TEN LECTURES OF NON-STABLE K-THEORY

Abstract. This note is based on Larry Brown's ten lectures on non-stable K-theory given at Purdue University during fall semester 2007.

1. Extremal richness of C^*-algebra

Proposition 1. *If $T \in B(H)$, then $ran(T) = ran(|T|)$.*

Proof. Since a partial isometry V in a polar decomposition of T sends $ran(|T|)$ onto $ran(T)$, if $\mathrm{ran}(T)$ is closed, then its inverse image $\mathrm{ran}(|T|)$ is closed. Coversely, since $\|T(x_n - x_m)\| = \||T|(x_n - x_m)\|$, if $T(x_n) \to y$, then $|T|(x_n)$ is a Cauchy sequence and closedness of $\mathrm{ran}(|T|)$ implies there is x such that $|T|(x_n) \to |T|(x)$. Hence $V|T|(x_n) = T(x_n) \to V|T|(x) = T(x)$. □

The following proposition is well-known but I was not able to find the reference.

Proposition 2. *$T \in B(H)$ has closed range if and only if 0 is an isolated point in $\sigma(|T|)$.*

Proof. In fact, we are going to show $|T|$ has a closed range iff 0 is an isolated point in $\sigma(|T|)$. Let 0 be an isolated point in $\sigma(|T|)$. Then there is a $\epsilon(>0)$ which is not in $\sigma(|T|) = \sigma_{ap}(|T|)$. Therefore there is $C > 0$ such that $\|(|T| - \epsilon)(x)\| \geq C\|x\|$. Let $|T|(x_n) \to y$. There is N such that $\||T|(x_n - x_m)\| \leq C\epsilon$ for $n, m \geq N$.

$$\begin{aligned}\|x_n - x_m\| &\leq \frac{1}{C}\|(|T| - \epsilon)(x_n - x_m)\| \\ &\leq \frac{1}{C}(\||T|(x_n - x_m)\| + \epsilon\|x_n - x_m\|)\end{aligned}$$

Consequently,

$$\leq \frac{C\epsilon}{C - \epsilon} < \epsilon$$

x_n is a Cauchy sequence hence $x_n \to x$ therefore $|T|(x_n) \to |T|(x) = y$ Conversely, assume 0 is an accumulation point in $\sigma(|T|)$. If $\mathrm{ran}(|T|)$ is closed, there is a C such that $\||T|(x)\| \geq C\|\overline{x}\|$ where $\|\overline{x}\| = \inf_{z \in Ker|T|}\{\|x + z\|\}$. There is $C/4 > \lambda(>0) \in \sigma_{ap}(|T|)$ by the assumtion which implies a sequence x_n such that $\|x_n\| = 1$(therefore $\|\overline{x}\| \leq 1$) such that $\|(|T| - \lambda)(x_n)\| \to 0$. If we choose x_n such that $\|(|T| - \lambda)(x_n)\| < \frac{C}{4}$, then

$$\frac{C}{4} + \frac{C}{4} > \|(|T| - \lambda)(x_n)\| + \lambda\|x_n\| \geq \||T|(x_n)\|$$

which is contradiction. □

Thoroughout this article A will denote a unital C^*-algebra, and A_1 denote the closed unit ball of A. Whenever convenient we will regard A as an algebra of operators on its universal Hilbert space H_u. Let $\mathcal{E}(A)$, or just $\mathcal{E}$, be the set of extreme points of A_1 and $\mathcal{U}(A)$ be the set of unitaries in A. Recall the characterization by R.V.Kadison that elements in $\mathcal{E}$ are the partial isometries V such that

$$(1 - VV^*)A(1 - V^*V) = 0.$$

Given a partial isometry V, we call $1 - VV^*$ and $1 - V^*V$ the defect projections of V. Thus the defect projections of V in $\mathcal{E}$ are centrally orthogonal.

Theorem 1. *The follwoing statements are equivalent.*

1. *$T \in A$ has closed range and T can be written as $V|T|$ where $V \in \mathcal{E}$ such that $kerV = kerT$.*
2. *There is an orthogonal pair of ideals I, J of A, such that T is left-invertible modulo I and T is right-invertible modulo J.*
3. *For every irreducible representation $\pi : A \to \mathcal{B}(H_\pi)$, $\pi(T)$ is either left-invertible or right-invertible, and there is $\epsilon > 0$, such that*

$$\|\pi(T)x\| \geq \epsilon\|x\| \quad x \in H_\pi$$
$$\|\pi(T^*)x\| \geq \epsilon\|x\| \quad x \in H_\pi$$

Proof. $(1) \to (2)$: Let I, J be the ideals generated by $1 - V^*V$, $1 - VV^*$ respectively. Then I, J are orthogonal. Also, since T therefore $|T|$ has closed range, $ran(|T|) = ker(|T|)^\perp$. This implies $|T| + I - V^*V$ is invertible in A, and therefore $|T| + I$ is invertible in A/I. Hence $T + I$ is left-invertible. Similarly, using $T^* = V^*|T^*|$, we can show $T^* + J$ is left-invertible which is equal to $T + J$ is right-invertible.
$(2) \to (3)$: Let π be an irreducible representation, then $ker\pi$ is prime ideal. Since $I \cap J = 0$, it follows $I \subset ker\pi$ or $J \subset ker\pi$. On the first case, π is well-defined on A/I and $\sigma(\pi(|T|+I)) \subset \sigma(|T| + I) \subset [\epsilon, \infty)$ for some $\epsilon > 0$. The other case is similar.
$(3) \to (1)$: Conditions of (3) implies that $(0. \epsilon) \cap \sigma(|T|) = \emptyset$. i.e., 0 is an isolated point in $\sigma(|T|)$. Let $e(t) = 1/t$ if $t > \epsilon$ with $e(0) = 0$. Then $V = Te(|T|) \in A$ and $T = V|T|$. □

The element satisfying one of the conditions in Theorem 1 is called *quasi-invertible*. We denote by A_q^{-1} the set of quasi-invertible elements. We say A is extremally rich if the set A_q^{-1} is dense in A.
If V_1, V_2 are partial isometries, we say V_2 extends V_1 (write $V_1 \lesssim V_2$.) if $V_2^*V_2 \geq V_1^*V_1$. i.e., $V_1 = V_2V_1^*V_1$, If we consider polar decomplosition of T as $V|T|$, we do not expect, in general, $V \in A$ (But in A''). For every $\delta > 0$, let E_δ and F_δ be the spectral projection of $|T|$ and $|T^*|$, respectively, corresponding to the open interval (δ, ∞). Note that $VE_\delta = F_\delta V$ is a partial isometry. Now we are interested in finding an extension of VE_δ.

