Adam Clouse

Math 527 : Homework #11

p. 532 : 3, 4, 7 ; p. 542 : 2, 8, 10, 19 ; p. 551 : 11, 15, 16

533.3 $f(x) = \begin{cases} 1 & a < x < b \\ 0 & \text{otherwise} \end{cases}$ ⇒ find $\hat{f}(w)$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \int_a^b (1) e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \frac{e^{-iwx}}{-iw} \Big|_a^b$$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \cdot \frac{-1}{iw} \left(e^{-iwb} - e^{-iwa}\right) = \boxed{\frac{i}{w\sqrt{2\pi}} \left(e^{-iwb} - e^{-iwa}\right)}$$

533.4 $f(x) = \begin{cases} e^{kx} & x < 0 \quad (k > 0) \\ 0 & x > 0 \end{cases}$ ⇒ find $\hat{f}(w)$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{kx} e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{k-iw} e^{x(k-iw)} \Big|_{-\infty}^{0}$$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{k-iw} \left(e^{0} - e^{-\infty(k-iw)} \right)$$

$e^{0} \to 1$; e^{-iw} → unit of 1; $k > 0$, therefore $e^{-\infty} \to \phi$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{k-iw} \frac{(k+iw)}{(k+iw)} (1 - \phi)$$

$$\boxed{\hat{f}(w) = \frac{k+iw}{k^2+w^2} \cdot \frac{1}{\sqrt{2\pi}}}$$

533.7 $f(x) = \begin{cases} x & 0 < x < a \\ 0 & \text{otherwise} \end{cases}$ ⇒ find $\hat{f}(w)$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_0^a x e^{-iwx} dx$$

let $u = x$, $dv = e^{-iwx}$; $du = 1$, $v = -\frac{1}{iw} e^{-iwx}$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \left[\frac{-x}{iw} e^{-iwx} \Big|_0^a + \frac{1}{iw} \int_0^a e^{-iwx} dx \right]$$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \left[\frac{-a}{iw} e^{-iwa} + \frac{1}{w^2} e^{-iwx} \Big|_0^a \right]$$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \left[\frac{ai}{w} e^{-iwa} + \frac{1}{w^2}\left(e^{-iwa} - 1\right) \right] = \boxed{\frac{1}{\sqrt{2\pi}\, w^2} \left[(wai+1) e^{-iwa} - 1 \right]}$$

542.2 wave equation: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

$u = x^2 + t^2 \Rightarrow \frac{\partial u}{\partial t} = 2t \Rightarrow \frac{\partial^2 u}{\partial t^2} = 2$

$\Rightarrow \frac{\partial u}{\partial x} = 2x \Rightarrow \frac{\partial^2 u}{\partial x^2} = 2$

$2 = c^2 (2) \Rightarrow c^2 = 1 \Rightarrow \boxed{c = 1}$

542.8 heat equation: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

$u = e^{-9t} \sin wx \Rightarrow \frac{\partial u}{\partial t} = \sin wx\,(-9 e^{-9t})$

$\Rightarrow \frac{\partial u}{\partial x} = e^{-9t}(\cos wx)\,w \Rightarrow \frac{\partial^2 u}{\partial x^2} = e^{-9t}(-\sin wx)\,w^2$

$\sin wx\,(-9)\,e^{-9t} = e^{-9t}(-\sin wx)\,w^2\,(c^2) \Rightarrow \sqrt{c^2} = \sqrt{\frac{9}{w^2}} \Rightarrow \boxed{c = \frac{3}{w}}$

542.10 laplace equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \phi$

(a)

$u = e^x \cos y \Rightarrow \frac{\partial u}{\partial x} = e^x \cos y \Rightarrow \frac{\partial^2 u}{\partial x^2} = e^x \cos y$

$\Rightarrow \frac{\partial u}{\partial y} = e^{-x}(-\sin y) \Rightarrow \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$

$e^x \cos y - e^x \cos y = \phi$ $\blacksquare$

(b)

$u = e^x \sin y \Rightarrow \frac{\partial u}{\partial x} = e^x \sin y \Rightarrow \frac{\partial^2 u}{\partial x^2} = e^x \sin y$

$\Rightarrow \frac{\partial u}{\partial y} = e^x \cos y \Rightarrow \frac{\partial^2 u}{\partial y^2} = e^x(-\sin y)$

