Chapter 8 Matrix Eigenvalue Problems

8.1, 8.3, 8.4, and 8.5

§8.1 The Matrix Eigenvalue Problem.

$$A_{n\times n}\vec{x} = \underbrace{\lambda}_{\text{eigenvalue or characteristic value}}\ \vec{x} \sim \text{eigenvector}$$

spectrum of $A = \{\lambda \mid A\vec{x} = \lambda\vec{x}\}$

spectral radius of $A = \max\{|\lambda| \mid A\vec{x} = \lambda\vec{x}\}$

Calculation

$$(A - \lambda I)\vec{x} = \vec{0}$$

(1) eigenvalues $\det(A - \lambda I) = 0$ ∼ characteristic eq. $\Rightarrow \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$

(2) eigenvectors for each λ_i, solving $(A - \lambda_i I)\vec{x} = \vec{0}$.

Ex. #2 $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ #3 $\begin{bmatrix} 5 & -2 \\ 9 & -6 \end{bmatrix}$ ~~#14~~

Eigenspace of A corresponding to λ

$$\Lambda_A(\lambda) = \{\vec{x} \mid A\vec{x} = \lambda\vec{x}\} \cup \{\vec{0}\}$$

$$A(\alpha\vec{x}_1 + \beta\vec{x}_2) = \alpha A\vec{x}_1 + \beta A\vec{x}_2 = \lambda(\alpha\vec{x}_1 + \beta\vec{x}_2)$$

Ex. 2 $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} -(\lambda+2) & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} \longrightarrow \begin{bmatrix} -(\lambda+2) & 2(\lambda+3) & \lambda^2+2\lambda-3 \\ 2 & -(\lambda+3) & -2(\lambda+3) \\ -1 & 0 & 0 \end{bmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} 2(\lambda+3) & \lambda^2+2\lambda-3 \\ -(\lambda+3) & -2(\lambda+3) \end{vmatrix} = -(\lambda+3)\left[4(\lambda+3) - (\lambda^2+2\lambda-3)\right]$$

$$= -(\lambda+3)\left[-\lambda^2+2\lambda+15\right] = (\lambda+3)(\lambda+3)(\lambda-5)$$

$\lambda_1 = 5, \quad \lambda_2 = \lambda_3 = -3 \qquad M_{-3} = 2$ ~ M_λ = algebraic multiplicity of $\lambda = -3$

$\lambda_1 = 5, \quad \vec{x}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ $\qquad m_\lambda$ = geometric multiplicity

$\lambda = -3, \quad \vec{x}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \ \vec{x}_3 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \dim \Lambda_A(-3) = 2$

In general $m_\lambda \le M_\lambda$. $\Delta_\lambda = M_\lambda - m_\lambda$ ~ defect of λ.

Thrm $A_{n\times n}$ and A^t have the same eigenvalues.

Ex. (complex eigenvalues...) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

HW p329 #4, 11, 12, 24.

§8.3 Symmetric, Skew-Symmetric, and Orthogonal Matrices

$A_{n\times n}$ — real square matrix

symmetric $\quad A^t = A \quad$ i.e., $a_{jk} = a_{kj}$

skew-symmetric $\quad A^t = -A \quad$ .. $\quad a_{jk} = -a_{kj}$

orthogonal $\quad A^t = A^{-1}$.

Ex. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ sym $\qquad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ skew-sym $a_{ii} = 0 \qquad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ orthogonal $\qquad A^t = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$$A = \underbrace{\tfrac{1}{2}(A + A^t)}_{R} + \underbrace{\tfrac{1}{2}(A - A^t)}_{S}$$

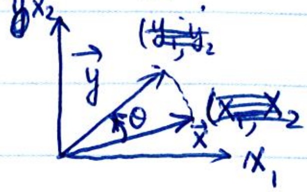
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{5}{2} & 4 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$$

Thrm (1) A is sym $\Longrightarrow$ $\lambda(A)$ is real

(2) A is skew-sym $\Longrightarrow$ $\lambda(A)$ is zero or pure imaginary.