Theorem 2. *Let $\alpha(T) = dist(T, A^{-1})$. If $\delta > \alpha(T)$, then VE_δ has an extension in $\mathcal{U}$. Furthermore, if $\delta < \alpha(T)$, then no such extension exists in $\mathcal{U}$.*

More generally, we have

Theorem 3. *Let $\alpha_q(T) = dist(T, A_q^{-1})$. If $\delta > \alpha_q(T)$, then VE_δ has an extension in $\mathcal{E}$. Furthermore, if $\delta < \alpha_q(T)$, then no such extension exists in $\mathcal{E}$.*

Corollary 1. *If $T = V|T|$ is the polar decomposition of an element of A, then each element of $Vf(|T|)$ has an extremal decomposition $Uf(|T|) = Vf(|T|)$, with $U \in \mathcal{E}(or\, \mathcal{U})$, provided that f is a continuous function on $\sigma(|T|)$ vanishing on $[0, \delta]$ for some $\delta > \alpha_q(T)(or\, \alpha(T))$.*

Proof. Note that $VE_{\delta'}f(|T|) = Vf(|T|)$ for $\delta > \delta' > \alpha_q(T)$. By applying Theorem 3 to $VE_{\delta'}$, we get the conclusion. □

Corollary 2. *If V is a partial isometry in A, then $\alpha_q(V) = 1$, or else $\alpha_q(V) = 0$, in which case $V = UV^*V = VV^*U$.*

Proof. If $\alpha_q(V) < \delta < 1$, then let $f(t) = \min\{\frac{t-\delta}{1-\delta}, 0\}$. By corollary 1, there is $U \in \mathcal{E}$ such that

$$V = Vf(P) = Uf(P) = UV^*V.$$

Since $U(P + \epsilon I) \in A_q^{-1}$ for any ϵ, it follows that $\alpha_q(V) = 0$. □

Corollary 3. *A unitial C^*-algebra A is extremally rich, then every partial isometry in A has an extremal extension.*

Theorem 4. *A unital C^*-algebra A of real rank zero is extremally rich if and only if every partial isometry in A has an extremal extension.*

Proof. For "if", it is enough to show that $\alpha_q(T) = 0$ for T in A of real rank zero with finite spectrum. In this case, 0 is isolated point $\sigma(|T|)$. and therfore $(0, \epsilon) \cap \sigma(|T|) = \emptyset$. Let $e(t) = 1/t$ if $t > \epsilon$ with $e(0) = 0$. Then $V = Te(|T|) \in A$ is a partial isometry and $T = V|T|$. By assumption, then we have an extremal extension $U \in \mathcal{E}$ of V. In addition, we have

$$T = V|T| = VV^*V|T| = UV^*V|T| = U|T|.$$

Note that $U(|T| + \epsilon I) \in A_q^{-1}$ for any $\epsilon > 0$, hence $\alpha_q(T) = 0$. We are done. □

Theorem 5. *A unital C^*-algebra is purely infinite and simple if and only if it is real rank zero and every partial isometry in A has an isomtric or co-isometric extension.*

Proof. If A is purely infinite and simple, it is real rank zero. In addition, if V is a non-zero partial isometry in A, if we let $P = V^*V$ and $Q = VV^*$, then $I - P$ and $I - Q$ are non-zero projections. Hence there is a partial isometry W such that $W^*W = I - P$ and $WW^* \leq I - Q$. $V + W$ is is an isometry which extends V. For the converse, it is enough to show every non-zero projection is infinite. Let P be a non-zero projection in A. Since P itself is a partial isometry, by the assumtion, it can have isometric extension W in A. Then

$$P = WP = PW^*$$
$$P = WPW^*$$
$$WP = PW$$

It follows that $PWW^*P \lneq P$ is Murray - von Neumann euqivalent to $W^*PW = P$, and we are done. □

Corollary 4. *A unital C^*-algebra is purely infinite and simple, then it is extremally rich.*

Proof. It follows from Theorem 4 and Theorem 5. □

In fact, since extremal richness was motivated by M. Rieffel's definition of topological stable rank one: invertibles are dense. The following characterization of stable rank one in view of extremal richness is expected.

Theorem 6. *A is a unital C^*-algebra of stable rank one if and only if it is extremally rich and $\mathcal{E}(A) = \mathcal{U}(A)$.*

Proof. It is enough to show that if V is an extreme point in A with $dist(V, A^{-1}) < 1$ then $V \in \mathcal{U}$. By applying Theorem 2 to V for $1 > \delta > dist(V,\, A^{-1})$, we have $VP = UP = V$ where $P = V^*V$. Hence $Q = VV^* = UPPU^* = UPU^*$. By assumption, $I - Q$ and $I - P$ are centrally orthogonal, and therefore $(I-Q)U(I-P) + QU(I-P) = 0$. Now $U = UP + U(I-P) = UP + (I-Q)U(I-P) + QU(I-P) = V$. We are done. □

Theorem 7. *Extremal richness passes to quotient, direct sums, direct products, and hereditary subalgebras(in particular, ideals).*

