Name: ____________________

PUID#: ____________________

Midterm 1A (On-Campus)– Math 527 (10/03/12)

Books, notes, calculators, and any electronic devices are NOT allowed during the exam. The exam is one hour long.

Please write your answers in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Answer	C	D	E	C ?	D	C	A	E	E	A	D	C

105	96	87	79	69	53
105	96	87	78	69	
105	96	87	78	69	
105	96	87	78	69	
	96	87	78	69	
	96	87	78	69	
	96	87	78	69	
	96	87	78	69	
	96	87	78	62	
	96	87	78	62	
	96	80	71	61	
	96		71	61	
			71	61	
			70	60	
			70	60	
			70		
			70		

$4+12+11+17+15+1=60$

1. (9pts) Which of the following are vector spaces?

(i) The set of all skew-symmetric 3×3 matrices.

(ii) The set of all vectors in R^2, $[v_1, v_2]$, such that $v_1 \geq v_2$.

(iii) The set of all 3×2 matrices with first column being any multiple of $[3, 0, 5]^t$.

A. (iii) only;
B. (i) and (ii)
C. (i) and (iii)
D. (ii) and (iii)
E. (i), (ii), and (iii).

(i) $V = \{ A_{3\times3} \mid A^t = -A \}$

$\forall A, B \in V \Rightarrow (\alpha A + \beta B)^t = \alpha A^t + \beta B^t$

$= -\alpha A - \beta B = -(\alpha A + \beta B)$

$\Rightarrow \alpha A + \beta B \in V \Rightarrow V$ is a vector space

(ii) $V = \{ [v_1, v_2] \mid v_1 \geq v_2 \}$

$[v_1, v_2] \in V \Rightarrow v_1 \geq v_2$

if V is a vector space $\Rightarrow -[v_1, v_2] \in V$ but $-v_1 \leq -v_2$

(iii) $V = \left\{ \left(\alpha \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}, \vec{a}_2, \vec{a}_3 \right) \middle| \alpha \in R, \vec{a}_2, \vec{a}_3 \in R^3 \right\}$

it is a vector space

2. (9pts) Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ and let $B = A^{-1}$. Then the entry b_{13} of A^{-1} is

A. -2;
B. -1;
C. 0;
D. 1;
E. 2.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \Longrightarrow b_{13} = 1$$

3. (9pts) The vectors $[1, 2, 1]$, $[3, 4, 5]$, and $[2, -2, k]$ are linearly **dependent** if k equals

A. -5;
B. -1;
C. 0;
D. 4;
E. 8.

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 5 \\ 2 & -2 & k \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 2 \\ 0 & -6 & k-2 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 2 \\ 0 & 0 & k-8 \end{bmatrix} \Longrightarrow k=8$$

4. (9pts) The homogeneous linear system $A\mathbf{x} = \mathbf{0}$, with coefficient matrix A , has only the trivial solution. Which of the following statements must be true?
A. A is a square matrix and $\det A \neq 0$;
B. A is a square matrix and $\det A = 0$;
C. The rank of A equals the number of columns of A;
D. The rank of A equals the number of rows of A;
E. The row-echelon form of A is $\mathbf{0}$.

$$r(A) + \underbrace{\text{nullity of } A}_{=0} = n \text{ — \# of columns}$$

$$\Updownarrow$$

no nontrivial solution

5. (9pts) The product of the eigenvalues of the matrix $M = \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}$ is
A. 2;
B. 3;
C. 4;
D. 5;
E. 6.

$$\begin{vmatrix} 1-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} = (\lambda-1)(\lambda-4)+1$$
$$= \lambda^2 - 5\lambda + 5$$

$$\Rightarrow \lambda = \frac{5 \pm \sqrt{25-20}}{2} = \frac{5 \pm \sqrt{5}}{2}$$

$$\frac{5+\sqrt{5}}{2} \cdot \frac{5-\sqrt{5}}{2} = \frac{1}{4}(25-5) = 5$$

6. (9pts) The matrix $M = \begin{bmatrix} 1 & -2 \\ 5 & -5 \end{bmatrix}$ has eigenvlaue $-2+i$. Which of the following is an eigenvector of A?

$A. \begin{bmatrix} 5 \\ 5-i \end{bmatrix}$, $B. \begin{bmatrix} 5 \\ 3-i \end{bmatrix}$, $C. \begin{bmatrix} 3+i \\ 5 \end{bmatrix}$, $D. \begin{bmatrix} 5-i \\ 3 \end{bmatrix}$, $E. \begin{bmatrix} 3+i \\ 3-i \end{bmatrix}$,

$$\begin{bmatrix} 1-(-2+i) & -2 \\ 5 & -5-(-2+i) \end{bmatrix} = \begin{bmatrix} 3-i & -2 \\ 5 & -3-i \end{bmatrix}$$

$$(3-i)x_1 - 2x_2 = 0$$

$$x_1 = \frac{2}{3-i}x_2 = \frac{2(3+i)}{10}x_2 = \frac{3+i}{5}x_2$$

7. (9pts) The eigenvalues of the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & -2 \\ 2 & 0 & 5 \end{bmatrix}$ are 3 and 5. One of the associated eigenspaces has dimension one. This eigenspace has basis:

$A. \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $B. \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $C. \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $D. \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $E. \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$,

$\underline{\lambda = 3}$

$$A - \lambda I = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & -2 \\ 2 & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 + x_3 = 0$ and x_2 is arbitrary

