Jose Quinones-Torres DUE 11/14/2012

Homework # 10

Sec. 11.6 p 509

Showing the details, develop ('Develop' means to write the given function as a Fourier-Legendre series)

(1.) $63x^5 - 90x^3 + 35x$, $f(x) = \sum_{m=0}^{\infty} a_m P_m(x)$, $a_m = \frac{2m+1}{2}\int_{-1}^{1} f(x)P_m(x)dx$

$$a_0 = \frac{1}{2}\int_{-1}^{1}(63x^5 - 90x^3 + 35x)dx = \frac{1}{2}\left[\frac{63}{6}x^6 - \frac{90}{4}x^4 + \frac{35}{2}x^2\right]_{-1}^{1} = 0 \text{ (even)}$$

$$a_1 = \frac{3}{2}\int_{-1}^{1}(63x^5 - 90x^3 + 35x)x\,dx = \frac{3}{2}\left[\frac{63}{7}x^7 - \frac{90}{5}x^5 + \frac{35}{3}x^3\right]_{-1}^{1}$$
$$= \frac{3}{\cancel{2}}\left[\frac{63}{7}\cdot\cancel{2} - \frac{90}{5}\cancel{2} + \frac{35}{3}\cdot\cancel{2}\right] = 3\left[\frac{63}{7} - \frac{90}{5} + \frac{35}{3}\right] = 8$$

$$a_2 = \frac{5}{2}\int_{-1}^{1}(63x^5 - 90x^3 + 35x)\left(\frac{3}{2}x^2 - \frac{1}{2}\right)dx$$
$$= \frac{5}{2}\left[\frac{3}{2}\int_{-1}^{1}(63x^7 - 90x^5 + 35x^3)dx - \frac{1}{2}\int_{-1}^{1}(63x^5 - 90x^3 + 35x)dx\right] = 0 \text{ (even)}$$

$$a_3 = \frac{7}{2}\int_{-1}^{1}(63x^5 - 90x^3 + 35x)\left(\frac{5}{2}x^3 - \frac{3}{2}x\right)dx$$
$$= \frac{7}{2}\left[\frac{5}{2}\int_{-1}^{1}(63x^8 - 90x^6 + 35x^4)dx - \frac{3}{2}\int_{-1}^{1}(63x^6 - 90x^4 + 35x^2)dx\right]$$
$$= \frac{7}{2}\left[\frac{5}{2}\left[\frac{63}{9}x^9 - \frac{90}{7}x^7 + \frac{35}{5}x^5\right]_{-1}^{1} - \frac{3}{2}\left[\frac{63}{7}x^7 - \frac{90}{5}x^5 + \frac{35}{3}x^3\right]_{-1}^{1}\right]$$
$$= \frac{7}{2}\left[\frac{5}{\cancel{2}}\left(\frac{63}{9}\cdot\cancel{2} - \frac{90}{7}\cdot\cancel{2} + \frac{35}{5}\cdot\cancel{2}\right) - 8\right] = -8$$

$a_4 = 0$ (even) , $a_5 = \frac{11}{2}\int_{-1}^{1}(63x^5 - 90x^3 + 35x)\left[\frac{1}{8}(63x^5 - 70x^3 + 15x)\right]dx$

$$a_5 = \frac{11}{2}\left(\frac{1}{8}\right)\left[63\int_{-1}^{1}(63x^{10} - 90x^8 + 35x^6)dx - 70\int_{-1}^{1}(63x^8 - 90x^6 + 35x^4)dx + 15\int_{-1}^{1}(63x^6 - 90x^4 + 35x^2)dx\right] = 8$$

$$f(x) = 8P_1(x) - 8P_3(x) + 8P_5(x)$$

(3) $1-x^4$, $f(x)=\sum_{m=0}^{\infty} a_m P_m(x)$, $a_m = \frac{2m+1}{2}\int_{-1}^{1} f(x)P_m(x)\,dx$

$$a_0 = \frac{1}{2}\int_{-1}^{1}(1-x^4)\,dx = \frac{1}{2}\left[x-\frac{1}{5}x^5\right]_{-1}^{1} = \frac{1}{\cancel{2}}\left[\cancel{2}-\frac{1}{5}\cdot\cancel{2}\right] = \frac{4}{5}$$

