (\infty, \infty)$ or $(u_1^{'}, \cdots, u_n^{'}, u_0^{'})$ for $X = \mathbb{T}$ is what we want.

■

Remark 5.8. In the above, in fact, we proved the following characterizations.

(i) $[\mathbf{p}] = [\mathbf{q}]$ if and only if $\sum k_i = \sum l_i$

(ii) If $\text{rank}(p_i(x)) = \text{rank}(q_i(x)) = \infty$, then there exist u_i's such that ${u_i}^* u_i = p_i$, $u_i {u_i}^* = q_i$, and $u_i(x_i) - u_{i-1}(x_i) \in K$ for $i \geq 1$ and $u_0^* u_0 = q_0$, $u_0(u_0)^* = q_0$, $u_0(x_0) - u_n(x_0) \in K$ when applicable if and only if $k_i = l_i$.

Corollary 5.9. $[\mathbf{p}]_0$ *is lifted to an element via the map* $K_0(M(I)) \to K_0(\mathcal{C}(I))$ *if and only if* $\mathbf{p} \oplus 1_n \oplus 0_n$ *is lifted to a projection for some integer* n.

Proof. Since $K_0(M(I)) = 0$, we know $[\mathbf{p}]_0$ is liftable if and only if $[\mathbf{p}]_0 = 0$. The latter is equivalent to $\sum k_i = 0$. Thus, if $[\mathbf{p}]_0$ is liftable, or $[\mathbf{p} \oplus 1_n \oplus 0_n]_0$ is liftable, $\mathbf{p} \oplus 1_n \oplus 0_n$ is liftable to a projection by Proposition 4.3 and Lemma 5.5.
Conversely, if $\mathbf{p}$ or $\mathbf{p} \oplus 1_n \oplus 0_n$ is lifted to a projection $p \in M(I)$, obviously $\pi_*([p]) = [\mathbf{p}] = 0$. ■

Proposition 5.10. $\mathbf{p} \sim \mathbf{q}$ *in* $\mathcal{C}(I)$ *if and only if there exist* t_i*s such that*

(i) $\text{rank}(p_i(x)) = t_i + \text{rank}(q_i(x))$ *for* $x \in X_i$

(ii) $t_i - t_{i-1} = k_i - l_i$ *for* $i \geq 1$ *and* $t_0 - t_n = k_0 - l_0$ *in the circle case.*

(iii) $t_i = 0$ *if* X_i *contains infinite end-point.*

for some local liftings $(p_0, \cdots, p_n)$ *and* $(q_o, \cdots, q_n)$ *of* $\mathbf{p}$ *and* $\mathbf{q}$ *respectively.*

Proof. "only if": For the first condition, observe that if $u_i(x)$ is a Fredholm operator from $p_i(x)H$ to $q_i(x)H$ and $\text{rank}(p_i(x))$ is finite, then $\text{rank}(q_i(x))$ is also finite and $\text{Ind}(u_i(x)) = t_i = \text{rank}(p_i(x)) - \text{rank}(q_i(x))$. Moreover, if $\text{rank}(p_i(x))$ is infinite, by applying the same reasoning to $q_i(x)$, $\text{rank}(q_i(x))$ is also infinite. Therefore, $\text{rank}(p_i(x)) = \text{rank}(q_i(x)) + t_i$ holds.
The second condition follows from the equations (5.4).
"if": If $\text{rank}(p_i(x)) = \text{rank}(q_i(x)) + t_i$ and $t_i \geq 0$ for each i, then $\text{rank}(p_i(x)) \geq t_i$, and therefore there is norm continuous projection valued function $p \leq p_i$ such that $rank(p(x)) = t_i$. Consider $p_i^{'} = p_i - p$. It is easily checked that $\text{rank}(p_i^{'}(x)) =$

rank$(q_i(x))$. So there is again a (double) strongly continuous function u_i on each X_i such that ${u_i}^*u_i = p_i^{'}, u_i{u_i}^* = q_i$. Furthermore,

$$ {u_i}^*u_i - p_i \in C(X_i) \otimes K $$
$$ u_i{u_i}^* - q_i \in C(X_i) \otimes K $$

implies that $u_i(x)$ is a Fredholm operator and $\mathrm{Ind}(u_i(x)) = t_i$ for every $x \in X_i$.
Similarly, we can construct such a u_i for the case $t_i < 0$ since we can have a perturbation $q_i^{'}$ of q_i with $\mathrm{rank}(q_i^{'}(x)) = \mathrm{rank}(p_i(x))$.
Note that

