MA 527 Advanced Mathematics for Engineers and Physicists I

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Blackboard
- record grades
- submission of off-campus homework
- answer questions

Grading
- Homework 100 pts
- 2 midterms (10/3 and 11/9 in class) 2×100 = 200 pts
- Final 200 pts

Topics

Chapter 7:	Linear Algebra: Basics	6 hrs
Chapter 8:	Linear Algebra: Eigenvalue Problems	3 hrs
Chapter 4:	Systems of ODEs	5 hrs
Chapter 6:	Laplace Transforms	6 hrs
Chapter 11:	Fourier Analysis	8 hrs
Chapter 12:	Partial Differential Equations	7 hrs
		35 hrs

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Chapter 7 Linear Algebra : matrices, vectors, determinants, linear systems

7.1–7.2 matrices and vectors — operations
7.3–7.5 solving systems of linear equations
7.6–7.7 determinants
7.8 inverses of matrices
7.9 vector spaces

§7.1 Matrices and Vectors : Addition & Scalar Multiplication

Linear Systems of Equations m equations and n unknowns

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \quad \cdots\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases} \implies \vec{A}\vec{x} = \vec{b}$$

$$\vec{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \cdots & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}_{m\times n}, \quad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}, \quad \vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

matrix column vector

$$= \left[a_{jk}\right]_{m\times n}$$

- square matrix : $m = n$
- main diagonal of A : $a_{11}, a_{22}, \cdots, a_{nn}$

- **Equality of Matrix**

$$A = [a_{jk}]_{m\times n} = [b_{jk}]_{m\times n} \iff a_{jk} = b_{jk} \quad \forall j = 1, \cdots, m \quad \forall k = 1, \cdots, n$$

- **Addition**

$$A + B = [a_{jk}]_{m\times n} + [b_{jk}]_{m\times n} = [a_{jk} + b_{jk}]_{m\times n}$$

- **Scalar Multiplication** $c \in R$, $A \in R^{m\times n}$

$$cA = c[a_{jk}]_{m\times n} = [c\, a_{jk}]_{m\times n}$$

- **Rules** commulative and associative

$$A + B = B + A \qquad c(A+B) = cA + cB$$

$$(A+B)+C = A+(B+C) \qquad (c+k)A = cA + kA$$

$$c(kA) = (ck)A$$

HWs p261 #9, 12, 13

§7.2 Matrix Multiplication

e.g. linear transformation $\vec{y} = A\vec{x}$, $\vec{z} = B\vec{y}$ $> \vec{z} = \boxed{BA}\vec{x}$

$$C_{m\times p} = A_{m\times n} B_{n\times p} = \begin{bmatrix} & \cdots & & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ & \cdots & & \end{bmatrix}_{m\times n} \begin{bmatrix} \vdots & b_{1k} & \vdots \\ & b_{2k} & \\ & \vdots & \\ & b_{nk} & \end{bmatrix}_{n\times p}$$

$$= [c_{jk}]_{m\times p}, \quad \underset{\text{row vector}}{\vec{a}_j = [a_{j1} \cdots a_{jn}]_{1\times n}}, \quad \vec{b}_k = \begin{bmatrix} b_{1k} \\ \vdots \\ b_{nk} \end{bmatrix}_{n\times 1}$$

with $c_{jk} = a_{j1} b_{1k} + a_{j2} b_{2k} + \cdots + a_{jn} b_{nk} = \sum_{l=1}^{n} a_{jl} b_{lk} = \vec{a}_j \vec{b}_k$

Examples $\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}3+4+3\end{bmatrix} = \begin{bmatrix}10\end{bmatrix}$

$$\begin{bmatrix}3\\2\\1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix} = \begin{bmatrix}3 & 6 & 9\\2 & 4 & 6\\1 & 2 & 3\end{bmatrix}$$

$\underset{n\times n}{A}\underset{n\times n}{B} \neq \underset{n\times n}{B}\underset{n\times n}{A}$ in general

example $A = \begin{bmatrix}1 & 1\\100 & 100\end{bmatrix}$, $B = \begin{bmatrix}-1 & 1\\1 & -1\end{bmatrix}$ $AB = 0$ and $BA = \begin{bmatrix}99 & 99\\-99 & -99\end{bmatrix}$