$\underline{\lambda = 5}$

$$A - \lambda I = \begin{bmatrix} -2 & 0 & 0 \\ -2 & -2 & -2 \\ 2 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = 0 \\ x_2 + x_3 = 0 \end{cases} \longrightarrow x = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

8. (9pts) A real matrix $A_{2\times 2}$ has eigenvalues $1+i$ and $1-i$ with corresponding eigenvectors $\begin{bmatrix} i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -i \\ 1 \end{bmatrix}$. If $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is the general solution of $\mathbf{y}' = A\mathbf{y}$, and $y_2(t) = e^t\,(c_1 \cos t + c_2 \sin t)$, then $y_1(t)$ is

A. $e^{-t}\,(c_2 \cos t - c_1 \sin t)$;

B. $e^{-2t}\,(c_1 \sin t - c_2 \cos t)$;

C. $e^{t}\,(c_1 \sin t - c_2 \cos t)$;

D. $e^{t}\,(c_1 \sin t - 2c_2 \cos t)$;

E. $e^{t}\,(c_2 \cos t - c_1 \sin t)$.

$$e^{(1+i)t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$= e^{t}\left(\cos t + i \sin t\right)\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$= e^{t}\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + i e^{t}\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

$$\vec{y}(t) = c_1 e^{t}\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + c_2 e^{t}\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

$$= e^{t}\begin{bmatrix} -c_1 \sin t + c_2 \cos t \\ c_1 \cos t + c_2 \sin t \end{bmatrix}$$

9. (9pts) If a fundamental matrix for $\mathbf{y}' = A\mathbf{y}$ is $Y(t) = \begin{bmatrix} -e^{-2t} & 0 \\ e^{-2t} & e^{2t} \end{bmatrix}$, then the general solution to the system of ODEs $\mathbf{y}' = A\mathbf{y} + \begin{bmatrix} 0 \\ e^t \end{bmatrix}$ is

A. $\begin{bmatrix} -e^{-2t} & 0 \\ e^{-2t} & e^{2t} \end{bmatrix} \left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix} \right\}$,

B. $\begin{bmatrix} -e^{-2t} & 0 \\ e^{-2t} & e^{2t} \end{bmatrix} \left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} - \begin{bmatrix} 0 \\ e^{t} \end{bmatrix} \right\}$,

C. $\begin{bmatrix} -e^{-2t} & 0 \\ e^{-2t} & e^{-2t} \end{bmatrix} \left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix} \right\}$,

D. $\begin{bmatrix} -e^{-2t} & 0 \\ e^{-2t} & e^{-2t} \end{bmatrix} \left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} - \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix} \right\}$,

E. $\begin{bmatrix} -e^{-2t} & 0 \\ e^{-2t} & e^{2t} \end{bmatrix} \left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} - \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix} \right\}$.

$$Y^{-1} = \frac{1}{-1} \begin{bmatrix} e^{2t} & 0 \\ -e^{-2t} & -e^{-2t} \end{bmatrix} = \begin{bmatrix} -e^{2t} & 0 \\ e^{-2t} & e^{-2t} \end{bmatrix}$$

$$\vec{u}' = Y^{-1}\vec{g} = \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -e^{-t} \end{bmatrix}$$

$$\vec{y} = Y\vec{u} = Y\left(\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} - \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix} \right)$$

10. (9pts) For the system $\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \mathbf{y}$, the origin is

A. an unstable node;
B. a stable node;
C. a saddle point;
D. a stable spiral point;
E. an unstable spiral point.

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = (\lambda-1)(\lambda-4)+2$$
$$= \lambda^2 - 5\lambda + 6$$
$$= (\lambda-2)(\lambda-3)$$

$\lambda_1 = 2, \lambda_2 = 3$

11. (7pts) The number of critical points for the system $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} -4y_1 + y_2 \\ y_1^3 - y_2 \end{bmatrix}$ is

A. 0;
B. 1;
C. 2;
D. 3;
E. 4.

$$\begin{cases} -4y_1 + y_2 = 0 \\ y_1^3 - y_2 = 0 \end{cases} \Rightarrow y_2 = 4y_1$$

$$\Rightarrow 0 = y_1^3 - 4y_1 = y_1(y_1^2 - 4) = y_1(y_1+2)(y_1-2)$$

$$\Rightarrow y_1 = -2, 0, 2$$
$$y_2 = -8, 0, 8$$

$$(-2, -8), (0, 0), (2, 8)$$

12. (8pts) For the system $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} -4y_1 + y_2 \\ y_1^3 - y_2 \end{bmatrix}$, the critical point $(2, 8)$ is

A. an unstable node;
B. a stable node;
C. a saddle point;
D. a stable spiral point;
E. an unstable spiral point.

$$\nabla f(2,8) = \begin{bmatrix} -4 & 1 \\ 3y_1^2 & -1 \end{bmatrix}_{(2,8)} = \begin{bmatrix} -4 & 1 \\ 12 & -1 \end{bmatrix}$$

$$\begin{vmatrix} -4-\lambda & 1 \\ 12 & -1-\lambda \end{vmatrix} = (\lambda+1)(\lambda+4) - 12 = \lambda^2 + 5\lambda + 4 - 12$$
$$= \lambda^2 + 5\lambda - 8$$

$$\lambda = \frac{-5 \pm \sqrt{25+32}}{2} = \frac{-5 \pm \sqrt{57}}{2}$$

$$\lambda_1 = \frac{-5 - \sqrt{57}}{2} < 0, \quad \lambda_2 = \frac{-5 + \sqrt{57}}{2} > 0$$