$$ \begin{aligned} \mathrm{Ind}(q_i(x_i)u_{i-1}(x_i)p_i^{'}(x_i) &= -[q_i(x_i) : q_{i-1}(x_i)] + \mathrm{Ind}(u_{i-1}(x_i)) + [p_i^{'}(x_i) : p_{i-1}(x_i)] \\ &= -l_i + t_{i-1} + k_i - t_i \\ &= 0 \end{aligned} $$

and $q_i(x_i)u_{i-1}(x_i)p_i^{'}(x_i) - u_{i-1}(x_i) \in K$.
As we have seen before in Theorem 5.7, this implies that we can have a perturbation $(u_0^{'}, \cdots u_n^{'})$ of $(u_0, \cdots, u_n)$ such that $u_i^{'}(x_i) - u_{i-1}^{'}(x_i) \in K$. ■

Since $\mathbf{p} \sim_u \mathbf{q}$ is equivalent to $\mathbf{p} \sim \mathbf{q}$ and $\mathbf{1} - \mathbf{p} \sim \mathbf{1} - \mathbf{q}$, we get the following statement immediately from Proposition 5.10.

Corollary 5.11. $\mathbf{p} \sim_u \mathbf{q}$ *if and only if there exist* t_i*s and* s_i*s such that*

(i) $\mathrm{rank}(1 - p_i(x)) = s_i + \mathrm{rank}(1 - q_i(x)) x \in X_i$

(ii) $\mathrm{rank}(p_i(x)) = t_i + \mathrm{rank}(q_i(x)) x \in X_i$

(iii) $t_i - t_{i-1} = k_i - l_i$ *for* $i \geq 1$ *and* $t_0 - t_n = k_0 - l_0$ *in the circle case.*

(iv) $s_i + t_i = s_{i-1} + t_{i-1}$

(v) t_i *and* s_i *are vanishing if* X_i *contains infinite end-point.*

for some local liftings $(p_0, \cdots, p_n)$, $(q_o, \cdots, q_n)$ *of* $\mathbf{p}$ *and* $\mathbf{q}$ *respectively.*

It is said that a (non-unital)C^*-algebra I has *good index theory* if whenever I is embedded as an ideal of in a unital C^*-algebra A and u is a unitary in A/I such that $\partial_1([u]) = 0$ in $K_0(I)$, there is a unitary in A which lifts u. Equivalently, If $u \in \mathcal{U}(A/I)$, $\alpha \in K_1(A)$, and α lifts $[u]$, then u lifts to $\mathcal{U}(A)$(See p2-3 in [7]). It was proved that stable rank one C^*-algebra has good index theory by G.Nagy [11]. In general, $\mathbf{p} \sim_h \mathbf{q}$ if and only if $\mathbf{upu}^* = \mathbf{q}$ where $\mathbf{u}$ is connected to 1 in the unitary group of $\mathcal{C}(I)$. Since I is stable rank one, I has good index theory. Also, recall that the unitary group of the multiplier algebra of a stable C^*-algebra is path connected(even contractible). Thus $\mathbf{p} \sim_h \mathbf{q}$ if and only if $\mathbf{upu}^* = \mathbf{q}$ where $\mathbf{u}$ has trivial K_1-class.

Corollary 5.12. $\mathbf{p} \sim_h \mathbf{q}$ *if and only if there exist t_is and s_is such that*

(i) $\operatorname{rank}(1 - p_i(x)) = s_i + \operatorname{rank}(1 - q_i(x))$ *for* $x \in X_i$

(ii) $\operatorname{rank}(p_i(x)) = t_i + \operatorname{rank}(q_i(x))$ *for* $x \in X_i$

(iii) $t_i - t_{i-1} = k_i - l_i$ *for* $i \geq 1$ *and* $t_0 - t_n = k_0 - l_0$ *in the circle case.*

(iv) $s_i + t_i = s_{i-1} + t_{i-1} = 0$

(v) t_i *and* s_i *are vanishing if* X_i *contains infinite point.*

for some local liftings $(p_0, \cdots, p_n)$ *and* $(q_o, \cdots, q_n)$ *of* $\mathbf{p}$ *and* $\mathbf{q}$ *respectively.*