Proof. We only prove a hereditary C^*-subalgebra of extremally rich C^*-algebra is extremally rich. Let B be a hereditary C^*-subalgebra of A. Let $\widetilde{B} = \mathbb{C} + B$ and consider an element of the form $I + T$ where $T \in B$. Given ϵ, choose δ such that

$$2\delta < 1, \quad 4\delta\|T\| < 1, \quad 4\|T\|^2\delta < \epsilon$$

Since A is extremally rich, there is an element $I + S \in A_q^{-1}$ such that $\|T - S\| < \delta$. Now $(I + T - S)^{-1}$ exists and $\|D = I - (I + T - S)^{-1}\| = \|\sum_{k=1}(S-T)^k\| < \dfrac{\delta}{1-\delta} < 2\delta$. Note that

$$\begin{aligned}(I+T-S)^{-1}(I+S)(I-DT)^{-1} &= (I-D)(I+S-T+T)(I-DT)^{-1}\\ &= (I+(I-D)T)(I-DT)^{-1}\\ &= (I-DT+T)(I-DT)^{-1}\\ &= I + T\sum_{k=0}(DT)^k\\ &= I + T + TDT + TDTDT + \cdots\end{aligned}$$

Hence $C = (I+T-S)^{-1}(I+S)(I-DT)^{-1} \in \widetilde{B} \cap A_q^{-1}$.

$$\begin{aligned}\|C - (I+T)\| &= \|TDT + TDTDT + \cdots\|\\ &\le \|T\|^2\|D\|\frac{1}{1-\|DT\|}\\ &\le 2\|T\|^2\|D\|\\ &< 4\delta\|T\|^2 < \epsilon\end{aligned}$$

Since $\mathcal{E} \cap \widetilde{B} \subset \mathcal{E}(\widetilde{B})$, $C \in \widetilde{B}_q^{-1}$, and therefore we have shown $\widetilde{B}_q^{-1}$ is dense in $\widetilde{B}$. □

When p and q are projections in A, pAq can be seen as Hibert C^*-(bi)module. In fact, any Hilbert C^*-(bi)module is of the form pAq. We can extend the extremal richness to Hibert C^*-bimodule via the Sakai's characterization of extreme points in pAq:

$$U \in \mathcal{E}(pAq) \quad \text{if and only if} \quad (p - UU^*)A(q - U^*U) = 0.$$

Proposition 3. *If A is a unital extremally rich C^*-algebra, and if $1 - p \sim 1 - q$, then pAq is extremally rich.*

Proof. Let U be a partial isometry in A such that $UU^* = I - p$ and $U^*U = I - q$. If $X \in pAq \subset A$, then there is $V \in \mathcal{B}(qH, pH)$ and $|X| \in qAq$ such that $X = V|X|$. We are going to use the orthogonality of $(I-p)A(I-q) \subset \mathcal{B}((I-q)H, (I-p)H)$ and $pAq \subset \mathcal{B}(qH, pH)$. Let $S = U + X$ where $U \in (I-p)A(I-q)$ and $X \in pAq$. Then S has a polar decomposition as $(U+V)(U^*U+|X|) = (U+V)(I-q+|X|)$. Then for arbitrary $\delta > 0$, $(U+V)E_\delta^{'}$ has an extremal decomposition $U+W$. Consequently, W extends VE_δ. $\square$

Proposition 4. *If A is a unital extremally rich C^*-algebra, and p and q are defect projections, then pAq is extremally rich.*

Proof. Let $p = I - UU^*$, $p' = I - U^*U$ and $q = I - V^*V$, $q' = I - VV^*$ for some $U, V \in \mathcal{E}$. Then the ideals $\mathcal{I}$ and $\mathcal{I}^{'}$ generaged by defect projections of U are orthogonal and the ideals $\mathcal{J}$ and $\mathcal{J}^{'}$ generaged by defect projections of V are othogonal. Hence $(\mathcal{I}\cap\mathcal{J})(\mathcal{I}^{'}+\mathcal{J}^{'}) = 0$. Then the surjective map $pAq \to pAq/\mathcal{I}^{'}+\mathcal{J}^{'} = \bar{p}(A/\mathcal{I}^{'}+\mathcal{J}^{'})\bar{q}$ is an isomorphism. The latter is extremally rich by Theorem7 and Proposition 3 with the fact $\bar{p}$ and $\bar{q}$ are Murray-von Neumann equivalent. $\square$

The following theorem says that extremal richness is *stable* property.

Theorem 8. *If a unital C^*-algebra A is extremally rich, then so is $A\otimes M_n$ more generally $A\otimes\mathcal{K}$.*

Proof. Consider an element

$$\alpha = \begin{pmatrix} T & S \\ R & Q \end{pmatrix} \in M_2(A)$$

Since T is approximated by a quasi-invertible UH where $U \in \mathcal{E}$, $H \in A^{-1}$,

$$\begin{pmatrix} UH & S \\ R & Q \end{pmatrix}\begin{pmatrix} H^{-1} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} U & S \\ RH^{-1} & Q \end{pmatrix} = \begin{pmatrix} U & D \\ R_1 & Q \end{pmatrix}$$

$$\begin{pmatrix} I & 0 \\ -R_1U^* & I \end{pmatrix}\begin{pmatrix} U & S \\ R_1 & Q \end{pmatrix}\begin{pmatrix} I & -U^*S \\ 0 & I \end{pmatrix} = \begin{pmatrix} U & (I-UU^*)S \\ R_1(I-U^*U) & Q_1 \end{pmatrix}$$

Let p_u, q_u be defect projections of U and note that $p_uAq_u = 0$. Since Q_1 is approximated by a quasi-invertible VG where $V \in \mathcal{E}$, $G \in A^{-1}$,

$$\begin{pmatrix} U & p_uS \\ R_1q_u & VG \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & G^{-1} \end{pmatrix} = \begin{pmatrix} U & p_uS_1 \\ R_1q_u & V \end{pmatrix}$$

If p_v, q_v be the defect projections of V, recall that V is a partial isometry.

$$\begin{pmatrix} I & -p_uS_1V^* \\ 0 & I \end{pmatrix}\begin{pmatrix} U & p_uS_1 \\ R_1q_u & V \end{pmatrix}\begin{pmatrix} I & 0 \\ -V^*R_1q_u & I \end{pmatrix} = \begin{pmatrix} U & p_uS_1q_v \\ p_vR_1q_u & V \end{pmatrix}$$

