Jose Quinones-Torres DUE: 11/28/2012

Homework # 11

Sec. 11.9 p. 532

Find the Fourier transform of f(x) (without using Table III in Sec. 11.10). Show details.

(3.) $f(x) = \begin{cases} 1 & \text{if } a < x < b \\ 0 & \text{if otherwise} \end{cases}$

$$\mathcal{F}(f) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_a^b (1)\, e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \left(\frac{1}{-i\omega}\right) e^{-i\omega x} \Big|_a^b = \frac{i}{\omega\sqrt{2\pi}} \left(e^{-i\omega b} - e^{-i\omega a}\right)$$

(4.) $f(x) = \begin{cases} e^{kx} & \text{if } x < 0 \quad (k>0) \\ 0 & \text{if } x > 0 \end{cases}$

$$\mathcal{F}(f) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{kx} \cdot e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(k-i\omega)x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \left(\frac{1}{k-i\omega}\right) e^{(k-i\omega)x} \Big|_{-\infty}^{0} = \frac{1}{\sqrt{2\pi}} \cdot \left(\frac{1}{k-i\omega}\right)$$

⑦ $f(x) = \begin{cases} x & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$

$$\mathcal{F}(f) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^a x e^{-i\omega x} dx \qquad \begin{matrix} u = x & dv = e^{-i\omega x} dx \\ du = dx & v = \frac{i}{\omega} e^{-i\omega x} \end{matrix}$$

$$\hat{f}(\omega) = \left[\frac{i}{\omega} x e^{-i\omega x} \Big|_0^a - \frac{i}{\omega} \int_0^a e^{-i\omega x} dx \right] \cdot \frac{1}{\sqrt{2\pi}}$$

$$= \left[\frac{i}{\omega} a e^{-i\omega a} - \frac{i}{\omega} \cdot \left(\frac{1}{-i\omega} \right) e^{-i\omega x} \Big|_0^a \right] \cdot \frac{1}{\sqrt{2\pi}}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{i}{\omega} a e^{-i\omega a} - \frac{i^2}{\omega^2} \left(e^{-i\omega a} - 1 \right) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{i}{\omega} a e^{-i\omega a} + \frac{1}{\omega^2} e^{-i\omega a} - \frac{1}{\omega^2} \right)$$

$$= \frac{1}{\omega^2 \sqrt{2\pi}} \left(i\omega a e^{-i\omega a} + e^{-i\omega a} - 1 \right)$$

Sec. 12.1 p. 542

Verify (by substitution) that the given function is a solution of the PDE. Sketch or graph the solution as a surface in space.

Wave Equation (1) with suitable c

(2) $u = x^2 + t^2$, $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

$\frac{\partial u}{\partial t} = 2t$, $\frac{\partial^2 u}{\partial t^2} = 2$ $\qquad$ $\frac{\partial u}{\partial x} = 2x$, $\frac{\partial^2 u}{\partial x^2} = 2$

$2 = c^2 2 \Rightarrow c^2 = 1 \therefore c = \pm 1$

Heat Equation (2) with suitable c

(8) $u = e^{-9t} \sin \omega x$, $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

$\frac{\partial u}{\partial t} = -9 e^{9t} \sin \omega x$ $\qquad$ $\frac{\partial u}{\partial x} = e^{-9t} \omega \cos \omega x$, $\frac{\partial^2 u}{\partial x^2} = -e^{-9t} \omega^2 \sin \omega x$

$-9 e^{9t} \sin \omega x = -c^2 \omega^2 e^{-9t} \sin \omega x \Rightarrow 9 = c^2 \omega^2$

$c^2 = 9/\omega^2 \therefore c = \pm 3/\omega$

Laplace Equation (3)