$e^x \sin y + e^x(-\sin y) = e^x \sin y - e^x \sin y = \phi$ $\blacksquare$

$u_y + u y^2 = \phi$ solve for $u(x,y)$

543.19

$$\frac{\partial u}{\partial y} + y^2 u = \phi \Rightarrow \frac{\partial u}{\partial y} = -y^2 u \Rightarrow \frac{\partial u}{u} = -y^2 \partial y$$

$$\ln(u) = -\frac{1}{3}y^3 + C(x) \Rightarrow u = e^{-\frac{1}{3}y^3} \cdot + e^{C(x)} \Rightarrow u = C(x) e^{-\frac{y^3}{3}}$$

551.11

$$f(x) = \begin{cases} 0 & \text{for } 0 < x < L/4 \\ \frac{2K}{L}(x - L/4) & L/4 < x < L/2 \\ \frac{2K}{L}(\frac{3L}{4} - x) & L/2 < x < 3L/4 \\ 0 & 3L/4 < x < L \end{cases}$$

$L = 1,\ c^2 = 1 \Rightarrow c = 1$

$$g(x) = u_t(x,0) = \phi \Rightarrow B_n^* = \phi$$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t) \sin \frac{n\pi}{L} x$$

$$\lambda_n = \frac{cn\pi}{L} = \frac{1\, n\pi}{1} = n\pi$$

$$u(x,t) = \sum_{n=1}^{\infty} (B_n \cos n\pi t) \sin n\pi x$$

$$B_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx = \frac{2}{L}\left[\int_{L/4}^{L/2} \frac{2K}{L}(x - L/4) \sin \frac{n\pi x}{L}\, dx + \int_{L/2}^{3L/4} \frac{2K}{L}(\frac{3L}{4} - x) \sin \frac{n\pi x}{L}\, dx\right]$$

$$\text{Let } A_1 = \frac{2}{L}\int_{L/4}^{L/2} \frac{2K}{L}(x - L/4) \sin \frac{n\pi x}{L}\, dx$$

$$B_n = A_1 + A_2$$

$$\text{Let } A_2 = \frac{2}{L}\int_{L/2}^{3L/4} \frac{2K}{L}(\frac{3L}{4} - x) \sin \frac{n\pi x}{L}\, dx$$

$$A_1 = \frac{4K}{L^2}\left[\int_{L/4}^{L/2} x \sin \frac{n\pi}{L} x\, dx + -\frac{L}{4}\int_{L/4}^{L/2} \sin \frac{n\pi}{L} x\, dx\right]$$

$$A_1 = \frac{4k}{L^2}\left[\left.\frac{-x\cos\frac{n\pi}{L}x}{\frac{n\pi}{L}}\right|_{L/4}^{L/2} + \int_{L/4}^{L/2} \frac{\cos\frac{n\pi}{L}x}{\frac{n\pi}{L}}\,dx + \frac{L}{4}\left.\frac{\cos\frac{n\pi}{L}x}{\frac{n\pi}{L}}\right|_{L/4}^{L/2} \right]$$

$$= \frac{4k}{L^2}\left[\frac{L}{n\pi}\left(-\frac{L}{2}\cos\left(n\frac{\pi}{2}\right)^{\nearrow 0} + \frac{L}{4}\cos\left(n\frac{\pi}{4}\right)\right) + \frac{L^2}{n^2\pi^2}\left(\sin\frac{n\pi}{2} - \sin\frac{n\pi}{4}\right) + \frac{L^2}{4n\pi}\left(\cos\frac{n\pi}{2}^{\nearrow 0} - \cos\frac{n\pi}{4}\right) \right]$$

cancel

$$= \frac{4k}{L^2}\left(\frac{L^2}{n^2\pi^2}\left(\sin\frac{n\pi}{2} - \sin\frac{n\pi}{4}\right)\right) = \frac{4k}{n^2\pi^2}\left(\sin\frac{n\pi}{2} - \sin\frac{n\pi}{4}\right) = A_1$$

$$A_2 = \frac{4k}{L^2}\left[\frac{3L}{4}\int_{L/2}^{3L/4} \sin\frac{n\pi}{L}x\,dx - \int_{L/2}^{3L/4} x\sin\frac{n\pi}{L}x\,dx\right]$$