Since $p_uA_1q_v$, $p_vA_1q_u$ are extremally rich by Proposition 4, we can approximate $p_uS_1q_v$, $p_vR_1q_u$ by YN, XM where X and Y are extreme point, and N and M are invertibles in q_vAq_v and q_uAq_u repectively

$$\begin{pmatrix} U & YN \\ XM & V \end{pmatrix}\begin{pmatrix} M^{-1}+U^*U & 0 \\ 0 & N^{-1}+V^*V \end{pmatrix} = \begin{pmatrix} U & Y \\ X & V \end{pmatrix} = Z$$

It is easy to check that Z is an extremal partial isometry in $M_2(A)$ such that

$$\begin{pmatrix} I-UU^*-YY^* & 0 \\ 0 & I-VV^*-XX^* \end{pmatrix} \text{ and } \begin{pmatrix} I-U^*U-X^*X & 0 \\ 0 & I-V^*V-Y^*Y \end{pmatrix}$$

are defect projections. $\square$

Corollary 5. *If A is a unital extremally rich C^*-algebra, and $[p]=[q]$ in $K_0(A)$ where p, q are projections in A, then pAq is extremally rich.*

Proof. Since $[p] = [q]$, $P = 1_r \oplus p \sim 1_r \oplus q = Q$, then $1_{r+1} - (1_{r+1} - P) \sim 1_{r+1} - (1_{r+1} - Q)$. Since $(1_{r+1} - P)M_{r+1}(A)(1_{r+1} - Q) \simeq pAq$, by Theorem 3 and Theorem 8, pAq is extremally rich. □

Theorem 9. *Let $\mathcal{I}$ be an ideal of A. Suppose $\mathcal{I}$ and $A/\mathcal{I}$ are extremally rich, then A is extremally rich if and anly if*

(i) $\mathcal{E}(A/\mathcal{I}) = \pi(\mathcal{E})$ *where* $\pi : A \to A/\mathcal{I}$.
(ii) pAq *are extremally rich for every defect projection* p *in* A *and defect projection* q *in* $\widetilde{\mathcal{I}}$.

We denote by $\mathcal{D}(A)$ *the defect ideal* of A which is defined as the ideal generated by all defect projections of extremal partial isometries in A. Note that if A is extremally rich, $\mathcal{D}(A)$ is the smallest ideal such that $A/\mathcal{D}(A)$ has stable rank one.(*proof.* From above Theorem, $A/\mathcal{D}(A)$ is extremally rich, and $\mathcal{E}(A/\mathcal{D}(A)) = \pi(\mathcal{E}) = \mathcal{U}(A/\mathcal{D}(A))$ by the definition of $\mathcal{D}(A)$. Hence $A/\mathcal{D}(A)$ has stable rank one by Theorem6. Furtheremore, If $\mathcal{I}$ is an ideal such that $A/\mathcal{I}$ has stable rank one, then again $\pi(\mathcal{E}) = \mathcal{U}(A/\mathcal{I})$. If we consider a defect projection p of the form $I - U^*U$ for some $U \in \mathcal{E}$, then $\bar{p} = \pi(I - U^*U) = 0$ which is equivalent to $p \in \mathcal{I}$. Hence $\mathcal{D}(A) \subset \mathcal{I}$.)

2. Weak cancellation

Definition 1. *A C^*-algebra A has weak cancellation if any pair of projections p, q in A that generate the same closed ideal $\mathcal{I}$ of A and have the same image in $K_0(\mathcal{I})$ must be Murray-von Neumann equivalent in A(hence in $\mathcal{I}$). If $M_n(A)$ has weak cancellation for every n, equivalently $A \otimes \mathbb{K}$ has weak canellation, we say A has* stable weak cancellation.

Example 1. *A purely infinite simple C^*-algebra has weak cancellation. Here is the sketch of the proof based upon Cuntz. First, note that if A is simple then a non-zero projection p is infinite if and only if p is properly infinite. Using the fact that $[p] = [q]$ in $K_0(A)$ implies that $p \sim q$ for two full properly infinite projections p, q and every non-zero projection in a purely infinite simple C^*-algebra is infinite, we can deduce the conclusion.*

Example 2. *A C^*-algebra with stable rank 1 has weak cancellation.*

Brown and Pedersen have proved a theorem which generalize both cases. Before stating the theorem, we need some technical lemmas.

Lemma 1. *The ideal generated by pAq is the intersection of ideals generated by p, q respectively.*

Proof. $\overline{span(ApAqA)} = \overline{span(ApAAqA)} = \overline{span(ApA)span(AqA)}$ □

Lemma 2. *If $U \in \mathcal{E}(pAq)$, then the ideal generated by U is equal to the ideal generated by pAq. Consequently, the ideal generated by U is equal to the intersection of ideals generated by p, q respectively.*

Proof. Recall that S.Sakai's criterion for $U \in \mathcal{E}(pAq)$, i.e., $(p - UU^*)A(q - U^*U) = 0$. Hence $\overline{span(pAq)} \subset \overline{span(AUA)}$. □

Lemma 3. *Let A be extremally rich C^*-algebra. Suppose p, q are full projections in A such that $[p] = [q]$ in $K_0(A)$, and there is $U \in \mathcal{E}(pAq)$ such that projections $p_1 = p - UU^*$ and*

$q_1 = q - U^*U$ are in the same ideal $\mathcal{I}$ of A. Furthermore, if $eAe/e\mathcal{I}e$ has K_1- surjectivity for every full projection e in A. then

$$p \sim p_2 \oplus e_2 \quad \textit{and} \quad q = q_2 + e_2$$

for some full projection e_2 in A and projections p_2, q_2 in $\mathcal{I}$, with $[p_2] = [q_2]$ in $K_0(\mathcal{I})$.

