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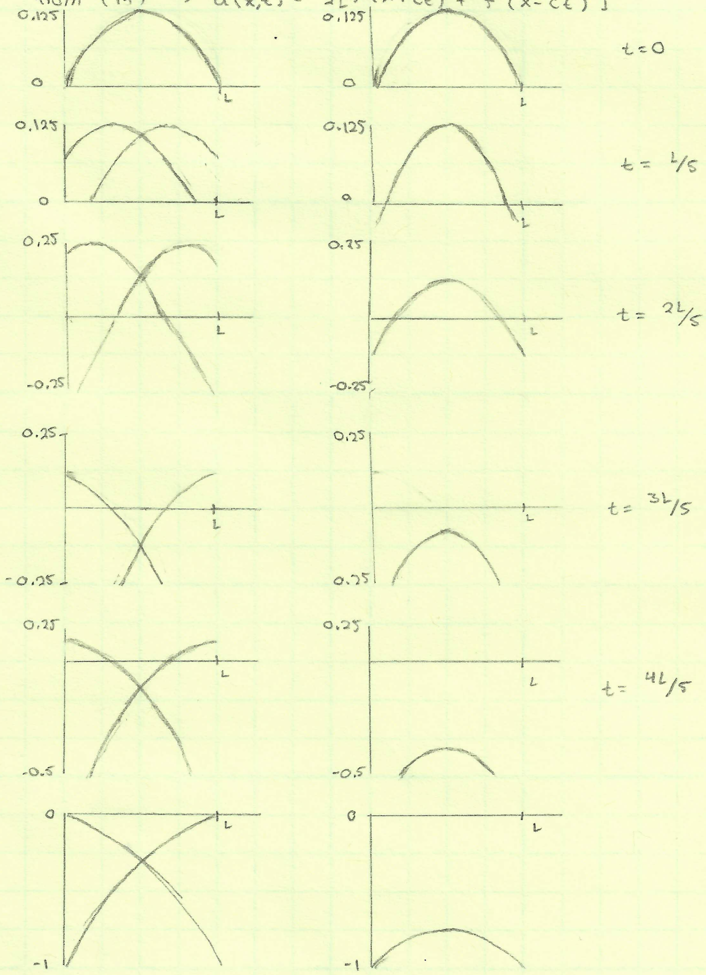
Problem # 8 : Sketch deflection $u(x,t)$ of vibrating string starting with initial velocity and initial deflection

$L = 1$, $c = 1$, $k = 0.01$

$f(x) = kx(1-x) \rightarrow kx - kx^2$

from (13) $\rightarrow u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$



Problem # 19 : Vibrations in the direction of the x-axis are modeled by the wave equation $u_{tt} = c^2 u_{xx}$, $c^2 = E/\rho$. If the rod is fastened at one end $x = 0$ and free at the other, $x = L$, we have $u(0,t) = 0$ and $u_x = (L,t)$. Show that the motion corresponding to initial displacement $u(x,0) = f(x)$ and initial velocity zero is

$$u = \sum^{\infty} A_n \sin(p_n x) \cos(p_n ct)$$

$$A_n = \frac{2}{L}\int_0^L f(x) \sin(p_n x)\, dx \qquad p_n = \frac{(2n+1)\pi}{2L}$$

From wave equation $\rightarrow \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ $\qquad c^2 = E/\rho$

Determine Boundary Conditions

$u(0,t) = 0$, $u_x(L,t) = 0$, $u(x,0) = f(x)$, $u_t(x,0) = 0$

$\rightarrow$ for all t $\qquad \rightarrow 0 < x < L$

Let $u(x,t) = F(x)\,G(t)$

$$\frac{\partial^2 u}{\partial t^2} = F\ddot{G}, \qquad \frac{\partial^2 u}{\partial x^2} = \ddot{F}G$$