Proof. "Only if": Since a unitary $\mathbf{u}$ which has trivial class in $K_1(\mathcal{C}(I))$ can be lifted to a unitary u in $M(I)$, there is a unitary-valued function u in I such that $up_iu^* - q_i \in C(X_i) \otimes K$ for each i. Hence $q_i(x)u(x)p_i(x)$ and $(1-q_i(x))u(x)(1-p_i(x))$ are Fredholm operators from $p_i(x)H$ to $q_i(x)H$ and from $(1-p_i(x))H$ to $(1-q_i(x))H$ respectively. Using matrix decomposition of u_i and the fact that $(1-q_i)up_i, q_iu_i(1-p_i) \in C(X_i) \otimes K$(vanishing at infinite end-points), if we let $t_i = \operatorname{Ind}(q_i(x)u(x)p_i(x))$ and $s_i = \operatorname{Ind}((1 - q_i(x))u(x)(1 - p_i(x))$ we can deduce that

$$t_i + s_i = \operatorname{Ind}(q_i(x)u_i(x)p_i(x)) + \operatorname{Ind}((1 - q_i(x))u_i(x)(1 - p_i(x)) = 0$$

"If": As we have shown in the proof of Proposition 5.10, from conditions (i),(ii),(iii),(iv) we can construct a (double) strongly continuous function v_i on X_i such that ${v_i}^*v_i = p_i^{'} \leq p_i, u_i{u_i}^* = q_i$ with $v_i(x_i) - v_{i-1}(x_i) \in K$ and a (double) strongly continuous function w_i on X_i such that $w_i{w_i}^* = q_i^{'} \leq 1 - q_i, {w_i}^*w_i = 1 - p_i$ with $w_i(x_i) - w_{i-1}(x_i) \in K$. Then $u_i = v_i + w_i$ is a function such that $u_ip_iu_i^* - q_i \in C(X_i) \otimes K$. Hence $\mathbf{u}$ that is determined from the local representation $(u_0, \cdots, u_n)$ satisfies that $\mathbf{upu}^* = \mathbf{q}$. Recall that $K_0(\mathcal{C}(I)) = 0$ if $I = C(X) \otimes K$ where X contains ∞ or ∞. Thus in this case $[\mathbf{u}]_1$ is trivial so that we have $\mathbf{p} \sim_h \mathbf{q}$. Moreover, we also know that $[\mathbf{u}]_1 = \mathrm{Ind}(v_i + w_i) = 0$ via the map $K_1((C)(I)) \to K_0(I) \to K_0(K)$ where the second map is induced by the evaluation at 1 if $X = [0, 1]$ or $\mathbb{T}$. Thus we have $\mathbf{p} \sim_h \mathbf{q}$ as desired. ■

For the following examples we frequently use Proposition 2.7 and Corollary 2.9 to construct projection valued functions with desired rank properties.

Example 5.13. Consider a partition $\{x_1, x_2\}$ in $(-\infty, \infty)$. In other words, we divide $(-\infty, \infty)$ into three intervals X_0, X_1, X_2. We can construct projection valued functions p_0, p_1, p_2 such that

$$\begin{aligned}
&\mathrm{rank}(p_0(x_1)) = 5, \\
&\mathrm{rank}(p_1(x_1)) = \mathrm{rank}(1 - p_1(x_2)) = 2, \\
&\mathrm{rank}(1 - p_2(x_2)) = 1 \quad \text{where } p_1(x_2) \leq p_2(x_2)
\end{aligned}$$

Then we can easily check that $k_1 = -3$, $k_2 = 1$. Suppose $\mathbf{p}$ which is represented by above (p_0, p_1, p_2) is liftable to a projection in $M(I)$. By Theorem 4.4, we must have l_0, l_1, l_2 such that $l_0 = 0 = l_2$, $l_1 - l_0 = -k_1 = 3$. However, since $\mathrm{rank}(1 - p_1(x_2)) = 2$, we have a restriction $l_1 \leq 2$. This is a contradiction to $l_1 - l_0 = l_1 = 3$. Note that if we change $\mathbf{p}$ into $\mathbf{p} \oplus 0$, k_i are preserved. Since $\mathrm{rank}(1 - p_1 \oplus 1) = \mathrm{rank}(1 - (p_1 \oplus 0)) = \infty$, we have no restriction on choosing l_1 as 3. Hence $\mathbf{p} \oplus 0$ is liftable.

