Name: ________________
PUID#: ________________

Midterm 1B (On-Campus)– Math 527 (10/03/12)

Books, notes, calculators, and any electronic devices are NOT allowed during the exam. The exam is one hour long.

Please write your answers in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Answer	B	A	B	B	D	E	B	C	B	B	C	E

105	96	96	87	80	79	71	69	61	53
105	96	96	87		78	71	69	61	
105	96		87		78	71	69	61	
105	96		87		78	70	69	60	
	96		87		78	70	69	60	
	96		87		78	70	69		
	96		87		78	70	69		
	96		87		78		62		
	96		87		78		62		
	96		87		78		62		

$4 + 12 + 11 + 17 + 15 + 1 = 60$

1. (9pts) Which of the following are vector spaces?

(i) The set of all vectors in R^2, $[v_1, v_2]$, such that $v_1 \geq v_2$.

(ii) The set of all skew-symmetric 3×3 matrices.

(iii) The set of all 3×2 matrices with first column being any multiple of $[3,\ 0,\ 5]^t$.

A. (i) and (iii)
B. (ii) and (iii)
C. (i) and (ii)
D. (iii) only
E. (i), (ii), and (iii).

(i) $V = \{ (v_1, v_2) \mid v_1 \geq v_2 \}$

$(v_1, v_2) \in V \Rightarrow v_1 \geq v_2 \Rightarrow -v_1 \leq -v_2 \Rightarrow -(v_1, v_2) \notin V$

(ii) $V = \{ A_{3\times 3} \mid A^t = -A \}$

$(\alpha A + \beta B)^t = -(\alpha A + \beta B)$

(iii) $V = \left\{ (\vec{a}_1, \vec{a}_2, \vec{a}_3) \;\middle|\; \vec{a}_1 = \alpha \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} \ \forall \alpha \in R,\ \vec{a}_2, \vec{a}_3 \in R^3 \right\}$

$\gamma(\vec{a}_1, \vec{a}_2, \vec{a}_3) + \beta(\vec{b}_1, b_2, b_3)$

$= (\gamma\vec{a}_1 + \beta\vec{b}_1,\ \gamma\vec{a}_2 + \beta\vec{b}_2,\ \gamma\vec{a}_3 + \beta\vec{b}_3)$

$\gamma\vec{a}_1 + \beta\vec{b}_1 = (\gamma\alpha_1 + \beta\alpha_2)\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$, $\quad \gamma\vec{a}_2 + \beta\vec{b}_2 \in R^3$, $\quad \gamma\vec{a}_3 + \beta\vec{b}_3 \in R^3$

2. (9pts) The vectors $[1,2,1]$, $[3,4,5]$, and $[2,-2,k]$ are linearly **dependent** if k equals
A. 8;
B. 4;
C. 0;
D. -1;
E. -5.

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 5 \\ 2 & -2 & k \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 2 \\ 0 & -6 & k-2 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 2 \\ 0 & 0 & k-8 \end{bmatrix} \Rightarrow k=8$$

3. (9pts) The homogeneous linear system $A\mathbf{x} = \mathbf{0}$, with coefficient matrix A , has only the trivial solution. Which of the following statements must be true?
A. The rank of A equals the number of rows of A; B. The rank of A equals the number of columns of A;
C. The row-echelon form of A is $\mathbf{0}$;
D. A is a square matrix and $\det A \neq 0$;
E. A is a square matrix and $\det A = 0$.

$$r(A) + \underbrace{\text{nullity of } A}_{= 0} = n \sim \#\text{ of columns}$$

$$\Updownarrow$$

no non-trivial solution

4. (9pts) Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ and let $B = A^{-1}$. Then the entry b_{12} of A^{-1} is

A. -2;
B. -1;
C. 0;
D. 1;
E. 2.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$b_{12} = -1$$

5. (9pts) The matrix $M = \begin{bmatrix} 1 & -2 \\ 5 & -5 \end{bmatrix}$ has eigenvlaue $-2+i$. Which of the following is an eigenvector of A?

$A. \begin{bmatrix} 5 \\ 5-i \end{bmatrix}$, $B. \begin{bmatrix} 5 \\ 3-i \end{bmatrix}$, $C. \begin{bmatrix} 5-i \\ 3 \end{bmatrix}$, $D. \begin{bmatrix} 3+i \\ 5 \end{bmatrix}$, $E. \begin{bmatrix} 3+i \\ 3-i \end{bmatrix}$,

$$\begin{bmatrix} 1-(-2+i) & -2 \\ 5 & -5-(-2+i) \end{bmatrix} = \begin{bmatrix} 3-i & -2 \\ 5 & -3-i \end{bmatrix}$$

$$(3-i)x_1 - 2x_2 = 0$$

$$x_1 = \frac{2}{3-i} x_2 = \frac{3+i}{5} x_2$$

6. (9pts) The eigenvalues of the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & -2 \\ 2 & 0 & 5 \end{bmatrix}$ are 3 and 5. One of the associated eigenspaces has dimension one. This eigenspace has basis:

$A. \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $B. \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $C. \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $D. \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $E. \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$,

$\underline{\lambda=5}$ $\quad A-\lambda I = \begin{bmatrix} -2 & 0 & 0 \\ -2 & -2 & -2 \\ 2 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$$\begin{cases} x_1 = 0 \\ x_1+x_2+x_3=0 \end{cases} \Longrightarrow \begin{cases} x_1 = 0 \\ x_2+x_3=0 \end{cases} \Longrightarrow \vec{x} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