$$a_1 = \frac{3}{2}\int_{-1}^{1}(1-x^4)x\,dx = \frac{3}{2}\left[\frac{1}{2}x^2-\frac{1}{6}x^6\right]_{-1}^{1} = 0$$

$$a_2 = \frac{5}{2}\int_{-1}^{1}(1-x^4)\left(\frac{3}{2}x^2-\frac{1}{2}\right)dx = \frac{5}{2}\left[\frac{3}{2}\left(\frac{1}{3}x^3-\frac{1}{7}x^7\right)-\frac{1}{2}\left(x-\frac{1}{5}x^5\right)\right]_{-1}^{1}$$
$$= \frac{5}{2}\left[\frac{3}{\cancel{2}}\left(\frac{1}{3}\cdot\cancel{2}-\frac{1}{7}\cdot\cancel{2}\right)-\frac{1}{\cancel{2}}\left(\cancel{2}-\frac{1}{5}\cdot\cancel{2}\right)\right] = \frac{5}{2}\left[3\left(\frac{1}{3}-\frac{1}{7}\right)-\frac{4}{5}\right]$$
$$= -\frac{4}{7}$$

$$a_3 = 0,\ a_4 = \frac{9}{2}\int_{-1}^{1}(1-x^4)\left[\frac{1}{8}(35x^4-30x^2+3)\right]dx$$
$$= \frac{9}{2}\cdot\frac{1}{8}\int_{-1}^{1}(-35x^8+30x^6+32x^4-30x^2+3)\,dx$$
$$= \frac{9}{2}\left(\frac{1}{8}\right)\cdot\left[-\frac{35}{9}x^9+\frac{30}{7}x^7+\frac{32}{5}x^5-\frac{30}{3}x^3+3x\right]_{-1}^{1}$$
$$= \frac{9}{8}\left[3-10+\frac{32}{5}+\frac{30}{7}-\frac{35}{9}\right] = -\frac{8}{35}$$

$$f(x) = \frac{4}{5}P_0(x) - \frac{4}{7}P_2(x) - \frac{8}{35}P_4(x)$$

⑤ Prove that if $f(x)$ is even (is odd, respectively), its Fourier-Legendre series contains only $P_m(x)$ with even m (only $P_m(x)$ with odd m, respectively). Give examples.

Even: Given an even function multiplied by an even term, we get an even term. (i.e $x^2 \cdot x^4 = x^6$)
Given an even function multiplied by an odd term, we get an odd term. (i.e. $x^2 \cdot x^3 = x^5$)

Integrating over $[-1,1]$ even terms will yield non-zero results.

$\therefore$ Fourier-Legendre series only contain even terms (m is even)

Example: $f(x) = x^2$ $\quad a_0 = \frac{1}{2}\int_{-1}^{1} x^2 dx = 1/3 \quad a_1 = \frac{3}{2}\int_{-1}^{1} x^3 dx = 0$

$f(x) = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x) \quad a_2 = \frac{5}{4}\int_{-1}^{1} x^2(3x^2-1)dx = \frac{2}{3} \quad a_3 = \frac{7}{4}\int_{-1}^{1} x^2(5x^3-3x)dx = 0$

$a_4 = 0$

Odd: $\text{odd}(x^3) \cdot \text{odd}(x) = \text{even}(x^4)$
$\text{odd}(x^3) \cdot \text{even}(x^2) = \text{odd}(x^5)$

Only integrating even terms over $[-1,1]$ give non-zero result
$\therefore$ Fourier-Legendre series only contain odd terms (m is odd)

Example: $f(x) = x$ $\quad a_0 = \frac{1}{2}\int_{-1}^{1} x\,dx = 0 \quad a_1 = \frac{3}{2}\int_{-1}^{1} x^2 dx = 1$

$f(x) = 1 \cdot P_1(x) \quad a_2 = \frac{5}{2}\int_{-1}^{1} x^3 dx = 0 \quad a_3 = 0$

Sec. 11.7 p 517

Show that the integral represents the indicated function. Hint. Use (5), (10), or (11); the integral tells you which one, and its value tells you what function to consider. Show your work in detail.