Similarly, if we construct projection valued functions p_0, p_1, p_2 such that

$$\begin{aligned}
&\operatorname{rank}(1 - p_0(x_1)) = 4,\\
&\operatorname{rank}(1 - p_1(x_1)) = 1 \text{where } p_0(x_1) \leq p_1(x_1)\\
&\operatorname{rank}(p_1(x_2)) = 2,\\
&\operatorname{rank}(p_2(x_2)) = 1
\end{aligned}$$

then we can see that $k_1 = 3$, $k_2 = -1$. If $\mathbf{p}$ which is represented by above (p_0, p_1, p_2) is liftable to a projection in $M(I)$, by Theorem 4.4, we must have l_0, l_1, l_2 such that $l_0 = 0 = l_2$, $l_1 - l_0 = l_1 = -k_1 = -3$ which is impossible because we have restriction on $l_1 \geq -\operatorname{rank}(p_1(x_2)) = -2$. Thus $\mathbf{p}$ cannot be liftable. Similarly, we can check $\mathbf{p} \oplus 1$ is liftable.

If we have one more partition point x_3 in $(-\infty, \infty)$ and keep p_0, p_1 as above and construct p_2, p_3 such that

$$\begin{aligned}
&\operatorname{rank}(1 - p_2(x_3)) = 1,\\
&\operatorname{rank}(1 - p_3(x_3)) = 3
\end{aligned}$$

then $\mathbf{p}$ which is represented by (p_0, p_1, p_2, p_3) cannot be lifted to a projection by the same observation as we have seen above, but K_0-class of $\mathbf{p}$ is liftable since $\sum k_i = 0$.

Example 5.14. Consider projection valued functions $p = 1$ and q on $[0, 1]$ which satisfies $\operatorname{rank}(q(x)) = 1$. Then $\pi(p) = \mathbf{p}$ and $\pi(q) = \mathbf{q}$ has trivial K_0-class. But we cannot find t such that $\operatorname{rank}(p(x)) = \operatorname{rank}(q(x)) + t$. Thus this example shows that there is $\mathbf{p}$ and $\mathbf{q}$ such that $[\mathbf{p}] = [\mathbf{q}]$ but $\mathbf{p} \nsim \mathbf{q}$.

Example 5.15. Even if we are given non-trivial K_0-data for $\mathbf{p}$ and $\mathbf{q}$, we can find a example such that $\mathbf{p} \nsim \mathbf{q}$. Consider (p_0, p_1) and (q_0, q_1) on $(-\infty, \infty)$ such that $\operatorname{rank}(p_1(x_1)) - \operatorname{rank}(p_0(x_1)) = k_1 = l_1 = \operatorname{rank}(q_1(x_1)) - \operatorname{rank}(q_0(x_1))$ but $\operatorname{rank}(p_i(x)) \neq \operatorname{rank}(q_i(x))$. Moreover, this type of example give us projections $\mathbf{p}$ and $\mathbf{q}$ such that $\mathbf{p} \nsim \mathbf{q}$ but $\mathbf{1} - \mathbf{p} \sim \mathbf{1} - \mathbf{q}$. Define p_0 on $(-\infty, x_1]$ such that

$\text{rank}(p_0(x)) = 1$ and p_1 on $[x_1, \infty)$ such that $\text{rank}(p_1(x)) = 2$. Similarly, we define q_0 and q_1 such that $\text{rank}(q_0(x)) = 2, \text{rank}(q_1(x)) = 3$. Note that $\text{rank}(1 - p_i(x)) = \text{rank}(1 - q_i(x)) = \infty$ but $\text{rank}(p_i(x)) \neq \text{rank}(q_i(x))$. Since $t_0 = t_1 = 0 = s_0 = s_1$ for this case, by Proposition 5.10, $\mathbf{1} - \mathbf{p} \sim \mathbf{1} - \mathbf{q}$ but $\mathbf{p} \nsim \mathbf{q}$ hence $\mathbf{p} \nsim_u \mathbf{q}$ by Corollary 5.11.

Example 5.16. In $[0, 1]$, if one division point is given, we can take projection valued function p_0, q_0 on $[0, x_1]$ such that

$$\begin{cases} \text{rank}(p_0(0)) & = 1, \\ \text{rank}(q_0(0)) & = 0, \\ \text{rank}(p_0(x)) = \text{rank}(q_0(x)) & = \infty \quad \text{if } x \neq 0, \\ \text{rank}(1 - p_0(x_1)) = \text{rank}(1 - q_0(x_1)) & = 2 \end{cases}$$

Also, we can construct p_1, q_1 on $[x_1, 1]$ such that

$$\begin{cases} \text{rank}(1 - p_1(1)) = \text{rank}(1 - q_1(1)) = 0 & , \\ \text{rank}(1 - p_1(x)) = \text{rank}(1 - q_1(x)) = 2 & \text{if } x \neq 1, \\ p_1(x_1) = p_0(x_1) & , \\ q_1(x_1) = q_0(x_1) \end{cases}$$

Then, we have $k_1 = l_1 = 0$, $t_0 = 1$, $s_1 = 0$, thus if we take $t_1 = 1$, $s_0 = 0$ then $\mathbf{p} \sim_u \mathbf{q}$ but $\mathbf{p}$ cannot be homotopic to $\mathbf{q}$ since $t_1 + s_1 \neq 0$.