7. (9pts) The sum of the eigenvalues of the matrix $M = \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}$ is

A. 6;
B. 5;
C. 4;
D. 3;
E. 2.

$$\begin{vmatrix} 1-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} = (\lambda-1)(\lambda-4)+1 = \lambda^2-5\lambda+5$$

$$\lambda_{1,2} = \frac{5 \pm \sqrt{25-20}}{2}$$

$$\lambda_1+\lambda_2 = \frac{5+\sqrt{5}}{2} + \frac{5-\sqrt{5}}{2} = 5$$

8. (9pts) If a fundamental matrix for $\mathbf{y}' = A\mathbf{y}$ is $Y(t) = \begin{bmatrix} -e^{-2t} & 0 \\ e^{-2t} & e^{2t} \end{bmatrix}$, then the general solution to the system of ODEs $\mathbf{y}' = A\mathbf{y} + \begin{bmatrix} 0 \\ e^{t} \end{bmatrix}$ is

A. $\begin{bmatrix} -e^{-2t} & 0 \\ e^{-2t} & e^{2t} \end{bmatrix} \left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix} \right\}$,

B. $\begin{bmatrix} -e^{-2t} & 0 \\ e^{-2t} & e^{2t} \end{bmatrix} \left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} - \begin{bmatrix} 0 \\ e^{t} \end{bmatrix} \right\}$,

C. $\begin{bmatrix} -e^{-2t} & 0 \\ e^{-2t} & e^{2t} \end{bmatrix} \left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} - \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix} \right\}$,

D. $\begin{bmatrix} -e^{-2t} & 0 \\ e^{-2t} & e^{-2t} \end{bmatrix} \left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} - \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix} \right\}$,

E. $\begin{bmatrix} -e^{-2t} & 0 \\ e^{-2t} & e^{-2t} \end{bmatrix} \left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix} \right\}$.

$$Y^{-1} = -\begin{bmatrix} e^{2t} & 0 \\ -e^{-2t} & -e^{-2t} \end{bmatrix}, \quad \vec{u}' = Y^{-1}\vec{g} = \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -e^{-t} \end{bmatrix}$$

$$\vec{y} = Y\vec{u} = Y\left(\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} - \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix} \right)$$

9. (9pts) For the system $\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \mathbf{y}$, the origin is

A. a stable node;
B. an unstable node;
C. a saddle point;
D. a stable spiral point;
E. an unstable spiral point.

$$\begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = (\lambda-1)(\lambda-4)+2 = (\lambda-2)(\lambda-3)$$

$\lambda_1 = 2,\ \lambda_2 = 3$

10. (9pts) A real matrix $A_{2\times 2}$ has eigenvalues $1+i$ and $1-i$ with corresponding eigenvectors $\begin{bmatrix} i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -i \\ 1 \end{bmatrix}$. If $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is the general solution of $\mathbf{y}' = A\mathbf{y}$, and $y_2(t) = e^t\,(c_1 \cos t + c_2 \sin t)$, then $y_1(t)$ is

A. $e^{-t}\,(c_2 \cos t - c_1 \sin t)$;
B. $e^t\,(c_2 \cos t - c_1 \sin t)$;
C. $e^t\,(c_1 \sin t - c_2 \cos t)$;
D. $e^t\,(c_1 \sin t - 2c_2 \cos t)$;
E. $e^{-2t}\,(c_1 \sin t - c_2 \cos t)$.

$$e^{(1+i)t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$= e^t(\cos t + i \sin t)\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$= e^t\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + i e^t \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

$$\vec{y}(t) = c_1 e^t \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + c_2 e^t \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

$$= e^t \begin{bmatrix} -c_1 \sin t + c_2 \cos t \\ c_1 \cos t + c_2 \sin t \end{bmatrix}$$

11. (7pts) The number of critical points for the system $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} -4y_1 + y_2 \\ y_1^3 - y_2 \end{bmatrix}$ is

A. 1;
~~B. 2;~~
C. 3;
D. 4;
E. 5.

$$\begin{cases} -4y_1 + y_2 = 0 \\ y_1^3 - y_2 = 0 \end{cases} \Rightarrow \begin{cases} y_2 = 4y_1 \\ \Rightarrow 0 = y_1^3 - 4y_1 = y_1(y_1+2)(y_1-2) \end{cases}$$

$y_1 = -2, 0, 2$

$(-2,-8), (0,0), (2,8)$

12. (8pts) For the system $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} -4y_1 + y_2 \\ y_1^3 - y_2 \end{bmatrix}$, the critical point $(2, 8)$ is

A. a stable node;
B. an unstable node;
C. a stable spiral point;
~~D. an unstable spiral point;~~
E. a saddle point.

$$\nabla f(2,8) = \begin{bmatrix} -4 & 1 \\ 3y_1^2 & -1 \end{bmatrix}_{(2,8)} = \begin{bmatrix} -4 & 1 \\ 12 & -1 \end{bmatrix}$$

$$\begin{vmatrix} -4-\lambda & 1 \\ 12 & -1-\lambda \end{vmatrix} = (\lambda+4)(\lambda+1) - 12 = \lambda^2 + 5\lambda - 8$$

$$\lambda = \frac{-5 \pm \sqrt{25+32}}{2} = \frac{-5 \pm \sqrt{57}}{2}$$

$$\lambda_1 = \frac{-5-\sqrt{57}}{2} < 0, \quad \lambda_2 = \frac{-5+\sqrt{57}}{2} > 0$$