Insert into wave equation

$$F\ddot{G} = c^2 \ddot{F} G$$

Rearrange

$$\ddot{F}/F = \ddot{G}/c^2 G = \text{constant} = \beta^2$$

$\rightarrow \ddot{F} - \beta^2 F = 0$

$\rightarrow \ddot{G} - c^2\beta^2 G = 0$

Solution for F and G

$u(0,t) = F(0)\,G(t) = 0$ Let $F(0) = 0$

$u_x(L,t) = F'(L)\,G(t) = 0$ Let $F'(L) = 0$

General solution for $\ddot{F} + \beta^2 F = 0$

$\rightarrow F(x) = A\cos(\beta x) + B\sin(\beta x)$

but $F(0) = 0 \rightarrow A\cos(\beta 0) + B\sin(\beta 0) \rightarrow A = 0$

$F'(L) = 0 \rightarrow B\beta\cos(\beta L) = 0$ but $B\beta \neq 0 \rightarrow \cos(\beta L) = 0$

$\therefore \beta L = \dfrac{(2n+1)\pi}{2} \rightarrow \beta = \dfrac{(2n+1)\pi}{2L}$ where n is an integer

Set $B = 1$ to simplify

$\rightarrow F_n(x) = \sin(\beta x) = \sin\left(\dfrac{(2n+1)\pi}{2L}x\right)$ where n is an integer

General solution for $\ddot{G} - c^2\beta^2 G$

$\rightarrow G(t) = B_n\cos(c\beta t) + B_n^*\sin(c\beta t)$

$\rightarrow u(x,t) = \Sigma(B_n\cos(c\beta t) + B_n^*\sin(c\beta t))\sin(\beta x)$ from section 12.3 (12)

from Boundary conditions

$u(x,0) = f(x) = \Sigma(B_n\cos(0) + B_n^*\sin(0))\sin(\beta x)$

$\rightarrow B_n\sin(\beta x) = f(x)$

Solve for B_n

$B_n = \dfrac{2}{L}\int_0^L f(x)\sin(\beta x)\,dx$ from section 12.3 (14)

$\rightarrow u_t(x,t) = \Sigma((-B_n\beta c\sin(\beta c t) + B_n^*\beta c\cos(c\beta t))\sin(\beta x)$

from Boundary Condition

$u_t(x,0) = \Sigma((-B_n\beta c\sin(0) + B_n^*\beta c\cos(0))\sin(\beta x) = 0$

$\rightarrow B_n^*\beta c\sin(\beta x) = 0$

Solve for B_n^*

$B_n^* = \dfrac{2}{\beta}\int_0^L g(x)\sin(\beta x)\,dx$ from section 12.3 (15)

Given initial velocity, $g(x) = 0$

$\rightarrow B_n^* = 0$

Substituting into $u(x,t) = \Sigma(B_n\cos(c\beta t) + B_n^*\sin(c\beta t))\sin(\beta x)$

$\rightarrow u(x,t) = \Sigma(B_n\cos(c\beta t) + 0)\sin(\beta x)$

$u(x,t) = \Sigma B_n\cos(c\beta t)\sin(\beta x)$ where $\beta = \dfrac{(2n+1)\pi}{2L}$

$B_n = \dfrac{2}{L}\int_0^L f(x)\sin(\beta x)\,dx$

Let $B_n = A_n$, Let $\beta = p_n$

$$\boxed{\therefore u(x,t) = \sum_{n=0}^{\infty} A_n\cos(p_n t)\sin(p_n x) \quad \text{where} \quad A_n = \frac{2}{L}\int_0^L f(x)\sin(p_n x)\,dx, \quad p_n = \frac{(2n+1)\pi}{2L}}$$

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Problem # 7: Find the temperature $u(x,t)$ in a bar of silver of length 10 cm and constant cross section area 1 cm^2 (density 10.6 g/cm^3, thermal conductivity 1.04 cal/(cm sec °C), specific heat 0.056 cal/(g°C) that is perfectly insulated laterally, with ends kept at temperature 0°C and initial temperature $f(x)$°C where