- **Transposition**

$A = \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}$ $A^T = \begin{bmatrix}1 & 4\\2 & 5\\3 & 6\end{bmatrix}$

$A = [a_{jk}]_{m\times n}$ $A^T = [a_{kj}]_{n\times m} = \begin{bmatrix}a_{11} & \cdots & a_{m1}\\a_{12} & & a_{m2}\\\vdots & & \vdots\\a_{1n} & \cdots & a_{mn}\end{bmatrix}_{n\times m}$

Rules $(A^T)^T = A$, $(A+B)^T = A^T + B^T$, $(cA)^T = cA^T$, $(AB)^T = B^T A^T$

Special Matrices

symmetric $A = A^T$

skew-symmetric $A^T = -A$

upper triangular matrix $\begin{bmatrix}1 & 2\\0 & 4\end{bmatrix}$

lower " " $\begin{bmatrix}1 & 0\\3 & 4\end{bmatrix}$

diagonal matrix $D = \begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{bmatrix}$

unit matrix $I = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

Applications

Ex. 12 Weight Watching ($A x$)

$$\begin{array}{c} \\ M \\ W \\ F \\ S \end{array}\begin{array}{c} W \quad\quad B \quad\quad J \\ \begin{bmatrix} 1.0 & 0 & 0.5 \\ 1.0 & 1.0 & 0.5 \\ 1.5 & 0 & 0.5 \\ 2.0 & 1.5 & 1.0 \end{bmatrix} \end{array} \begin{array}{c} \text{cal/hr} \\ \begin{bmatrix} 350 \\ 500 \\ 950 \end{bmatrix} \end{array} = \begin{bmatrix} 825 \\ 1325 \\ 1000 \\ 2400 \end{bmatrix}$$

hrs

HW p271 #12, 14, 17, 29

§7.3 Linear Systems of Equations. Gauss Elimination.

$$\vec{A}\vec{x} = \vec{b}$$

$$\vec{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

solution

coefficient matrix   unknowns

- homogeneous system : $\vec{b} = \vec{0}$

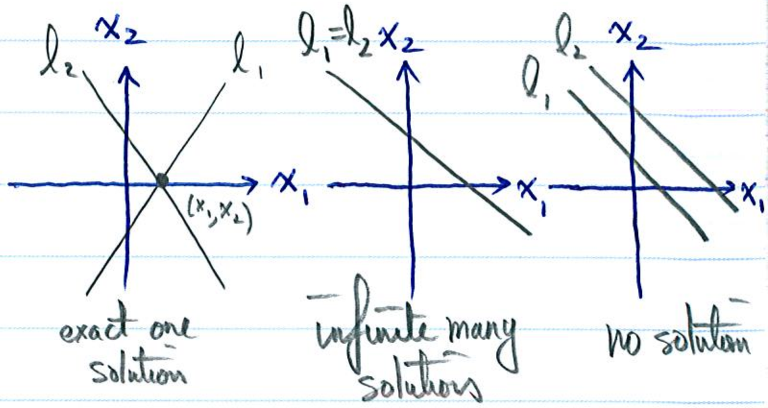
nonhomogeneous system : $\vec{b} \neq \vec{0}$

$$\tilde{A} = [\vec{A} \,\vdots\, \vec{b}]$$ augmented matrix

$m > n$	overdetermined
$m = n$	determined
$m < n$	underdetermined
consistent	at least one solution
inconsistent	no solutions

Ex. 1 Geometric Interpretation

$$\begin{aligned} \text{line } l_1: & \begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases} \\ \text{line } l_2: & \end{aligned}$$



- Gauss Elimination and Back Substitution

(179 CE "The Nine Chapters on the Mathematical Art"
1670 Newton, 1810 Gauss)

(a unique solution)

Ex. $\begin{cases} 2x_1 + 5x_2 = 2 \\ -4x_1 + 3x_2 = -30 \end{cases}$ ② $\left[\begin{array}{cc:c} 2 & 5 & 2 \\ -4 & 3 & -30 \end{array}\right] \xrightarrow{GE} \left[\begin{array}{cc:c} 2 & 5 & 2 \\ 0 & 13 & -26 \end{array}\right]$ BS $\begin{array}{l} x_1 = 6 \\ x_2 = -2 \end{array}$

Ex.2 $\begin{cases} x_1 - x_2 + x_3 = 0 \\ -x_1 + x_2 - x_3 = 0 \\ \quad 10x_2 + 25x_3 = 90 \\ 20x_1 + 10x_2 \quad = 80 \end{cases}$ $\left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right] \rightarrow \left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & \cancel{30} & \cancel{-20} & 80 \end{array}\right]$

$-95 \quad -190$

- Elementary Row Operations. (not for Columns)

(1) Interchange two rows — equation

(2) Addition of a const multiple of one row to another row

(3) Multiplication of a row by a nonzero const. c

Def. Linear systems S_1 and S_2 are row-equivalent.