Proof. The element $[p_1] - [q_1]$ in $K_0(\mathcal{I})$ is sent to 0 by the map $K_0(\mathcal{I}) \to K_0(A)$ since $[p] = [p_1] + [UU^*] = [q_1] + [U^*U] = [q]$ in $K_0(A)$. Then there is an $\alpha \in K_1(A/\mathcal{I})$ such that $\partial_1(\alpha) = [p_1] - [q_1]$ where $\partial_1 : K_1(A/\mathcal{I}) \to K_0(\mathcal{I})$. Note that $e = U^*U$ is full projection by Lemma 2, and therefore $\pi(e)$ is full projection in $A/\mathcal{I}$. Since $\pi(e)A/\mathcal{I}\pi(e) = eAe/e\mathcal{I}e$ is full hereditary C^*-algebra of $A/\mathcal{I}$, $K_1(A/\mathcal{I}) = K_1(eAe/e\mathcal{I}e)$ which statisfies K_1-surjectivity. Hence there is an $\widetilde{V} \in \mathcal{U}(A/\mathcal{I})$ such that $\alpha = [\widetilde{V}]$. Since $\widetilde{V}$ is liftable to a partial isometry V in $e\mathcal{I}e$ by Theorem 9, $\partial_1(\alpha) = [e-V^*V]-[e-VV^*]$ in $K_0(\mathcal{I})$. Hence we have $[e-V^*V]-[e-VV^*] = [p_1] - [q_1]$ which is equivalent to $[e - V^*V] + [q_1] = [p_1] + [e - VV^*]$. Let $e_2 = V^*V$, $q_2 = q-e_2$ and $p_2 = p_1+U(e-VV^*)U^*$. Then $p = p_1+U(e-VV^*)U^*+UVV^*U^* \sim p_2 \oplus e_2$. Evidently, $[p_2] = [q_2]$ □

Lemma 4. *Let $\mathcal{I}$ be the ideal generated by projections $\{p_1, p_2, \cdots\}$. Then*
- (i) *If $q \in \mathcal{I}(or \mathcal{I} \otimes \mathbb{K})$, then $q \precsim \bigoplus p_{n_i}$.*
- (ii) *If $[p] = [q]$ in $K_0(\mathcal{I})$, then $p \oplus \bigoplus p_{n_i} \sim q \oplus \bigoplus p_{n_i}$.*

Theorem 10. *For an extremally rich C^*-algebra A the following conditions are equivalent.*
- (i) *A has weak cancellation.*
- (ii) *If $B = pAp$ for some projection p in A and $W \in \mathcal{E}(M_2(B))$, there is a projection q in B such that*
$$q \sim_0 (p \oplus p) - WW^*$$
- (iii) *If $B = pAp$ for some projection p in A, and $\{U_1, U_2, \cdots, U_n\}$ is a finite subset of $\mathcal{E}(B)$, there is a projection q in B such that*
$$q \sim_0 \bigoplus_{i=1}^{n} (p - U_iU_i^*) \quad \textit{in} \quad M_n(B)$$

Proof. *(i)* $\Rightarrow$ *(ii)* In view of Theorem 8, we may assume $W \in \mathcal{E}(M_2(B))$ is of the form

$$\begin{pmatrix} U & Y \\ X & V \end{pmatrix}$$

where U, V are in $\mathcal{E}(B)$, $Y \in \mathcal{E}(p_uAq_v)$, and $X \in \mathcal{E}(p_vAq_u)$. Hence

$$(p \oplus p) - WW^* = p - UU^* - YY^* \oplus p - VV^* - XX^*.$$

Then $p - XX^*$ and $p - X^*X$ has the same image in $K_0(I)$ for any ideal $\mathcal{I}$ containing p. To show they generate same ideal in A it suffices to show they generate B inside B(Why? $\overline{span(pAp(p - X^*X)pAp)} = \overline{span(pAp)} = \overline{span(pAp(p - XX^*)pAp)}$ implies that $\overline{span(Ap(p - XX^*)pA)}$ contains $p - X^*X$ and $\overline{span(Ap(p - X^*X)pA)}$ contains $p - XX^*$.) Note that $p - XX^* \geq p - (p - UU^*) = UU^*$ and UU^* is full in B by the Lemma 2. Similarly, $p - Y^*Y \geq p - (p - V^*V) = V^*V$ and V^*V is full in B. Therefore $p - XX^*$ and $p - X^*X$ are full in B. By the assumtion, there is a partial isometry v such that

$$v^*v = p - YY^* \quad \text{and} \quad vv^* = p - Y^*Y.$$

Thus $e = v(p_v - XX^*)v^*$ is equivalent to to $p_v - XX^*$,and since $e \leq vv^* = p - X^*X$ is centrally orthogonal $q_u - X^*X$ it follows that $e \leq (p - X^*X) - (q_u - X^*X) = p - q_u = U^*U$. Then $q = p_u - YY^* + UeU^*$ which is equivalent $p \oplus p - WW^*$.
$(ii) \Rightarrow (iii)$ Use induction on n. Case $n = 1$ is obvious. Now apply the assumtion to $\{U_1^2, \cdots, U_{n-1}^2\}$ and we can find find q_1 in B such that $q_1 \sim_0 \bigoplus_{i=1}^{n-1}(p - U_i^2(U_i^*)^2) = \bigoplus_{i=1}^{n-1}(p - U_iU_i^*) \oplus \bigoplus_{i=1}^{n-1}(p - U_iU_i^*)$. There is $q \geq q_2 \sim \bigoplus_{i=1}^{n-1}(p - U_iU_i^*)$ and $q \geq q_3 \sim \bigoplus_{i=1}^{n-1}(p - U_iU_i^*)$ such that $q_2 + q_3 = q$. In particular, $q_2 \sim q_1 - q_2 \leq p - q_2 = p_1$, and therefore $q_2 \precsim p_1$. Set $B_1 = p_1Ap_1$. Since $p = p_1 + q_2 \precsim 2p_1$, there is a partial isometry w such that $w^*w = p \oplus 0$ and $e = ww^* \leq p_1 \oplus p_1$. It follows that if we define $W = p_1 \oplus p_1 - e + w(U_n \oplus 0)w^*$ then $W \in \mathcal{E}(M_2(B_1))$. By (ii) there is $q_0 \sim_0 (p_1 \oplus p_1) - WW^* = w(p - U_nU_n^* \oplus 0)w^* \sim_0 p - U_nU_n^*$. Now $q_2 + q_0$ is what we are looking for.
$(iii) \Rightarrow (ii)$ Consider projections p and q which generate the same ideal $\mathcal{I}$ such that $[p] = [q]$ in $K_0(\mathcal{I})$. Note that if $p \in \mathcal{I}$ then $pAp = p\mathcal{I}p$. Therefore we may assume p and q are full projections in A (if necessary, change A into $\mathcal{I}$ in condition (iii)). Since $[p] = [q]$ in $K_0(A)$, pAq is extremally rich by Corollary 5. Take $U \in \mathcal{E}(pAq)$ and define $p_1 = p - UU^*$ and $q_1 = q - U^*U$. Then $[p_1] = [q_1]$ in $K_0(A)$. If $\pi : A \to A/\mathcal{D}(A)$, we want to show p_1 and q_1 belong to $\mathcal{D}(A)$. Since $eAe/e\mathcal{D}(A)e$ which is stable rank one satisfies K_1-surjectivity, by Lemma 3,