6. Some example of lifting problems

Let $\mathcal{E}(A)$, or just $\mathcal{E}$, denote the set of extreme points of the unit ball of a unital C^*-algebra A. Recall that elements in $\mathcal{E}$ are characterized as the partial isometries u in A satisfying $(1-uu^*)A(1-u^*u)=0$ by R. V. Kadison [13]. We call them extremal partial 006006 isometries and call the projections $1-uu^*$, $1-u^*u$ defect projections. In [6], Brown and Pedersen defined the notion of extremal richness for C^*-algebra A which means quasi-invertible elements are dense in A as an analogue of stable rank one for possibly infinite C^*-algebras. (We say T in A is quasi-invertible if T has closed range and the kernel projections of T^* and T are centrally orthogonal in A, or if T is in $A^{-1}\mathcal{E}A^{-1}$. For more equivalent definitions, see Theorem 1.1 in [6].) As a result, stable rank one C^*-algebras are characterized within the class of extremally rich C^*-algebras by the property that all extreme points of the unit ball are unitaries. Suppose we have an extension of C^*-algebras.

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

It is well known that if we have an extension of an extremally rich C^*-algebra I by an extremally rich C^*-algebra B, we cannot deduce A is extremally rich even in the finite case. The obstacle, as in the analogous problem with stable rank one, can be expressed as a lifting problem but with special properties. In fact, Brown and Pedersen proved the following theorem in [4].

Theorem 6.1. *If A is an extension of I by B and both I and B are extremally rich, then A is extremally rich if and only if extremal partial isometries in B are liftable to extremal partial isometries in A and PAQ is extremally rich whenever P is projection of the form $1-uu^*$ where $u \in \mathcal{E}(A)$ and Q projection of the from $1-v^*v$ where $v \in \mathcal{E}(\tilde{I})$.*

However, there are some special ideals I for which the hypotheses in theorem 6.1 can be simplified:

Corollary 6.2. *Let I be a C^*-algebra of stable rank one. Then A is extremally rich if and only if B is extremally rich and extremal partial isometries in B lift.*

This corollary implies the following: Whenever I embedded as an ideal in a unital extremally rich C^*-algebra and u is an extremal partial isometry in a extremally rich C^*-algebra A/I, there is an extremal partial isometry in A which lifts u. Motivated by above example, but not necessarily related to extension, we can ask whether we can find examples of I such that whenever there exist A containing I as a closed ideal, certain lifting questions below from any quotient C^*-algebras $B = A/I$ to A are affirmative:

(1) Does every unitary in B lift to a unitary in A?

(2) Does every unitary in B lift to an extremal partial isometry in A?

(3) Does every extremal partial isometry in B lift to an extremal partial 055055 isometry in A?

We cannot expect a positive answer in great generality due to the nature of the problem; In fact, it is not very difficult to find counter examples for these questions. Interesting direction, however, is to find an affirmative answer. For example, it is well-known that if I has an approximate identity consisting of projections, then every unitary in B lifts to a partial isometry in A.

Remark 6.3. (i) When I and B are of stable rank one, an affirmative answer to (1) is equivalent to the stable rank one property for A.

(ii) When I and B are of stable rank one, an affirmative answer(2) is equivalent to extremal richness for A.

(iii) When I has stable rank one and B is extremally rich, an affirmative answer to (3) is equivalent to extremal richness for A.

6.1 Examples

In view of Remark 6.3-(i), it is still interesting if we restrict I to be the class of stable rank one C^*-algebras. But, even in this case, Question (1) is not true for every stable rank one C^*-algebras. In fact, there is a well-known counter example: the Toeplitz extension

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \longrightarrow C(S^1) \longrightarrow 0$$

where $\mathcal{T}$ is the C^*-algebra generated by the unilateral shift S on H is such an example by the remark 6.3-(i) and the fact $\mathrm{tsr}(\mathcal{T}) \neq 1$.(See example 4.13 in [15].) However, 6there is a C^*-algebra of stable rank one which serve as an answer for question (1).