(18) $u = e^x \cos y, e^x \sin y \quad , \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(i) $u = e^x \cos y$

$$\frac{\partial u}{\partial x} = e^x \cos y \, , \, \frac{\partial^2 u}{\partial x^2} = e^x \cos y \qquad \frac{\partial u}{\partial y} = e^x \sin y \, , \, \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$$

$$e^x \cos y + (-e^x \cos y) = e^x \cos y - e^x \cos y = 0$$

(ii) $u = e^x \sin y$

$$\frac{\partial u}{\partial x} = e^x \sin y \, , \, \frac{\partial^2 u}{\partial x^2} = e^x \sin y \qquad \frac{\partial u}{\partial y} = -e^x \cos y \, , \, \frac{\partial^2 u}{\partial y^2} = -e^x \sin y$$

$$e^x \sin y + (-e^x \sin y) = e^x \sin y - e^x \sin y = 0$$

Solve for $u = u(x,y)$ (when the PDE involves derivatives with respect to one variable only)

(19) $u_y + y^2 u = 0 \Rightarrow u_y = -y^2 u$

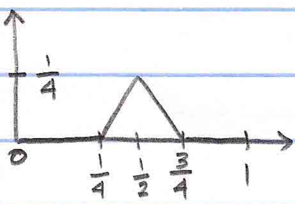
$$\frac{\partial u}{\partial y} = -y^2 u \qquad \frac{1}{u} du = -y^2 dy \quad \int \frac{1}{u} du = -\int y^2 dy$$

$$\ln u = -\frac{1}{3} y^3 \quad \therefore \; u = e^{-y^3/3} \Rightarrow u = c(x) e^{-y^3/3}$$

Sec. 12.3 p. 551

Find $u(x,t)$ for the string of length $L=1$ and $c^2=1$ when the initial velocity is zero and the initial deflection with small k (say, 0.01) is as follows.

(11)



$$\ddot{u} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad L=1 \quad c^2=1$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = \begin{cases} x - \frac{1}{4} & \frac{1}{4} < x < \frac{1}{2} \\ \frac{3}{4} - x & \frac{1}{2} < x < \frac{3}{4} \\ 0 & \text{otherwise} \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} \left(B_n \cos n\pi t + B_n^* \sin n\pi t \right) \sin n\pi x, \quad \lambda_n = n\pi$$

$$B_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L}x \, dx = 2\int_{1/4}^{1/2} (x - \tfrac{1}{4}) \sin n\pi x \, dx + 2\int_{1/2}^{3/4} (\tfrac{3}{4} - x) \sin n\pi x \, dx$$

$$= 2\int_{1/4}^{1/2} x \sin n\pi x \, dx - \frac{1}{2}\int_{1/4}^{1/2} \sin n\pi x \, dx + \frac{3}{2}\int_{1/2}^{3/4} \sin n\pi x \, dx - 2\int_{1/2}^{3/4} x \sin n\pi x \, dx$$

$$\int x \sin n\pi x \, dx \Rightarrow u = x \qquad dv = \sin n\pi x \, dx$$

$$du = dx \qquad v = \frac{-1}{n\pi}\cos(n\pi x)$$

$$\frac{-x}{n\pi}\cos n\pi x + \frac{1}{n\pi}\int \cos n\pi x \, dx = \frac{-x}{n\pi}\cos n\pi x + \frac{1}{n^2\pi^2}\sin n\pi x$$

$$B_n = 2\left[\frac{-x}{n\pi}\cos n\pi x \Big|_{1/4}^{1/2} + \frac{1}{n^2\pi^2}\sin n\pi x \Big|_{1/4}^{1/2} \right] + \frac{1}{2}\cdot\frac{1}{n\pi}\cos n\pi x \Big|_{1/4}^{1/2} -$$
$$\frac{3}{2}\cdot\frac{1}{n\pi}\cos n\pi x \Big|_{1/2}^{3/4} - 2\left[\frac{-x}{n\pi}\cos n\pi x \Big|_{1/2}^{3/4} + \frac{1}{n^2\pi^2}\sin n\pi x \Big|_{1/2}^{3/4} \right]$$

$$B_n = \frac{2}{(n\pi)^2}\left(2\sin\frac{n\pi}{2} - \sin\frac{n\pi}{4} - \sin\frac{3n\pi}{4}\right)$$

$$u(x,t) = \frac{2}{\pi^2}\sum_{n=1}^{\infty}\left(2\sin\frac{n\pi}{2} - \sin\frac{n\pi}{4} - \sin\frac{3n\pi}{4}\right)\cos n\pi t \sin n\pi x$$