$\iff$ S_1 can be obtained from S_2 by row-operations.

Thrm Row-equivalent systems have the same set of solutions.

Ex. 3 (infinite many solutions)

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 + x_2 = 1 \\ x_1 = 1 - x_2 \\ x_2 = x_2 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{array}{r} \\ -0.2 \\ -0.4 \end{array} \left[\begin{array}{cccc|c} 3 & 2 & 2 & -5.0 & 8 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{cases} 3x_1 + 2x_2 + 2x_3 - 5x_4 = 8 \\ x_2 + x_3 - 4x_4 = 1 \end{cases}$$

$$\begin{cases} 3x_1 + 2x_2 = 8 - 2x_3 + 5x_4 \\ x_2 = 1 - x_3 + 4x_4 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ -1 \end{pmatrix} + x_4 \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

Ex. 4 (no solution)

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 2 \end{array}\right] \Rightarrow \begin{cases} x_1 + x_2 = 1 \\ 0 = 2 \Rightarrow \text{no solution} \end{cases}$$

$$\begin{array}{r} \\ -\frac{2}{3} \\ -2 \end{array} \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{array}\right] \qquad 0 = 12 \quad \text{no solution}$$

- Row Echelon Form for A and $[A|b]$

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{array}\right]$$

reduced row echelom form $\begin{bmatrix} \boxed{1} & \frac{2}{3} & \frac{1}{3} \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$A\vec{x} = \vec{b}$ row-equivalent $R\vec{x} = \vec{f}$

$$[A|\vec{b}] \longrightarrow [R|\vec{f}] = \left[\begin{array}{cccccc|c} r_{11} & r_{12} & \cdots & & \cdots & r_{1n} & f_1 \\ & r_{22} & \cdots & & \cdots & r_{2n} & f_2 \\ & 0 & \ddots & & & & \vdots \\ & & & r_{rr} & \cdots & r_{rn} & f_r \\ & & & & & & \vdots \\ & & & 0 & & & f_m \end{array}\right]$$

where $r_{11} \neq 0$ and $r \leq m$.

- <u>Rank</u> $r(A) = r(R) =$ # of nonzero rows
- <u>consistent</u> $r = m$ or $r < m$ and $f_{r+1} = \cdots = f_m = 0$
- <u>No solution</u> $r < m$ and at least one of $f_{r+1}, \cdots, f_m$ is not zero.
- <u>Unique solution</u> consistent and $r = n$.
- <u>Infinitely many solutions</u> consistent and $r < n$.

<u>HW</u> p280 # 3, 9, 18

§7.4 Linear Independence. Rank of a Matrix. Vector Space.

fundamental linear algebra concepts $\begin{cases} \text{linear indep.} \\ \text{rank} \end{cases}$

Linear Indep. and Dep. of Vectors

$\vec{a}_1, \cdots, \vec{a}_m$ m vectors with the same number of components

- linear combination $c_1\vec{a}_1 + \cdots + c_m\vec{a}_m$ with c_i — any scalars
- linearly independent

$$c_1\vec{a}_1 + \cdots + c_m\vec{a}_m = \vec{0} \Rightarrow c_1 = \cdots = c_m = 0.$$

e.g. $\vec{a}_1 = (1,0)$ and $\vec{a}_2 = (0,1)$, then $c_1\vec{a}_1 + c_2\vec{a}_2 = \vec{0} \Rightarrow c_1 = c_2 = 0$ $(c_1, c_2) = c_1\vec{a}_1 + c_2\vec{a}_2$

- linearly dep. $\exists\, c_i$ not all zero s.t. $c_1\vec{a}_1 + \cdots + c_m\vec{a}_m = \vec{0}$

Ex. $\vec{a}_1 = \begin{pmatrix}1\\1\end{pmatrix}^t$ $\vec{a}_2 = \begin{pmatrix}2\\2\end{pmatrix}^t$ $\Rightarrow \vec{a}_1, \vec{a}_2$ are l. dep. since $\vec{a}_1 = 2\vec{a}_2$.