Definition 6.4. (Brown and Pedersen) We say a(non-unital) C^*-algebra I has good index theory if whenever I is embedded as an ideal in a unital C^*-algebra A and u is a unitary in A/I such that $\partial_1([u]) = 0$ where $\partial_1 : K_1(A/I) \to K_0(I)$, there is a unitary in A which lifts u.

It was noted by Brown and Pedersen that G. Nagy proved that any stable rank one C^*-algebra has good index theory [7]. Thus if we can show the existence of I with $\mathrm{tsr}(I) = 1$ such that $\partial_1 : K_1(B) \to K_0(I)$ is trivial, question (1) holds for I.

Example 6.5. Let $I = \{(f_n) \mid f_n \in C(\mathbb{T}, M_n(\mathbb{C}))\,\text{and}\, f_n \to f \,\text{in}\, C(\mathbb{T}, \mathcal{K})\}$
Here $f_n \to f \,\text{in}\, C(\mathbb{T}, \mathcal{K})$ means $\sup_{t\in\mathbb{T}} \|f_n(t) - f(t)\|$ goes to 0 as $n \to \infty$. Note that I has stable rank one. Let $C(\mathbb{T}, B(H)_{*-S})$ be the set of functions $m : \mathbb{T} \to B(H)$ which are continuous with respect to the $*$-strong operator topology on $B(H)$, and which satisfy $\|m\|_\infty := \sup_{t\in\mathbb{T}} \|m(t)\| < \infty$. It is not hard to show that $M(I) = \{(F_n) \mid F_n \in C(\mathbb{T}, M_n(\mathbb{C}))\,\text{and}\, F_n \rightrightarrows F \,\text{in}\, C(\mathbb{T}, B(H)_{*-S})\}$. Here $F_n \rightrightarrows F$ means $F_nT \to FT$ and $F_n^*T \to F^*T$ in A for each $T \in \mathcal{K}$: i.e., given $\epsilon > 0$, there is N such that if $n \geq N$, then $\sup_{t\in\mathbb{T}} \|F_n(t)T - F(t)T\| < \epsilon$ and $\sup_{t\in\mathbb{T}} \|F_n(t)^*T - F(t)^*T\| < \epsilon$ for each $T \in \mathcal{K}$.
Next we want to show $K_0(\iota) : K_0(I) \to K_0(M(I))$ is injective; since the sequence

$\alpha = (z_n)$ in $K_0(I)$ is eventually constant, we may assume I has only finite dimensional irreducible representations. Let π be such an irreducible representation and suppose $K_0(\iota)(\alpha) = 0$. Then $K_0(\bar{\pi})(K_0(\iota)(\alpha)) = 0$ where $\bar{\pi} : M(I) \to B(H)$ is a unique map such that $\bar{\pi} \circ \iota = \pi$. Note that we have $\pi(I) = \bar{\pi}(I)$ since π is finite dimensional representation. Consequently, we have $K_0(\pi)(\alpha) = 0$ in $K_0(\pi(I))$. By the Remark 5.12 in [4], $\alpha = 0$.
From the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \xrightarrow{j} & A & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I & \longrightarrow & \mathcal{M}(I) & \longrightarrow & \mathcal{C}(I) & \longrightarrow & 0140140 \end{array}$$

we have

$$\begin{array}{ccc} K_0(I) & \xrightarrow{K_0(j)} & K_0(A) \\ \Big\| & & \downarrow \\ K_0(I) & \xrightarrow[K_0(\iota)]{} & K_0(M(I))148148 \end{array} \quad 149149$$

It follows from that $K_0(j)$ is also injective. Thus from the exactness of six term exact sequence of K-theory we know the map $\partial_1 : K_1(B) \to K_0(I)$ is trivial.

Next we show there are counter-examples to question (2).