①
$$\int_0^\infty \frac{\cos xw + w\sin xw}{1+w^2}\,dx = \begin{cases} 0 & \text{if } x<0 \\ \pi/2 & \text{if } x=0 \\ \pi e^{-x} & \text{if } x>0 \end{cases}$$

(5) $f(x) = \int_0^\infty (A(w)\cos xw + B(w)\sin xw)\,dw$

$A(w) = \frac{1}{\pi}\int_0^\infty \pi e^{-x}\cos wx\,dx = \int_0^\infty e^{-x}\cos wx\,dx$ $\qquad u = e^{-x} \quad dv = \cos wx\,dx$

$\qquad du = -e^{-x}dx \quad v = \frac{1}{w}\sin wx$

$A(w) = \cancel{e^{-x}\frac{\sin wx}{w}}^{\,0}\Big|_0^\infty + \frac{1}{w}\int_0^\infty e^{-x}\sin wx\,dx$ $\qquad u = e^{-x} \quad dv = \sin wx\,dx$

$\qquad du = -e^{-x}dx \quad v = -\frac{1}{w}\cos wx$

$= \frac{1}{w}\left[-e^{-x}\frac{\cos wx}{w}\Big|_0^\infty - \frac{1}{w}\int_0^\infty e^{-x}\cos wx\,dx\right]$

$= \frac{1}{w^2}\left[-e^{-x}\cos wx\Big|_0^\infty - A(w)\right]$

$A(w)\left(1+\frac{1}{w^2}\right) = \frac{1}{w^2}\left[-e^{-\infty}\cos\infty + e^{0}\cos 0\right] = \frac{1}{w^2} \qquad A(w) = \frac{1}{w^2+1}$

$B(w) = \frac{1}{\pi}\int_0^\infty \pi e^{-x}\sin wx\,dx = \int_0^\infty e^{-x}\sin wx\,dx$ $\qquad u = e^{-x} \quad dv = \sin wx\,dx$

$\qquad du = -e^{-x}dx \quad v = -\frac{1}{w}\cos wx$

$= -e^{-x}\frac{\cos wx}{w}\Big|_0^\infty - \frac{1}{w}\int_0^\infty e^{-x}\cos wx\,dx$

$= \frac{1}{w}\left[1 - \frac{1}{w^2+1}\right] = \frac{1}{w}\left[\frac{w^2+1-1}{w^2+1}\right] = \frac{w}{w^2+1}$

$f(x) = \int_0^\infty \frac{1}{w^2+1}\cdot\cos wx + \frac{w}{w^2+1}\sin wx\,dw$

Represent f(x) as an integral (10).

(11.) $f(x) = \begin{cases} \sin x & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$

$$A(\omega) = \frac{2}{\pi}\int_0^{\pi} \sin x \cos \omega x \, dx \qquad u = \sin x \qquad dv = \cos \omega x dx$$

$$du = \cos x dx \qquad v = \frac{1}{\omega}\sin \omega x$$

$$A(\omega) = \left[\cancel{\frac{\sin x \sin \omega x}{\omega}}\Big|_0^{\pi} - \frac{1}{\omega}\int_0^{\pi} \cos x \sin \omega x \, dx \right] \qquad u = \cos x \qquad dv = \sin \omega x \, dx$$

$$du = -\sin x \, dx \qquad v = -\frac{1}{\omega}\cos \omega x$$

$$= -\frac{1}{\omega}\left[-\frac{\cos x \cos \omega x}{\omega}\Big|_0^{\pi} - \frac{1}{\omega}\int_0^{\pi} \sin x \cos \omega x \, dx \right]$$

$$= \frac{1}{\omega^2}\left[-\cos \omega\pi - 1 + A(\omega) \right] \Rightarrow \left(1 - \frac{1}{\omega^2}\right) A(\omega) = \frac{-1}{\omega^2}\left(1 + \cos \omega \pi\right)$$

$$A(\omega) \cdot \left(\frac{\omega^2 - 1}{\cancel{\omega^2}}\right) = \frac{-1}{\cancel{\omega^2}}\left(1 + \cos \omega \pi\right) \qquad A(\omega) = \frac{\cos \omega \pi + 1}{1 - \omega^2}$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \frac{\cos \omega\pi + 1}{1 - \omega^2} \cos \omega x \, d\omega$$

Represent f(x) as an integral (11).