$$p \sim p_2 \oplus e_2, \quad \text{and} \quad q \sim q_2 + e_2$$

for some full projection e_2 in A and projections p_2, q_2 in $\mathcal{D}(A)$. Then $e_2\mathcal{D}(A)e_2 = \mathcal{D}(e_2Ae_2)$ which is generated by $\{e_2 - V_iV_i^*\}$ is actually full hereditary C^*-sublagebra of $\mathcal{D}(A)$. Therefore the set $\{e_2 - V_iV_i^*\}$ generate $\mathcal{D}(A)$. By Lemma 4,

$$p_2 \precsim \bigoplus(e_2 - V_iV_i^*) \quad \text{and} \quad q_2 \precsim \bigoplus(e_2 - V_iV_i^*)$$

$$p_2 \bigoplus(e_2 - V_iV_i^*) \sim q_2 \bigoplus(e_2 - V_iV_i^*)$$

. Applying condition (iii) we find the projection $q_0 \sim_0 \bigoplus(e_2 - V_iV_i^*)$. Then $p \sim_0 p_2 \oplus q_0 \oplus e_2 - q_0 \sim_0 q_2 \oplus q_0 \oplus e_2 - q_0 \sim_0 q_2 + e_2 \sim_0 q$. □

Corollary 6. *If A is isometrically rich C^*-algebra, then it has weak cancellation.*

Proof. we will show condition (iii) holds for this case. Given $\{U_1, U_2, \cdots, U_n\}$, let $q = p - (\prod U_i)(\prod U_i)^*$. □

By direct application of this Corollary, we know that if A has stable rank one then A has weak cancellation. As we have promised at the beginning of this section, we also show purely infinite, simple C^*-algebra has weak cancellation.

Corollary 7. *If A is extremally rich, and if $\mathcal{D}(\mathcal{I}) = \mathcal{I}$ whenever $\mathcal{I}$ is the left defect ideal of some $u \in \mathcal{E}(A)$, then A has weak cancellation.*

In particular, the hypothesis of above Corollary is met if every defect projection is properly infinite or A is purely infinite in the Brown-Pedersen sense.
Another remarkable fact that Brown and Pedersen have proved is the the following theorem.

Theorem 11. *If A is extremally rich C^*-algebra of real rank zero, A has weak cancellation.*

Before giving a proof, we need some lemmas.

Lemma 5. *Let A be a unital C^*-algebra, and if p, q, q_0 be projections such that $q \sim q_0 \leq p$, and $1 - p \sim 1 - q \sim 1$, then $1 - q_0 \sim 1$.*

Proof. $1 - q_0 = 1 - p + p - q_0 \sim 1 \oplus p - q_0 \sim 1 - q + q \oplus p - q_0 \sim 1 - q \oplus q_0 \oplus p - q_0 \sim 1 - p + q_0 + p - q_0$. □

Lemma 6. *Let A be a unital C^*-algebra with $RR(A) = 0$. If two projections p, q satisfy $q_0 \sim q$ and $1 - q_0 \sim 1 - q \sim 1$ in A, then $1 - q \sim 1 - q_0$ in $\widetilde{\mathcal{I}}$ where $\mathcal{I}$ is the ideal generated by q_0 (hence q also).*

Proof. Let u, v be isometries such that range projections uu^*, vv^* are $1-q, 1-q_0$ respectively and w be a partial isometry from q_0 to q. If $\pi : A \to A/\mathcal{I}$, note that $\pi(u)$ and $\pi(v)$ are unitaries in $A/\mathcal{I}$. Hence $[\pi(uv^*vu^*)] = 0$ in $K_1(A/\mathcal{I})$. Since $A/\mathcal{I}$ is also real rank zero, it satisfies K_1-injectivity by H.Lin. There is an element $\pi(u_0)$ in $\mathcal{U}_0(A/\mathcal{I}) = \pi(\mathcal{U}_0(A))$ such that $[\pi(u_0)] = [\pi(uv^*vu^*)]$. Now $w_1 = u_0(uv^* + w)$ and consider $v_1 = vw_1u^*$. It is easy to check that $\pi(v_1) = 1$ and $v_1^*v_1 = uw_1^*v^*vw_1u^* = uw_1^*w_1u^* = uu^*, v_1v_1^* = vw_1u^*uw_1^*v^* = vv^*$. □