Rank of a Matrix

rank A or $r(A)$ = the max. number of l. indep row vector of A.

Ex. $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \Rightarrow r(A) = 1$

Thrm Row-equivalent matrices have the same rank.

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Thrm 2 Let $\vec{a}_1, \cdots, \vec{a}_p$ be vectors with n components, and let $A = \begin{bmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_p \end{bmatrix}$

(1) $r(A) = p \Rightarrow \{\vec{a}_1, \cdots, \vec{a}_p\}$ are l. indep.

(2) $r(A) < p \Rightarrow \{\vec{a}_1, \cdots, \vec{a}_p\}$ are l. dep.

Thrm 3 (1) $r(A)$ = the max # of l. indep. column vector of A.

(2) $r(A) = r(A^t)$ proof on p285

Thrm 4 Let $\vec{a}_1, \cdots, \vec{a}_p$ be vectors with n components.

$n < p \Rightarrow \{\vec{a}_1, \cdots, \vec{a}_p\}$ are l. dep.

Proof $A = \begin{bmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_p \end{bmatrix}_{p \times n}$ $\quad r(A) \leq p$

$r(A) \leq n < p \Rightarrow$ l. dep. #

Vector Space V

(1) V is a nonempty set of vectors with the same components

(2) $\forall \vec{a}, \vec{b} \in V, \ \forall \alpha, \beta \in R \Rightarrow \alpha\vec{a} + \beta\vec{b} \in V$

(3) addition rules $\vec{a} + \vec{b} = \vec{b} + \vec{a}, \ \vec{a} + \vec{0} = \vec{a}, \ \vec{a} + (-\vec{a}) = \vec{0}$

$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

(4) scalar multiplication rules $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}, \ (c+k)\vec{a} = c\vec{a} + k\vec{a}$

$c(k\vec{a}) = (ck)\vec{a}, \ 1\vec{a} = \vec{a}$.

<u>dimension of V</u>

$\dim(V)$ = the max. # of l. indep. vectors in V

<u>a basis of V</u>

a l. indgp. set in V consisting of a max possible # of vectors.

- $\text{span}\{\vec{a}_1, \cdots, \vec{a}_p\} = \left\{ \sum_{i=1}^{p} c_i \vec{a}_i \,\middle|\, c_i \in R \right\}$

<u>Thrm 5</u> R^n has dimension n.

$R^n = \text{span}\{[1,0,\cdots,0], \cdots, [0,\cdots,0,1]\}$

- row space of A = span{ rows of A } $= RS_A$
- column space of A = span { columns of A } $= CS_A$

<u>Thrm 6</u> dim (row space of A) = dim (column space of A) = $r(A)$

- null space of A $= \{\vec{x} \mid A\vec{x} = 0\} = N_A$
- the nullity of A $= \dim(N_A)$

rank (A) + nullity (A) = # of columns of A.

§7.5 Solutions of Linear Systems: Existence and Uniqueness

Thrm 1 (a) Existence $A_{m\times n}\vec{x}=\vec{b}$ is consistent $\Longleftrightarrow$ rank(A) = rank$(A|b)$

(b) Uniqueness $A_{m\times n}\vec{x}=\vec{b}$ has precisely one solution

$$\Longleftrightarrow \text{rank}(A)=\text{rank}(A|b)=n$$

(c) Infinitely many solutions

$$r(A)=r(A|b)<n \Longrightarrow \text{infinitely many solutions}$$

(d) Guass Elimination If solution exist, they can be obtained by Guass Elimination.

Proof (a) Let $A=[\vec{c}_1,\cdots,\vec{c}_n]$ $\vec{c}_i$ — column vectors of A.

$$\vec{b}=A\vec{x}=[\vec{c}_1,\cdots,\vec{c}_n]\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}=x_1\vec{c}_1+\cdots+x_n\vec{c}_n$$

$\Rightarrow$ $A\vec{x}=\vec{b}$ has a solution $\Longleftrightarrow$ $\vec{b}\in\text{span}\{\vec{c}_1,\cdots,\vec{c}_n\}$ $\Longleftrightarrow$ $r(A)=r(A|b)$

(b) rank$(A)=n$ $\Longleftrightarrow$ $\vec{c}_1,\cdots,\vec{c}_n$ l. indep.