(18) $$f(x) = \begin{cases} \cos x & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$$

$$B(\omega) = \frac{2}{\pi}\int_0^{\pi} \cos x \sin \omega x \, dx$$

$u = \cos x$ $\quad dv = \sin \omega x \, dx$

$du = -\sin x \, dx$ $\quad v = -\frac{1}{\omega}\cos \omega x$

$$B(\omega) = \left[-\frac{\cos x \cos \omega x}{\omega}\Big|_0^{\pi} - \frac{1}{\omega}\int_0^{\pi} \sin x \cos \omega x \, dx\right]$$

$u = \sin x$ $\quad dv = \cos \omega x \, dx$

$du = \cos x \, dx$ $\quad v = \frac{1}{\omega}\sin \omega x$

$$= \frac{-1}{\omega}\left[\cos x \cos \omega x \Big|_0^{\pi} + \frac{1}{\omega}\left(\cancel{\sin x \sin \omega x \Big|_0^{\pi}}^{\,0} - \int_0^{\pi} \cos x \sin \omega x \, dx\right)\right]$$

$$= \frac{-1}{\omega}\left[-\cos \omega\pi - 1 - \frac{1}{\omega} B(\omega)\right] = \frac{\cos \omega\pi + 1}{\omega} + \frac{1}{\omega^2} B(\omega)$$

$$\left(1 - \frac{1}{\omega^2}\right) B(\omega) = \frac{\cos \omega\pi + 1}{\omega}$$

$$\frac{\omega^2 - 1}{\omega^2} B(\omega) = \frac{\cos \omega\pi + 1}{\omega} \qquad B(\omega) = \frac{\omega(\cos \omega\pi + 1)}{\omega^2 - 1}$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \frac{\omega(\cos \omega\pi + 1)}{\omega^2 - 1} \cos \omega x \, d\omega$$

Sec. 11.8 p 522

① Find the cosine transform $\hat{f}_c(\omega)$ of $f(x) = 1$ if $0 < x < 1$, $f(x) = -1$ if $1 < x < 2$, $f(x) = 0$ if $x > 2$.

$$\mathcal{F}_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \cos \omega x \, dx - \sqrt{\frac{2}{\pi}} \int_1^2 \cos \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1}{\omega} \sin \omega x \Big|_0^1 - \frac{1}{\omega} \sin \omega x \Big|_1^2 \right]$$

$$= \frac{1}{\omega} \cdot \sqrt{\frac{2}{\pi}} \left[\sin \omega - \cancelto{0}{\sin 0} - \sin 2\omega + \sin \omega \right]$$

$$= \frac{1}{\omega} \cdot \sqrt{\frac{2}{\pi}} \left[2 \sin \omega - \sin 2\omega \right] = \hat{f}_c(\omega)$$

② Find f in Prob. 1 from the answer $\hat{f}_c(\omega)$

$$\mathcal{F}_c^{-1}(\hat{f}_c(\omega)) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(\omega) \cos \omega x \, d\omega$$

$$= \frac{2}{\pi} \int_0^\infty \left(\frac{2 \sin \omega}{\omega} - \frac{\sin 2\omega}{\omega} \right) \cos \omega x \, d\omega$$

$$= \frac{2}{\pi} \left[\int_0^\infty \frac{2 \sin \omega}{\omega} \cos \omega x \, d\omega - \int_0^\infty \frac{\sin 2\omega}{\omega} \cos \omega x \, d\omega \right]$$

From Prob. 7 of sec. 11.7 we have

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$$\frac{2}{\pi}\int_0^{\infty} \frac{\sin w \cos wx}{\omega}\, dw \Rightarrow f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$$

$\therefore$

$$\frac{4}{\pi}\int_0^{\infty} \frac{\sin w \cos wx}{\omega}\, dw \Rightarrow f_1(x) = \begin{cases} 2, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$$

$$\frac{2}{\pi}\int_0^{\infty} \frac{\sin 2w \cos wx}{\omega}\, dw \Rightarrow \frac{2}{\pi}\int_0^{\infty} \frac{2\sin\tilde{\omega}\cos\tilde{\omega}\frac{x}{2}}{\tilde{\omega}} \cdot \frac{1}{2}\, d\tilde{\omega}$$