Now we begin the proof of Theorem 11

Proof. We are going to prove the condition(ii) of Theorem 10. Let $B = pAp$ for some projeciton p in A. $W \in \mathcal{E}(M_2(B))$ is of the form

$$\begin{pmatrix} U & Y \\ X & V \end{pmatrix}$$

where U, V are in $\mathcal{E}(B)$, $Y \in \mathcal{E}(p_uAq_v)$, and $X \in \mathcal{E}(p_vAq_u)$. Hence

$$(p \oplus p) - WW^* = p - UU^* - YY^* \oplus p - VV^* - XX^*.$$

Now to construct q in B such that $q \sim_0 (p \oplus p) - WW^* = p - UU^* - YY^* \oplus p - VV^* - XX^*$ we will show that there is a projection p_0 such that $p - VV^* - XX^* \sim p_0 \leq p - (p - UU^*)$. Let $\mathcal{J}$ be ideal generated by $p - UU^* - YY^*, p - VV^* - XX^*$. Then by replacing A by $A/\mathcal{J}^\perp$ we can assume that $q_u - X^*X = 0$. i.e., $q_u = X^*X$ since $q_u - X^*X \perp \mathcal{J}$. Now let $q_0 = XX^*$. Then $q_u \sim q_0 \leq p_v$. If we let $\mathcal{I}$ be the ideal generated by q_u(hence q_0), and $\pi : B \to B/\mathcal{I}^\perp$, then $\pi(p) - \pi(p_v) \sim \pi(p) - \pi(q_u) \sim 1$ since $p_u \in \mathcal{I}^\perp$ and $q_v \in \mathcal{I}^\perp$. Hence $1 - \pi(q_0) \sim 1$ by the Lemma. Now we apply second Lemma to $1 - \pi(q_u)$ and $1 - \pi(q_0)$, we can conclude $1 - \pi(q_u) \sim 1 - \pi(q_0)$ in $\pi(\widetilde{\mathcal{I}})$, and therefore $p - q_u \sim p - q_0$ in $\widetilde{\mathcal{I}}$. If we let W be the partial isometry between $p - q_u$ and $p - q_0$ then

$$p_v - XX^* = p_v - q_0 \sim W(p_v - q_0)W^* \leq p - q_u \sim p - p_u$$

□

Lemma 7. *Let A be extremally rich C^*-algebra and $\mathcal{I}$ be an ideal of A. Suppose $A/\mathcal{I}$, $\mathcal{I}$ have weak cancellation and $eAe/e\mathcal{I}e$ has K_1-surjectivity for every projection e in A. Then A has weak cancellation.*

Proof. Let p, q be projections such that $id(p) = id(q) = \mathcal{J}$ and $[p] = [q]$ in $K_0(\mathcal{J})$. We must show $p \sim q$. Since weak cancellation passes to the ideal $\mathcal{J}$, we can assume that p, q are full projections. If $\pi : A \to A/\mathcal{I}$, $\pi(p), \pi(q)$ satisfy the same condition as p, q have. Hence $\pi(p) \sim \pi(q)$. In other words, there is $u \in A$ such that $\pi(u^*u) = p, \pi(uu^*) = q$. By Corollary 5, pAq are extremally rich and $\pi(u) \in \mathcal{E}(\pi(pAq))$. This implies we can take u as the element in $\mathcal{E}(pAq)$. Also, note that $p - u^*u$, $q - uu^*$ are in $\mathcal{I}$. By Lemma 3, we obtain a full projections e_2, projections p_2 and q_2 such that $[p_2] = [q_2]$ in $K_0(\mathcal{I})$, and such that

$$p \sim p_2 \oplus e_2, \quad q \sim q_2 + e_2.$$

By applying similar argument $(iii) \to (ii)$ in the proof of Theorem 10 to p, q, we can produce again a full projection e_3, and projections p_3, q_3 in $\mathcal{D}(\mathcal{I})$ such that $[p_3] = [q_3]$ in $K_0(\mathcal{D}(\mathcal{I}))$ and

$$p \sim p_3 \oplus e_3, \quad q \sim q_3 + e_3$$

(If we set $\rho : \mathcal{I} \to \mathcal{I}/\mathcal{D}(\mathcal{I})$, from $[\rho(p_2)] = [\rho(q_2)]$ and $\mathcal{I}/\mathcal{D}(\mathcal{I})$ has strong cancellation property, we know that $\rho(p_2) \sim \rho(q_2)$. Since $\mathcal{I}$ is extremally rich, there is $v \in \mathcal{E}(p_2 \mathcal{I} q_2)$ wich implements this equivalence. Thus

$$p_2 \sim p_3^{'} \oplus e_3^{'}, \quad q_2 \sim q_3^{'} + e_3^{'}$$

where for sine full projection $e_3^{'}$ in $\mathcal{I}$ and projections $p_3^{'}$, $q_3^{'}$ in $\mathcal{D}(\mathcal{I})$ with $[p_3^{'}] = [q_3^{'}]$ in $K_0(\mathcal{D}(\mathcal{I}))$.)

Fianlly we want to produce subprojection of e_3, say e_4 such that $p_3 \oplus e_4 \sim q_3 + e_4$. We can take $\mathcal{D}(\mathcal{I})$ as the direct limit of ideals finitely generated by defect projections from $\rceil_{\ni}\mathcal{I}\rceil_{\ni}$, therefore $[p_3] = [q_3]$ in $K_0(I_j)$ where I_j is the ideal generated by projections $e_3 - w_1 w_1^*, \cdots, e_3 - w_j w_j^*$. By the Lemma 4,

$$p_3 \oplus \bigoplus e_3 - w_i w_i^* \sim q_3 \oplus \bigoplus e_3 - w_i w_i^*.$$

Since $\mathcal{I}$ has weak cancellation, by (iii) of Theorem 10, there is e_4 in $e_3 \mathcal{I} e_3$ such that $e_4 \sim_0 \bigoplus e_3 - w_i w_i^*$. Therefore we have

$$p_3 \oplus e_4 \sim q_3 + e_4$$

as we want whence

$$p \sim p_3 \oplus e_3 \sim p_3 \oplus e_4 \oplus e_3 - e_4 \sim q_3 + e_4 + e_3 - e_4 \sim q_3 + e_3 \sim q.$$

□

3. K_1-surjectivity

Let A be a unital C^*-algebra and $\mathcal{I}$ be an ideal of A. We also let $\widetilde{\mathcal{I}} = \mathcal{I} + \mathbb{C}1_A$. We define B as the set of the matices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a \in A, b, c \in \mathcal{I}, d \in \widetilde{\mathcal{I}}$.