Orthogonal Transformation and Orthogonal Matrices.

$$\vec{y} = A\vec{x} \quad \text{where } A \text{ is orthogonal matrix}$$



rotation $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Thrm (Invariance of Inner Product) A – orthogonal $\vec{a}\cdot\vec{b} = \vec{a}^t\vec{b}$

$$\vec{u} = A\vec{a} \text{ and } \vec{v} = A\vec{b} \Longrightarrow \vec{u}\cdot\vec{v} = \vec{a}\cdot\vec{b}$$

Thrm $A_{n\times n}$ is orthogonal $\Longleftrightarrow$ column vectors $\vec{a}_1, \cdots, \vec{a}_n$ form an orthonormal system $\vec{a}_j \cdot \vec{a}_k = \delta_{jk} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$

Proof $A = [\vec{a}_1, \cdots, \vec{a}_n]$

$$\Rightarrow I = A^{-1}A = A^tA = \begin{bmatrix} \vec{a}_1^{\,t} \\ \vdots \\ \vec{a}_n^{\,t} \end{bmatrix} [\vec{a}_1, \cdots, \vec{a}_n] = \begin{bmatrix} \vec{a}_1^{\,t}\vec{a}_1 & \cdots & \vec{a}_1^{\,t}\vec{a}_n \\ \vdots & & \vdots \\ \vec{a}_n^{\,t}\vec{a}_1 & \cdots & \vec{a}_n^{\,t}\vec{a}_n \end{bmatrix}.$$

Thrm ~~det~~ A is orthogonal $\Longrightarrow \det A = \pm 1$

Proof $1 = \det I = \det(A^{-1}A) = \det(A^tA) = \det A^t \det A = (\det A)^2.$

Thrm A is orthogonal $\Longrightarrow \lambda(A)$ is real or complex conjugates in pairs and $|\lambda(A)| = 1$

§8.5 Complex Matrices and Forms

$a = \alpha + i\beta$ complex conjugate $\bar{a} = \alpha - i\beta$

$A = [a_{jk}]$, $\bar{A} = [\bar{a}_{jk}]$, $\bar{A}^t$ — conjugate transpose

		real matrices
Hermitian	$\bar{A}^t = A$	sym
skew-Hermitian	$\bar{A}^t = -A$	skew-sym
unitary	$\bar{A}^t = A^{-1}$	orthogonal.

$$\begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix} \qquad \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{2}i & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2}i \end{bmatrix}$$

Hermitian — skew-Hermitian, a_{jj} — pure imaginary — unitary.

eigenvalues

- $\lambda^2 - 11\lambda + 18 = 0$, $\lambda = 9, 2$
- $\lambda^2 - 2i\lambda + 8 = 0$, $\lambda = 4i, -2i$
- $\lambda^2 - i\lambda - 1 = 0$, $\frac{1}{2}\sqrt{3} + \frac{1}{2}i, -\frac{1}{2}\sqrt{3} + \frac{1}{2}i$

Thrm 1 (a) A is Hermitian $\Rightarrow \lambda(A)$ is real

(b) .. .. skew-H $\Rightarrow \lambda(A) = 0$ or pure imaginary

(c) .. .. unitary $\Rightarrow |\lambda(A)| = 1$

Thrm 2 A is unitary $\quad \vec{a} \cdot \vec{b} = \overline{\vec{a}}^t \vec{b}$

$\Rightarrow$ (a) $\vec{u} = A\vec{a}$, $\vec{v} = A\vec{b} \Rightarrow \vec{u} \cdot \vec{v} = \vec{a} \cdot \vec{b}$

(b) $A = [\vec{a}_1, \cdots, \vec{a}_n]$ is unitary $\Leftrightarrow \vec{a}_j \cdot \vec{a}_k = \delta_{jk}$ ~ unitary system

(c) $|\det A| = 1$

§8.4 Eigenbases. Diagonalization. Quadratic Forms

Thrm: $A_{n\times n}$ has n distinct eigenvalues

$\Rightarrow$ A has a basis of eigenvectors $\vec{x}_1, \cdots, \vec{x}_n$ for R^n.