Example 6.6. Let $I = \{(a_n) \mid a_n \in M_{2n}(\mathbb{C})$ such that $a_n \to \left(\begin{smallmatrix} D & 0 \\ 0 & E \end{smallmatrix}\right) \in M_2(K)\}$. If $B(H)_{*-s}$ denote $B(H)$ equipped with $*$-strong operator topology, then

$$M(I) = \{(T_n) \mid T_n \in M_{2n}(\mathbb{C}) \text{ such that } T_n \to \left(\begin{smallmatrix} M & 0 \\ 0 & N \end{smallmatrix}\right) \in M_2(B(H)_{*-s}) = B(H \oplus H)_{*-s}) \}.$$

Now let's fix the basis $\mathfrak{B}$ of Hilbert space H as

$$\{\cdots, w_n, \cdots, w_2, w_1, v_1, v_2, \cdots v_n, \cdots\}$$

Let $\theta_{x\,y}$ be a rank one projection such that $\theta_{x\,y}(z) :=< y, z > x$, and take $T_n = \sum_{i=1}^{n-1} \theta_{w_i\,w_{i+1}} + \sum_{j=1}^{n-1} \theta_{v_{j+1}\,v_j} + \theta_{w_n\,v_n}$. In fact, T_n is of the following form with respect to the basis $\mathfrak{B}$:

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 & \ddots & 0 & 0 \\ 0 & \dots & \dots & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

Then $1-T_n^*T_n = \theta_{w_1\,w_1}$, 1 $1-T_nT_n^* = \theta_{v_1\,v_1}$. Therefore $\pi((T_n))$ is a unitary element of $\mathcal{Q}(I)$ where π is the natural quotient map. But it cannot be liftable to an extremal partial isometry: Assume there is an 195195 element $(a_n) \in I$ such that $T_n + a_n$ is extremal partial isometry in $M(I)$. Since $M_{2n}(\mathbb{C})$ has stable rank one, $T_n + a_n$ must be a unitary, and it happens only when $a_n = \theta_{v_1\,w_1}$. Then $T_n + a_n$ cannot converge to the operator of the form $\left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)$.

More sophisticated example can be found as follows.

Example 6.7. Let D be the stabilization of cone or suspension. Assume there exist a projection $\mathbf{p}$ in the corona algebra of D such that which does not lift but its K_0-class does lift.(For the construction of such a projection, see the example.) If we let a be the self adjoint element which lifts $\mathbf{p}$ in $M(D)$, we take I to be the C^*-algebra generated by a so that the quotient I/D is isomorphic to $\mathbb{C}$. Then the Busby invariant is determined by sending 1 to $\mathbf{p}$, and we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D & \xrightarrow{j} & I & \longrightarrow & \mathbb{C} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow \mathbf{p} & & \\ 0 & \longrightarrow & D & \longrightarrow & M(D) & \longrightarrow & \mathcal{C}(D) & \longrightarrow & 0 \end{array}$$

By the long exact sequence, we have

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_0(I) & \longrightarrow & K_0(\mathbb{C}) & \xrightarrow{\partial_0} & K_1(D) \\ & & \downarrow & & \downarrow \mathbf{p} & & \| \\ 0 & \longrightarrow & K_0(M(D)) & \longrightarrow & K_0(\mathcal{C}(D)) & \xrightarrow{\partial_0} & K_1(D) \end{array}$$

Since $\partial_0([\mathbf{p}]_0) = 0$, $\partial_0 : K_0(\mathbb{C}) \to K_1(D)$ becomes trivial. Thus $K_0(I) \cong K_0(\mathbb{C})$. In particular, $K_0(I)$ is non-trivial. Consequently, we found a (stably) projectionless stable rank one C^*-algebra such that its K_0-group is non-trivial. Now stabilize this algebra and call it I again. We consider an extension of I by $C(\mathbb{T})$ corresponding to a unitary u in the coronal algebra with non-trivial class K_1-class. u can't be lifted to a unitary (If so, $[u]_1 = 0$ which is a contradiction), and it can't be lifted to a partial isometry either because there are no non-zero projections available to defect projections of the partial isometry.

On the other hand, we have a certain class of C^*-algebras(e.g. elementary C^*-algebras) such that question (2) is true.

Proposition 6.8. *Let I be the C^*-algebra such that $M(I)$ is extremally rich. Suppose a unital C^*-algebra A contains I as an ideal. Then any unitary u in A/I can be liftable to an extremal partial isometry.*

Proof From the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & A/I & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \tau & & \\ 0 & \longrightarrow & I & \longrightarrow & M(I) & \longrightarrow & C(I) & \longrightarrow & 0 \end{array}$$

we see $\tau(u)$ is also a unitary in $C(I)$. Since $M(I)$ is an extremally rich C^*-algebra, we can find an extremal partial isometry w which is an inverse image of $\tau(u)$ under the map π so that we have $\pi(w) = \tau(u)$. Hence (w, u) is the extremal partial isometry in A which lifts u since we can view A as the pullback construction of A/I and $M(I)$.
■

Remark 6.9. We do not know the result of Proposition 6.8 is true under the weaker hypothesis $C(I)$ is extremally rich. In addition, I in Example 6.7 and Example 6.6 are (stable rank one) C^*-algebras such that $M(I)$ is not extremally rich.