B_{00}^{-1} will denote the connected component of 1_A in $B \cap M_2(\widetilde{\mathcal{I}})$.

Lemma 8. *Suppose $\widetilde{\mathcal{I}}$ has weak cancellation. If $v \in \mathcal{E}(\widetilde{\mathcal{I}})$ and $v^*v + w^*w = 1$ for some $w \in \mathcal{I}$, and if $\begin{pmatrix} w \\ v \end{pmatrix}$ is the 2nd column of left invertible element of B, then we have $1 - ww^* \sim vv^*$ in $\widetilde{\mathcal{I}}$.*

Proof. Note that $[1 - ww^*] = [1] - [ww^*] = [1] - [w^*w] = [1] - [1 - v^*v] = [v^*v] = [vv^*]$ in $K_0(\widetilde{\mathcal{I}})$. Hence it is enough to show $1 - ww^*$ and vv^* are full projections in $\widetilde{\mathcal{I}}$. By the Lemma 2, vv^* is a full projection. For the fullness of $1 - ww^*$, we are going to show it is full in A. Let $\mathcal{J}$ be the ideal generated by $1 - ww^*$. If it is not A, we consider $A/\mathcal{J}$. In this quotient, $ww^* = 1$. But $w = ww^*w = w(1 - v^*v)$ is in the right defect ideal of v which is centrally orthogonal to the left defect ideal $id(1 - vv^*)$. Hence $(1 - vv^*)w(1 - v^*v) = 0 = (1 - v^*v)w = 0$, and therefore $(1 - vv^*)ww^* = 1 - vv^* = 0$ in $A/\mathcal{J}$. By the hypothesis, $\begin{pmatrix} w \\ v \end{pmatrix}$ is the 2nd column of left invertible element, there is a non-zero vector $(a\ b)$ such that $av + bw = 0$ in

$A/\mathcal{J}$. Using $ww^* = 1, vv^* = 1, vw^* = 0$, we can deduce that $a = 0$ and $b = 0$ which is contradiction. □

Theorem 12. *Let A be a unital extremally rich C^*-algebra. Assume $T^*T + S^*S$ is invertible, then there is $W \in \mathcal{E}(A)$ such that $T + W^*S \in A_q^{-1}$.*

Theorem 13. *If $\mathcal{I}$ is extremally rich with weak cancellation, and if b is a left invertible element in B, then there is an element b_0 in B_{00}^{-1} such that $b_0 b = \begin{pmatrix} a & 0 \\ 0 & 1_A \end{pmatrix}$.*

Proof. Claim: Given $b = \begin{pmatrix} * & y \\ * & x \end{pmatrix}$, there is c in B_{00}^{-1} such that $cb = \begin{pmatrix} * & w \\ * & v \end{pmatrix}$ where $v \in \mathcal{E}(\widetilde{\mathcal{I}})$ and $v^*v + w^*w = 1$.

Once we prove this claim, then we can take a unitary $u = w + w_1$ where w_1 is the partial isometry such that $w_1 w_1^* = 1 - ww^*$ and $w_1^* w_1 = v^*v$ by the Lemma 8. Hence

$$\begin{pmatrix} 1 & w_1 v^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} * & v \\ * & w \end{pmatrix} = \begin{pmatrix} * & u \\ * & v \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ (1-v)u^* & 1 \end{pmatrix} \begin{pmatrix} * & u \\ * & v \end{pmatrix} = \begin{pmatrix} * & u \\ * & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} * & u \\ * & 1 \end{pmatrix} = \begin{pmatrix} * & 0 \\ * & 1 \end{pmatrix}$$

Finally, by the row operation which is null-homotopic action, we can have the element b_0 in B_{00}^{-1} such that $b_0 b = \begin{pmatrix} a & 0 \\ 0 & 1_A \end{pmatrix}$.

Let's prove the claim. By the assumption, $b^*b = \begin{pmatrix} * & * \\ * & y^*y + x^*x \end{pmatrix}$ is invertible. Thereore for some $\epsilon > 0$ $b^*b \geq \epsilon 1$ which means

$$((b^*b)(h,k),(h,k))_{H\oplus H} \geq \epsilon((h,k),(h,k))_{H\oplus H} \quad \text{for all} \quad (h,k) \in H \oplus H$$

In particular, we have

$$((b^*b)(0,k),(0,k))_{H\oplus H} = ((y^*y+x^*x)k,k)_H \geq \epsilon((0,k),(0,k))_{H\oplus H} = (k,k)_H \quad \text{for all} \quad k \in H.$$

Therefore $y^*y + x^*x$ is invertible in $\widetilde{\mathcal{I}}$. However, we can take $y_1 \in \mathcal{I}$ such that $y_1^* y_1 + x^*x$ is invertible and $\|y_1 - y\|$ small. By the Theorem, there is $w \in \mathcal{E}(\widetilde{\mathcal{I}})$ such that $w^* y_1 + x \in \widetilde{\mathcal{I}}_q^{-1}$. Then we may assume x is in $\widetilde{\mathcal{I}}_q^{-1}$, and therefore write $x = |x^*|v = (|x^*| + 1 - vv^*)v$. Then

$$\begin{pmatrix} 1 & 0 \\ 0 & (|x^*| + 1 - vv^*)^{-1} \end{pmatrix} \begin{pmatrix} * & y \\ * & x \end{pmatrix} = \begin{pmatrix} * & y \\ * & v \end{pmatrix}$$

$$\begin{pmatrix} 1 & -yv^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} * & y \\ * & v \end{pmatrix} = \begin{pmatrix} * & y(1 - vv^*) \\ * & v \end{pmatrix}$$

□