Proof by Contradiction Assume $\exists$ 2 solutions $\vec{x}$ and $\vec{y}$

$$\Rightarrow \begin{array}{l}\vec{b}=A\vec{x}\\ \vec{b}=A\vec{y}\end{array} \Longrightarrow A(\vec{x}-\vec{y})=\vec{0} \Longleftrightarrow \sum_{i=1}^{n}(x_i-y_i)\vec{c}_i=0$$

$$\xrightarrow{\text{if } \vec{c}_1,\cdots,\vec{c}_n \text{ l. indep.}} x_i-y_i=0 \quad \forall i \Longrightarrow \vec{x}=\vec{y}.$$

(c) $r(A) = r(A|b) = r < n$.

Assume that $\vec{c}_1, \cdots, \vec{c}_r$ are l. indep. and that

$$\vec{c}_{r+1}, \cdots, \vec{c}_n \in \operatorname{span}\{\vec{c}_1, \cdots, \vec{c}_r\}$$

$$\Longrightarrow \quad A\vec{x} = \vec{b} \iff \sum_{i=1}^{n} x_i \vec{c}_i = \vec{b}$$

$x_{r+1}\vec{c}_{r+1} + \cdots + x_n\vec{c}_n$

$$\overset{\parallel}{\sum_{j=1}^{r} y_j \vec{c}_j} \quad \text{where } y_j = x_j + \beta_j$$

Homogeneous Linear System

Thrm 2 (a) $\underset{m\times n}{A}\,\underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{0}}$ always has the trivial solution $\underset{n\times 1}{\vec{x}} = \underset{m\times 1}{\vec{0}}$.

(b) $r = \operatorname{rank}(A) < n \iff A\vec{x} = \vec{0}$ has non-trivial solutions.

$\cancel{\iff \dim\{\vec{x} \mid A\vec{x} = 0\} = n - r}$

(c) Let $\operatorname{rank}(A) = r < n$, then

$$\underbrace{\dim\{\vec{x} \mid A\vec{x} = 0\}}_{\substack{\parallel \\ N_A}} = n - r = \overset{n}{\#\text{ of columns of } A} - r(A)$$

Non-homogeneous Linear System

If $A\vec{x} = \vec{b}$ is consistent $\Longrightarrow$ all solutions has of the form

$$\vec{x} = \underset{\text{particular solution}}{\vec{x}_0} + \underset{\text{homg. solution}}{\vec{x}_h}$$

Proof $\begin{matrix} A\vec{x} = \vec{b} \\ A\vec{x}_0 = \vec{b} \end{matrix} \Rightarrow A(\vec{x} - \vec{x}_0) = \vec{0}$

§7.6 2nd and 3rd – Order Determinants

2nd-Order $D = \det \vec{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

Cramer's Rule

$$A_{2\times2}\vec{x} = \vec{b}$$

$$x_1 = \frac{D_1}{D} \text{ and } x_2 = \frac{D_2}{D} \text{ where } D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}, D_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

3rd–Order

$$D = \det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \cdots$$

Cramer's Rule

$$A_{3\times3}\vec{x} = \vec{b}$$

$$x_i = \frac{D_i}{D}$$

§7.7 Determinants. Cramer Rules

$$D = \det A_{n\times n} = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

• $n=1$, $D = a_{11}$

• $n \geq 2$ $\quad D = a_{j1}C_{j1} + \cdots + a_{jn}C_{jn}$, $\quad j = 1, 2, \cdots, \text{or}, n$

or

$$D = a_{1k}C_{1k} + \cdots + a_{nk}C_{nk}, \quad k = 1, 2, \cdots, \text{or}, n$$

Here, $C_{jk} = (-1)^{j+k} M_{jk}$, where M_{jk} is the det. of the $(n-1)\times(n-1)$ submatrix of A by omitting j^{th}-row and k^{th}-column

cofactor of a_{jk} in D

the minor of a_{jk} in D

Calculation

(1) $\begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix}$, $\begin{vmatrix} -3 & 0 & 0 \\ 6 & 4 & 0 \\ -1 & 2 & 5 \end{vmatrix}$