$f(x) = x(10-x) = 10x - x^2$

$u(x,t) = \sum^{\infty} u_n(x,t) = \sum^{\infty} B_n\sin\left(\dfrac{n\pi x}{L}\right)e^{-\lambda_n^2 t}$ $\left(\lambda_n = \dfrac{cn\pi}{L}\right)$ from Section 12.6 (9)

from initial conditions

$u(x,0) = \Sigma B_n\sin\left(\dfrac{n\pi x}{L}\right)(1) = f(x)$

$B_n = \dfrac{2}{L}\int_0^L f(x)\sin\left(\dfrac{n\pi x}{L}\right)dx$ from Section 12.6 (10)

$\rightarrow \dfrac{2}{L}\int_0^L (10x - x^2)\sin\left(\dfrac{n\pi x}{L}\right)dx$ where $L = 10$

$\rightarrow \dfrac{2}{10}\int_0^{10}(10x - x^2)\sin\left(\dfrac{n\pi x}{10}\right)dx$ Let $u = (10x - x^2)$ $du = (10 - 2x)$

$dv = \sin\left(\dfrac{n\pi x}{10}\right)$ $v = -\dfrac{10}{n\pi}\cos\left(\dfrac{n\pi x}{10}\right)$

$$\to \frac{1}{5}\left[\left\{(10x-x^2)\frac{-10}{n\pi}\cos\left(\frac{n\pi x}{10}\right)\right\}_0^{10} - \int_0^{10}(10-2x)\frac{-10}{n\pi}\cos\left(\frac{n\pi x}{10}\right)dx\right]$$

Let $u=(10-2x)$ $du=-2$

$dv=\cos\left(\frac{n\pi x}{10}\right)$ $v=\sin\left(\frac{n\pi x}{L}\right)\frac{10}{n\pi}$

$$\to \frac{1}{5}\frac{-10}{n\pi}\left[\left\{(10x-x^2)\cos\left(\frac{n\pi x}{10}\right)\right\}_0^{10} - \left\{(10-2x)\sin\left(\frac{n\pi x}{10}\right)\right\}_0^{10} + \int_0^{10}\frac{-2(10)}{n\pi}\sin\left(\frac{n\pi x}{10}\right)dx\right]$$

$$\to \frac{1}{5}\frac{-10}{n\pi}\left[\{0\}-\{0\} + \frac{2(10)^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{10}\right)\Big|_0^{10}\right]$$

$$\to \frac{-2}{n\pi}\left[\frac{2(10)^2}{n^2\pi^2}\right]\{\cos(n\pi)-1\}$$

$$\to \frac{-4(100)}{n^3\pi^3}((-1)^n-1)$$

$\to$ when n is even $B_n=0$

when n is odd $B_n=\frac{800}{n^3\pi^3}$

$c^2 = K/(\sigma\rho) = (1.04 \text{ cal}/(\text{cm sec}^\circ C))/[(0.056 \text{ cal}/(g\,^\circ C))(10.6\ g/cm^3)] = 1.752\ cm^2/sec$

$\lambda_1^2 = \frac{c^2\pi^2}{10^2} = 1.752\pi^2/10$

$\lambda_3^2 = 9(1.752\pi^2/10)$

$\lambda_5^2 = 25(1.752\pi^2/10)$

Substitute into solution

$$\boxed{u(x,t)=\sum_{n=1}^{\infty}u_n(x,t) = \frac{800}{\pi^3}\left[\sin\left(\frac{\pi x}{10}\right)e^{-(1.752\pi^2/10)t} + \frac{1}{3^3}\sin\left(\frac{3\pi x}{10}\right)e^{-(9\cdot 1.752\pi^2/10)t} + \frac{1}{5^3}\sin\left(\frac{5\pi x}{10}\right)e^{-(25\cdot 1.752\pi^2/10)t} + \dots\right.}$$