$$= \frac{4k}{L^2}\left[\left.\frac{-3L}{4}\cdot\frac{L}{n\pi}\left(\cos\frac{n\pi}{L}x\right)\right|_{L/2}^{3L/4} + \left.\frac{x\cos\frac{n\pi}{L}x}{\frac{n\pi}{L}}\right|_{L/2}^{3L/4} - \frac{L}{n\pi}\int_{L/2}^{3L/4}\cos\frac{n\pi}{L}x\,dx \right]$$

$$= \frac{4k}{L^2}\left[\frac{-3L^2}{4n\pi}\left(\cos\frac{3n\pi}{4} - \cos\frac{n\pi}{2}^{\nearrow 0}\right) + \frac{L}{n\pi}\left(\frac{3L}{4}\cos\frac{3n\pi}{4} - \frac{L}{2}\cos\frac{n\pi}{2}^{\nearrow 0}\right) \right.$$

cancel

$$\left. - \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{L}x \right|_{L/2}^{3L/4} \Bigg]$$

$$A_2 = \frac{4k}{L^2}\left[-\frac{L^2}{n^2\pi^2}\left(\sin\frac{3n\pi}{4} - \sin\frac{n\pi}{2}\right)\right] = \frac{4k}{n^2\pi^2}\left(\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{4}\right) = A_2$$

$$B_n = A_1 + A_2 = \frac{4k}{n^2\pi^2}\left(\sin\frac{n\pi}{2} - \sin\frac{n\pi}{4} + \sin\frac{n\pi}{2} - \sin\frac{3n\pi}{4}\right)$$

$$B_n = \frac{4k}{n^2\pi^2}\left(2\sin\frac{n\pi}{2} - \sin\frac{n\pi}{4} - \sin\frac{3n\pi}{4}\right)$$

$$B_1 = \frac{4k}{\pi^2}\left(2(1) - \left(\frac{\sqrt2}{2}\right) - \left(\frac{\sqrt2}{2}\right)\right) = \frac{4k}{\pi^2}\left(2 - \sqrt2\right)$$

$$B_2 = \frac{4k}{4\pi^2}\left(2(\emptyset) - (1) - (-1)\right) = \emptyset$$

$$B_3 = \frac{4k}{3^2\pi^2}\left(2(-1) - \left(\frac{\sqrt2}{2}\right) - \left(\frac{\sqrt2}{2}\right)\right) = \frac{4k}{9\pi^2}\left(-2 - \sqrt2\right)$$

$$B_5 = \frac{4k}{5^2\pi}\left(2(1) - \left(\frac{-\sqrt2}{2}\right) - \left(-\frac{\sqrt2}{2}\right)\right) = \frac{4k}{25\pi^2}\left(2 + \sqrt2\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} \left(B_n \cos n\pi t\right) \sin n\pi x$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4k}{n^2\pi^2}\left(2\sin\frac{n\pi}{2} - \sin\frac{n\pi}{4} - \sin\frac{3n\pi}{4}\right)\cos n\pi t \sin n\pi x$$

$$= \left(\frac{4k}{\pi^2}(2-\sqrt{2})\cos \pi t \sin \pi x + \frac{4k}{9\pi^2}(-2-\sqrt{2})\cos 3\pi t \sin 3\pi x + \frac{4k}{25\pi^2}(2+\sqrt{2})\cos 5\pi t \sin 5\pi x + \dots\dots\right)$$

552.15

$$\frac{\partial^2 u}{\partial t^2} = -c^2 \frac{\partial^4 u}{\partial x^4} \Rightarrow u = F(x)\,G(t)$$

$$\frac{\partial u}{\partial t} = F(x)\,\dot{G}(t) \Rightarrow \frac{\partial^2 u}{\partial t^2} = F(x)\,\ddot{G}(t)$$

$$\frac{\partial u}{\partial x} = F'(x)\,G(t) \Rightarrow \frac{\partial^2 u}{\partial x^2} = F''(x)\,G(t) \Rightarrow \frac{\partial^3 u}{\partial x^3} = F'''(x)\,G(t) \Rightarrow \frac{\partial^4 u}{\partial x^4} = F^{(4)}(x)\,G(t)$$

Sub back into original equation:

$$F(x)\,\ddot{G}(t) = -c^2 F^{(4)}(x)\,G(t) \Rightarrow \frac{F^{(4)}(x)}{F(x)} = \frac{-\ddot{G}(t)}{c^2 G(t)} = \text{const} = k = \beta^4$$

the LHS is only a function of 'x' & the RHS is only a function of 't'. therefore for them to be equal, they must equal a constant