(2) By row operations $\quad A \to B$ by one row operation

(a) interchange two rows, $\boxed{\text{~~the determinant changes sign~~}}$ $|A| = -|B|$

(b) addition of a multiple of a row to another row, $|A| = |B|$

(c) multiplication of a row by a $c \neq 0$, $|B| = c|A|$

p300 #13 $\begin{vmatrix} 0 & 4 & -1 & 5 \\ -4 & 0 & 3 & -2 \\ 1 & -3 & 0 & 1 \\ -5 & 2 & -1 & 0 \end{vmatrix}$

Properties

- $|A^t| = |A|$
- $A_{n\times n}$ has a zero row/column $\Rightarrow |A| = 0$
- $A_{n\times n}$ has proportional rows/columns $\Rightarrow |A| = 0$
- $A_{m\times n}$ has rank $r \geq 1$ $\Leftrightarrow$ A has an $r\times r$ submatrix $R_{r\times r}$ ~~with a nonz~~ s.t. $|R| \neq 0$.

  $\Downarrow$

  "If $S_{q\times q}$ is a submatrix of A and $q > r$, then $|S_{q\times q}| = 0$"
- $r(A_{n\times n}) = n \Leftrightarrow |A| \neq 0$.

Cramer's Rule (a) $A_{n\times n}\vec{x} = \vec{b}$

$|A| \neq 0 \Rightarrow x_i = \dfrac{D_i}{D}$ where $D = |A|$ and $D_i = |A_i|$

$A_i = [\vec{a}_1, \cdots, \vec{a}_{i-1}, \vec{b}, \vec{a}_{i+1}, \cdots, \vec{a}_n]$

(b) $A_{n\times n}\vec{x} = \vec{0}$.

$|A| \neq 0 \Rightarrow \vec{x} = \vec{0}$.

§7.8 Inverse of a Matrix. Gauss-Jordan Elimination.

inverse of $A_{n\times n}$ $\quad AA^{-1} = A^{-1}A = I$

nonsingular $\quad A$ has an inverse. $\quad$ Singular $\quad A$ has no inverse.

Thrm $\quad A_{n\times n}$ is nonsingular $\iff r(A) = n \iff \det A \neq 0$

Calculation of A^{-1} by Guass-Jordan Method

$$AA^{-1} = I \xRightarrow{X = A^{-1}} AX = I$$

$$[A \,\vdots\, I] \xrightarrow{\text{row operations}} [I \,\vdots\, A^{-1}]$$

p308 #7

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$A^{-1}$$

Formulas for Inverses

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad A^{-1} = \frac{1}{\det A}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A}\left[C_{jk}\right]^{t} = \frac{1}{\det A}\begin{bmatrix} C_{11} & \cdots & C_{n1} \\ C_{12} & \cdots & C_{n2} \\ \vdots & & \vdots \\ C_{1n} & \cdots & C_{nn} \end{bmatrix}$$

Ex. $\quad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad A^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$$\begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{a_{nn}} \end{bmatrix}$$

<u>Properties</u> $\left(A^{-1}\right)^{-1} = A$

$$(AB)^{-1} = B^{-1}A^{-1}$$

<u>Warnings</u> $AB \neq BA$ (in general)

(In general) $AB = 0 \not\Longrightarrow A = 0$ or $B = 0$

<u>Thrm</u>. • $r(\underset{n\times n}{A}) = n$ and $AB = 0 \Longrightarrow B = 0$

• $\det(AB) = \det(BA) = \det A \cdot \det B$.

HW p308 #2, 5, 20.

§7.9 Vector Space, Inner Products Spaces, Linear Transformation

<u>Real Vector Space</u>

V — a non-empty set of elements

• vector addition: (1) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$, (2) $(\vec{a}+\vec{b}) + \vec{c} = \vec{a} + (\vec{b}+\vec{c})$,

(3) $\exists\, \vec{0} \in V$ s.t. $\vec{a} + \vec{0} = \vec{a}$, (4) $\forall\, \vec{a} \in V, \exists\, -\vec{a} \in V$ s.t. $\vec{a} + (-\vec{a}) = \vec{0}$

• scalar multiplication: (1) $c(\vec{a}+\vec{b}) = c\vec{a} + c\vec{b}$, (2) $(c+k)\vec{a} = c\vec{a} + k\vec{a}$,

(3) $c(k\vec{a}) = (ck)\vec{a}$, (4) $1\vec{a} = \vec{a}$.