Corollary 6.10. *Let I be the C^*-algebra such that $M(I)$ is extremally rich and any Busby invariant into $C(I)$ is extreme point preserving map. Suppose a unital C^*-algebra A contains I as an ideal. Then any extremal partial isometry u in A/I can be liftable to an extremal partial isometry.*

Proof If a map is extreme point preserving, it sends an extremal partial isometry to an extremal partial isometry. Thus the result follows from above diagram. ■

However, we do not know the example which satisfies the assumption of Corollary 6.10. The followings are counter-examples to question (3).

Example 6.11. Let

$$A = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_2(B(H)) \,\middle|\, B, C \in \mathcal{K} \right\}$$

Then $I = M_2(K)$ is an ideal of A. And $B = A/M_2(K)$ is isomorphic to $Q(H) \oplus Q(H)$. Now consider $\begin{pmatrix} S & 0 \\ 0 & T^* \end{pmatrix}$ where S and T are isometries such that $1 - SS^*$ amd $1 - TT^*$ are not compact. Then $\pi\left(\begin{pmatrix} S & 0 \\ 0 & T^* \end{pmatrix}\right)$ becomes an extremal partial isometry in $Q(H) \oplus Q(H)$ where $\pi : I \to B$. It follows that there is no extremal partial isometry which lifts $\pi\left(\begin{pmatrix} S & 0 \\ 0 & T^* \end{pmatrix}\right)$: Suppose there is $R \in M_2(K)$ such that $V = \begin{pmatrix} S & 0 \\ 0 & T^* \end{pmatrix} + R$ is an extremal partial isometry. Since A contains $M_2(K)$, either one of defect projections of V must be zero. But note that $1 - V^*V = \begin{pmatrix} 0 & 0 \\ 0 & 1 - TT^* \end{pmatrix}$ + compact $\in M_2(K)$. Therefore $1 - VV^* = 0$ implies $1 - TT^*$ is compact which is a contradiction. The other case is also similar.

The following clever example is due to Larry Brown although it was slightly generalized by the author.

Example 6.12. Let A be a unital C^*-algebra such that $RR(A) = 0, \text{tsr}(A) = 1$ and $K_1(A) = 0$. And it is known that $\text{tsr}(C[0,1] \otimes A) = 1$. Now let's consider I as

$C[0,1] \otimes (A \otimes \mathcal{K})$ which has stable rank one too. Note that $M(A \otimes \mathcal{K})_s$ is equipped with the strict topology. Then $M(I) = C([0,1], M(A \otimes K)_s)$. If we denote $[\mathcal{F}_A]$ by the set of Fredholm elements in $M(A \otimes K)$, there is Mingo's index map $\partial : [\mathcal{F}_A] \to K_0(A)$ which is actually an isomorphism. Suppose $K_0(A)$ is an ordered group and let e be an order unit. Then let $f_0 \in M(I)$ such that $f_0(1) = 1$ and $f_0(t)$ is an isometry with $\partial(f_0(t)) = -2e$ for $t > 0$. Let u be a co-isometry in $M(A \otimes K)$ with index e and f be $f_0 u$. Then we can see that $\pi(f)$ is an isometry in $\mathcal{C}(I)$. Assume there is a $k \in I$ such that $f + k$ is extremal partial isometry in $M(I)$. Since $M(A \otimes K)$ is a prime C^*-algebra, using index theory, $f(1) + k(1)$ must be co-isometry of index e. It follows that there is a unitary v such that $u + k(1) = uv$. Since $uv - u$ is in $A \otimes K$, u is a Fredholm element in $M(A \otimes K)$, and $v \in 1 + A \otimes K$. Finally, let $f_1 = fv$. let k_1 be $f + k - f_1 = f(1 - v) + k \in I$. Hence $k_1(1) = 0$. Note that $f + k = f_1 + k_1$ hence $f_1 + k_1$ must be an isometry for $t < 1$ since f has negative index for $t < 1$ and $f + k$ is an extremal partial isometry (isometry or co-isometry in this case). Since f_1 has non-trivial kernel, we know that $||k(t)|| \geq 1$ for $t < 1$ which is a contradiction.

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