LHS: $F^{(4)} - \beta^4 F = \phi \Rightarrow r^4 - \beta^4 = \phi \Rightarrow (r^2-\beta^2)(r^2+\beta^2) = \phi$

$(r+\beta)(r-\beta)(r+i\beta)(r-i\beta) = \phi$

$$F(x) = Ae^{\beta x} + Be^{-\beta x} + C\cos\beta x + D\sin\beta x$$

RHS: $\frac{-\ddot{G}(t)}{c^2 G(t)} = \beta^4 \Rightarrow \ddot{G}(t) + c^2\beta^4 G(t) = \phi \Rightarrow r^2 + c^2\beta^4 = \phi$

$(r + ic\beta^2)(r - ic\beta^2) = \phi$

~~$G(t) = Ee^{i\beta^2 ct} + Fe^{-ic\beta^2 t}$~~

$$G(t) = E\cos(c\beta^2 t) + F\sin(c\beta^2 t)$$

552.16

BCs: $u(\emptyset,t)=\emptyset$, $u(L,t)=\emptyset$

$u_{xx}(0,t)=\emptyset$, $u_{xx}(L,t)=\emptyset$

ICs: $u_t(x,0)=\emptyset$, $u(x,0)=f(x)$

$\emptyset = u(\emptyset,t) = F(\emptyset)\cdot G(t) \Rightarrow F(\emptyset)=\emptyset$

$\emptyset = u(L,t) = F(L)\cdot G(t) \Rightarrow F(L)=\emptyset$

$\emptyset = u_{xx}(0,t) = F''(\emptyset)\cdot G(t) \Rightarrow F''(\emptyset)=\emptyset$

$\emptyset = u_{xx}(L,t) = F''(L)\cdot G(t) \Rightarrow F''(L)=\emptyset$

$\emptyset = u_t(x,0) = F(x)\cdot \dot{G}(\emptyset) \Rightarrow \dot{G}(\emptyset)=\emptyset$

$f(x) = u(x,\emptyset) = F(x)\cdot G(\emptyset) \Rightarrow G(\emptyset)=1$

$G(t) = E\cos(c\beta^2 t) + F\sin(c\beta^2 t)$

$\dot{G}(t) = c\beta^2 E(-\sin(c\beta^2 t)) + c\beta^2 F\cos(c\beta^2 t)$

$\dot{G}(\emptyset) = \emptyset + c\beta^2 F = \emptyset \Rightarrow F=\emptyset$

$G(\emptyset) = E(1) = 1 \Rightarrow E=1$

$\cos(\emptyset)$

$G(t) = \cos c\beta^2 t$

$F(x) = A\cos\beta x + B\sin\beta x + C\cosh\beta x + D\sinh\beta x$

$F'(x) = -\beta A\sin\beta x + \beta B\cos\beta x + \beta C\sinh\beta x + \beta D\cosh\beta x$

$F''(x) = -\beta^2 A\cos\beta x - \beta^2 B\sin\beta x + \beta^2 C\cosh\beta x + \beta^2 D\sinh\beta x$

$F(\emptyset) = A(1)+\emptyset+C(1)+\emptyset = \emptyset \Rightarrow A+C=\emptyset \Rightarrow A=-C$ ⟩ $A=C=\emptyset$

$F''(\emptyset) = -\beta^2 A(1)+\emptyset+\beta^2 C(1)+\emptyset=\emptyset \Rightarrow -A+C=\emptyset \Rightarrow A=C$

$F(L) = B\sin(\beta L) + D\sinh(\beta L) = \emptyset$

$F''(L) = -\beta^2 B\sin(\beta L) + \beta^2 D\sinh(\beta L) = \emptyset$

$\sinh(a)=\emptyset$ only if $a=\emptyset \Rightarrow$ therefore $D=\emptyset$

$B\sin(\beta L)=\emptyset \Rightarrow \beta L = n\pi \Rightarrow \beta=\frac{n\pi}{L} \Rightarrow F(x)=\sin\frac{n\pi}{L}x$

$G(t) = \cos c\beta^2 t = \cos c\left(\frac{n\pi}{L}\right)^2 t \Rightarrow G(t) = \cos\left(\sqrt{\frac{EI}{\rho A}}\left(\frac{n\pi}{L}\right)^2 t\right)$