(Proof by contradiction)

Assume that $\vec{x}_1, \cdots, \vec{x}_n$ are l. indep. eigenvectors of A

$$\Rightarrow \vec{y} = A\vec{x} = A\sum c_i \vec{x}_i = \sum c_i \lambda_i \vec{x}_i .$$

Example $\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ eigenvalue $\lambda_1 = 8$, $\lambda_2 = 2$ eigenvector $\vec{x}_1 = [1, 1]^t$, $\vec{x}_2 = [1, -1]^t$

ex2 p327 $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ $\lambda_1 = 5$, $\lambda_2 = \lambda_3 = -3$ $\vec{x}_1 = [1, 2, -1]^t$, $\vec{x}_2 = [-2, 1, 0]^t$, $\vec{x}_3 = [3, 0, 1]^t$

ex3 p328 $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\lambda_1 = \lambda_2 = 0$ $\vec{x}_1 = [1, 0]^t$

Thrm: A is sym $\Rightarrow$ A has an orthonormal basis of eigenvectors for R^n

$A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ $\vec{x}_1 = \frac{1}{\sqrt{2}}[1, 1]^t$, $\vec{x}_2 = \frac{1}{\sqrt{2}}[1, -1]^t$.

Similarity of Matrices. Diagonalization

Def. $\hat{A}_{n\times n}$ is similar to $A_{n\times n} \Leftrightarrow \exists$ nonsingular P

s.t. $\underbrace{\hat{A} = P^{-1}AP}_{\text{similarity transformation}}$.

Thrm $\hat{A}$ is similar to A

$$\Rightarrow \quad \lambda(\hat{A}) = \lambda(A)$$

$$A\vec{x} = \lambda\vec{x} \iff (P^{-1}AP)P^{-1}\vec{x} = \lambda P^{-1}\vec{x} \iff \hat{A}\vec{y} = \lambda\vec{y}$$

$$\vec{y} = P^{-1}\vec{x}.$$

Thrm (Diagonalization of A)

$A_{n\times n}$ has a eigenbasis $\{\vec{x}_1, \cdots, \vec{x}_n\}$ corresponding to $\lambda_1, \cdots, \lambda_n$

$$\Rightarrow \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = X^{-1}AX \quad \text{where } X = [\vec{x}_1, \cdots, \vec{x}_n]$$

$$A\vec{x}_i = \lambda\vec{x}_i \iff AX = \Lambda X \Rightarrow \Lambda = X^{-1}AX.$$

$$\Rightarrow \Lambda^m = X^{-1}A^mX$$

Ex. Diagonalize $\begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}$

Quadratic Forms in the components $x_1, \cdots, x_n$

$$Q = \underbrace{\vec{x}^t A \vec{x}}_{\text{coefficient matrix}} = \sum_j \sum_k a_{jk} x_j x_k = a_{11} x_1^2 + \cdots + a_{1n} x_1 x_n$$

$$\cdots$$

$$+ a_{n1} x_1 x_n + \cdots + a_{nn} x_n^2$$

$A_{n\times n}$ – sym $\Rightarrow$ A has an orthonormal basis

$$\Leftrightarrow \quad A = XDX^{-1} \qquad X = [\vec{x}_1, \cdots, \vec{x}_n],\ D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

and $X^{-1} = X^t$

$$Q(\vec{x}) = \vec{x}^t \overset{\text{sym}}{A} \vec{x} = \vec{x}^t X D X^t \vec{x} = (X^t \vec{x})^t D (X^t \vec{x}) = \vec{y}^t D \vec{y}$$

$$= \underbrace{\sum_{i=1}^{n} \lambda_i y_i^2}_{\text{the principal axes form or \underline{canonical form}}} \qquad \vec{y} = X^t \vec{x}$$

Ex. (Conic Sections) $Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = \vec{x}^t \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix} \vec{x} = 128$

$\lambda_1(A) = 2,\ \lambda_2(A) = 32 \Rightarrow Q = 2y_1^2 + 32y_2^2 = 128$

$$\Rightarrow \frac{y_1^2}{8^2} + \frac{y_2^2}{2^2} = 1$$

normalized eigenvectors $\vec{x}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{x}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\Rightarrow \vec{x} = X\vec{y} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\vec{y}. \quad \text{— rotation of } 45^\circ.$$

p346 #24

#25