$2\omega = \tilde{\omega}$ $\qquad \frac{x}{2} = \tilde{x}$

$\omega = \frac{\tilde{\omega}}{2} \Rightarrow d\omega = \frac{1}{2}d\tilde{\omega}$ $\qquad x = 2\tilde{x}$

$$\frac{2}{\pi}\int_0^{\infty} \frac{\sin\tilde{\omega}\cos\tilde{\omega}\tilde{x}}{\tilde{\omega}}\, d\tilde{\omega} \Rightarrow f_2(\tilde{x}) = \begin{cases} 1, & 0 < \tilde{x} < 1 \\ 0, & \tilde{x} > 1 \end{cases}$$

$$f_2(x) = \begin{cases} 1, & 0 < \frac{x}{2} < 1 \\ 0, & \frac{x}{2} > 1 \end{cases} = \begin{cases} 1, & 0 < x < 2 \\ 0, & x > 2 \end{cases}$$

$$f_1(x) - f_2(x) = \begin{cases} 2, & 0 < x < 1 \\ 0, & x > 1 \end{cases} - \begin{cases} 1, & 0 < x < 2 \\ 0, & x > 2 \end{cases}$$

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ -1, & 1 < x < 2 \\ 0, & x > 2 \end{cases} \quad \checkmark$$

③ Find $\hat{f}_c(\omega)$ for $f(x) = x$ if $0 < x < 2$, $f(x) = 0$ if $x > 2$.

$$\mathcal{F}_c(f(x)) = \sqrt{\tfrac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx$$

$$= \sqrt{\tfrac{2}{\pi}} \int_0^2 x \cos \omega x \, dx \qquad u = x \qquad dv = \cos \omega x \, dx$$

$$du = dx \qquad v = \tfrac{1}{\omega} \sin \omega x$$

$$= \sqrt{\tfrac{2}{\pi}} \left[\tfrac{x}{\omega} \sin \omega x \Big|_0^2 - \tfrac{1}{\omega} \int_0^2 \sin \omega x \, dx \right]$$

$$= \sqrt{\tfrac{2}{\pi}} \left[\tfrac{2}{\omega} \sin 2\omega + \tfrac{1}{\omega^2} \cos \omega x \Big|_0^2 \right]$$

$$= \sqrt{\tfrac{2}{\pi}} \left[\tfrac{2}{\omega} \sin 2\omega + \tfrac{1}{\omega^2} \cos 2\omega - \tfrac{1}{\omega^2} \right] = \hat{f}_c(\omega)$$

⑤ Find $\hat{f}_c(\omega)$ for $f(x) = x^2$ if $0 < x < 1$, $f(x) = 0$ if $x > 1$.

$$\mathcal{F}_c(f(x) = \sqrt{\tfrac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx = \sqrt{\tfrac{2}{\pi}} \int_0^1 x^2 \cos \omega x \, dx$$

$$u = x^2 \qquad dv = \cos \omega x \, dx \qquad = \sqrt{\tfrac{2}{\pi}} \left[\tfrac{x^2}{\omega} \sin \omega x \Big|_0^1 - \tfrac{2}{\omega} \int_0^1 x \sin \omega x \, dx \right]$$

$$du = 2x\,dx \qquad v = \tfrac{1}{\omega} \sin \omega x$$

$$u = x \qquad dv = \sin \omega x \, dx$$

$$du = dx \qquad v = \tfrac{-1}{\omega} \cos \omega x$$

$$\hat{f}_c(\omega) = \sqrt{\tfrac{2}{\pi}} \left[\tfrac{1}{\omega} \sin \omega + \tfrac{2}{\omega} \left(\tfrac{x}{\omega} \cos \omega x \Big|_0^1 - \tfrac{1}{\omega} \int_0^1 \cos \omega x \, dx \right) \right]$$

$$= \sqrt{\tfrac{2}{\pi}} \left[\tfrac{1}{\omega} \sin \omega + \tfrac{2}{\omega^2} \left(\cos \omega - \tfrac{1}{\omega} \sin \omega \right) \right]$$

$$= \sqrt{\tfrac{2}{\pi}} \cdot \tfrac{1}{\omega^3} \left[(\omega^2 - 1) \sin \omega + 2\omega \cos \omega \right]$$