$$= \frac{2}{\pi^2}\Big(\left[2-\sqrt{2}\right]\cos\pi t\sin\pi x - \frac{1}{9}\left[2+\sqrt{2}\right]\cos 3\pi t\sin 3\pi x +$$

$$\frac{1}{25}\left[2+\sqrt{2}\right]\cos 5\pi t\sin 5\pi x - + \cdots\Big)$$

(15) Substituting $u = F(x)G(t)$ into (21), show that

$$F^{(4)}/F = -\ddot{G}/c^2G = \beta^4 = \text{const},$$

$$F(x) = A\cos\beta x + B\sin\beta x + C\cosh\beta x + D\sinh\beta x,$$

$$G(t) = a\cos c\beta^2 t + b\sin c\beta^2 t$$

(21) $\dfrac{\partial^2 u}{\partial t^2} = -c^2\dfrac{\partial^4 u}{\partial x^4}$ $\qquad$ $\dfrac{\partial^2 u}{\partial t^2} = F\ddot{G}$, $\dfrac{\partial^4 u}{\partial x^4} = F^{(4)}G$

$$F\ddot{G} = -c^2F^{(4)}G \Rightarrow F^{(4)}/F = \ddot{G}/c^2G = k$$

$F^{(4)} - kF = 0$ $\qquad$ $\ddot{G} - kc^2G = 0$

$s^4 - k = 0 \Rightarrow s = \pm\sqrt[4]{k}$ $\qquad$ $s^2 - kc^2 = 0 \Rightarrow s = \pm\sqrt{kc^2}$

$= \pm\sqrt[4]{\beta^4}$ $\qquad$ $= \pm\sqrt{\beta^4c^2}$

$= \pm\beta$ $\qquad$ $= \pm c\beta^2$

$$\therefore F(x) = A\cos\beta x + B\sin\beta x + C\cosh\beta x + D\sinh\beta x$$

$$G(t) = a\cos c\beta^2 t + b\sin c\beta^2 t$$

(16) Simply supported beam in Fig. 293A. Find solutions $u_n = F_n(x) G_n(x)$ of (21) corresponding to zero initial velocity and satisfying the boundary conditions.

$u(0,t) = 0$, $u(L,t) = 0$ (ends simply supported for all times t)

$u_{xx}(0,t) = 0$, $u_{xx}(L,t) = 0$ (zero moments, hence zero curvature, at the ends)

$$F(x) = A\cos\beta x + B\sin\beta x + C\cosh\beta x + D\sinh\beta x$$

$$F'(x) = -A\beta\sin\beta x + B\beta\cos\beta x + C\beta\sinh\beta x + D\beta\cosh\beta x$$

$$F''(x) = -A\beta^2\cos\beta x - B\beta^2\sin\beta x + C\beta^2\cosh\beta x + D\beta^2\sinh\beta x$$

$F(0) = A + C = 0$

$F''(0) = -A\beta^2 + C\beta^2 = 0$

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} A \\ C \end{bmatrix} = 0$$

$A = 0$

$C = 0$

$F(L) = B\sin\beta L + D\sinh\beta L = 0$

$F''(L) = -B\beta^2\sin\beta L + D\beta^2\sinh\beta L = 0$ $\quad B\beta^2\sin\beta L = D\beta^2\sinh\beta L$

$B\sin\beta L = D\sinh\beta L$

$2D\sinh\beta L = 0 \quad \therefore D = 0$

$B\sin\beta L = 0 \Rightarrow \sin\beta L = 0 \quad \beta L = n\pi \quad \beta = \frac{n\pi}{L}, \ n = 0, 1, 2, \ldots$

$$\therefore F_n(x) = \sin\frac{n\pi}{L}x$$

$$G_n(t) = a\cos c\frac{n^2\pi^2}{L^2}t + b\sin c\frac{n^2\pi^2}{L^2}t$$

$$u_n(x,t) = \left(a\cos c\frac{n^2\pi^2}{L^2}t + b\sin c\frac{n^2\pi^2}{L^2}t\right)\sin\frac{n\pi}{L}x$$