$\boxed{c, k \in R \text{ — real \#}}$

$\boxed{c, k \in C \text{ — complex}}$

<u>linear combination</u> $c_1 \vec{a_1} + \cdots + c_m \vec{a_m}$, $c_i \in R$ or C.

<u>l. indep. set</u> $\sum_{i=1}^{m} c_i \vec{a_i} = \vec{0} \implies c_1 = \cdots = c_m = 0$.

<u>dim</u> (V) = # of max # of l. indep. vectors.

<u>basis</u> if $\dim(V) = n$, then any n l. indep. vectors form a basis.

<u>Ex. 1</u> $\operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

<u>Ex. 2</u> $\operatorname{span}\{1, x, x^2\}$

<u>Inner Product Space</u> V — a real vector space

<u>inner product</u> $(\vec{a}, \vec{b})$

linearity (1) $\forall q_i \in R, \forall \vec{a}, \vec{b}, \vec{c} \in V \implies (q_1 \vec{a} + q_2 \vec{b}, \vec{c}) = q_1 (\vec{a}, \vec{c}) + q_2 (\vec{b}, \vec{c})$

symmetry (2) $(\vec{a}, \vec{b}) = (\vec{b}, \vec{a})$

pos.-def. (3) $\left.\begin{array}{l} (\vec{a}, \vec{a}) \geq 0 \\ (\vec{a}, \vec{a}) = 0 \Leftrightarrow \vec{a} = 0 \end{array}\right\}$

$$\vec{a} \perp \vec{b} \iff (\vec{a}, \vec{b}) = 0$$

$$\|\vec{a}\| = \sqrt{(\vec{a}, \vec{a})}$$

$|(\vec{a}, \vec{b})| \leq \|\vec{a}\| \, \|\vec{b}\|$ Cauchy-Schwarz Ineq.

$\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$ Triangle Ineq.

$\|\vec{a} + \vec{b}\|^2 + \|\vec{a} - \vec{b}\|^2 = 2(\|\vec{a}\|^2 + \|\vec{b}\|^2)$ Parallelogram eq.

<u>Exs</u> (1) $R^n = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}$

$$(\vec{a}, \vec{b}) = \vec{a}^t \vec{b} = \sum_{i=1}^{n} a_i b_i, \quad \|\vec{a}\| = \sqrt{a_1^2 + \cdots + a_n^2}$$

(2) $C = \{ f(x) \text{ defined on } [\alpha, \beta] \text{ and continuous} \}$

$$(f, g) = \int_\alpha^\beta f g \, dx, \quad \|f\| = \sqrt{\int_\alpha^\beta f^2 dx}.$$

<u>Linear Transformation</u> $F : X \to Y$ where X, Y — vector spaces

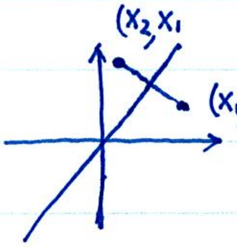
$$\text{image} \sim \vec{y} = F(\vec{x}) \in Y$$

<u>linear</u> $\begin{cases} F(\vec{v} + \vec{x}) = F(\vec{v}) + F(\vec{x}) \\ F(c\vec{x}) = cF(\vec{x}) \end{cases}$

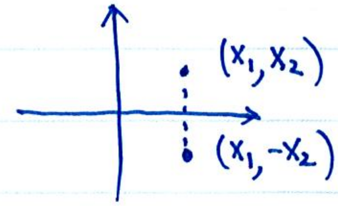
<u>Example</u> $X = R^n$ and $Y = R^m$

$F : R^n \to R^m$ is linear $\iff F = A_{m \times n}$

<u>Example</u> $\begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$



$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$



<u>Ex.</u> Find A s.t. $A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 - 5x_2 \\ 3x_1 + 4x_2 \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

<u>Find A^{-1}</u>

<u>Inverse Linear Transformation</u>

A is nonsingular $\Rightarrow A^{-1}$ exists and linear

<u>Composition of Linear Transformation</u>