Problem #11: Show that for a completely insulated bar, $u_x(0,t)=0$, $u_x(L,t)=0$, $u(x,t)=f(x)$ and separation of variables gives the following solution, with A_n given by (2) in sec 11.3

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right)e^{-(cn\pi/L)^2 t}$$

one dimensional heat equation $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ section 12.6 (1)

conditions $\to$ $u_x(0,t)=0$, $u_x(L,t)=0$, $u(x,t)=f(x)$

Let $u(x,t)=\sum_{n=0}^{\infty}X(x)\tau(t)$ be the solution of the one dimensional heat equation

$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \to X(x)\tau'(t) = c^2X''(x)\tau(t)$ by substitution

$\to \frac{\tau'(t)}{\tau(t)}\frac{1}{c^2} = \frac{X''(x)}{X(x)} = K = \text{constant}$

$K=0$ and $K>0$ lead to trivial solutions

for negative K Let $K=-z^2$

$\to \frac{\tau'(t)}{\tau(t)}\frac{1}{c^2} = \frac{X''(x)}{X(x)} = -z^2$

Rearrange using algebra

$X''(x)+X(x)z^2=0$ (1)

$\tau'(t)+z^2c^2\tau(t)=0$ (2)

Solving for (1) gives general solution

$X(x) = A\cos(zx) + B\sin(zx)$

condition $u_x(0,t)$ $\to$ $X'(0) = -zA\cancelto{0}{\sin(z0)} + zB\cos(z0) = 0$ but $z\neq 0$

$\therefore B=0$

condition $u_x(L,t)$ $\to$ $X'(L) = -zA\sin(zL) = 0$

To make $\sin(zL)=0$ $zL=n\pi$ $\to$ $z=\frac{n\pi}{L}$ for integer values n

Substitute for z in general solution: $X(x)=\cos\left(\frac{n\pi}{L}x\right)$ Assume $A=1$ for simplicity

solving for (2) given $z=\frac{n\pi}{L}$

$\tau'(t) + \left(\frac{n\pi}{L}\right)^2c^2\tau(t)=0$

Let $\lambda_n = \frac{cn\pi}{L}$ $\to$ $\tau'(t)+\lambda_n^2\tau(t)=0$

General solution $\to$ $\tau(t) = A_n e^{-\lambda_n^2 t}$

$\therefore u(x,t) = \sum_{n=0}^{\infty}X(x)\tau(t) = \sum_{n=0}^{\infty}A_n\cos\left(\frac{n\pi x}{L}\right)e^{-\lambda_n^2 t}$ where $\lambda_n=\frac{cn\pi}{L}$

Can be rewritten as $u(x,t) = A_0\cancelto{1}{\cos(0)}\,\cancelto{1}{e^{0}} + \sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi x}{L}\right)e^{-\lambda_n^2 t}$

$$\boxed{\to u(x,t) = A_0 + \sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi x}{L}\right)e^{-\lambda_n^2 t} \quad \text{where } \lambda_n = \frac{cn\pi}{L}}$$

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Problem # 18 : Find the potential in the rectangle $0 \le x \le 20$, $0 \le y \le 40$ whose upper side is kept at potential 110V and whose other sides are grounded.

Consider figure 296 on Page 564, from problem: $a = 20$, $b = 40$

From section 12.6 (18) $\rightarrow$ $u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} F_n(x)G_n(y) = \sum_{n=1}^{\infty} A_n^* \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)$

Applying initial conditions $u(x,b) = f(x) = \sum_{n=1}^{\infty} A_n^* \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi b}{a}\right)$

where $A^*_n = \frac{2}{a\sinh(n\pi b/a)} \int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx \rightarrow \frac{2}{20\sinh(40n\pi/20)} \int_0^{20} (110)\sin\left(\frac{n\pi x}{20}\right)dx$

$\rightarrow \frac{220}{20\sinh(2n\pi)} \int_0^{20} \sin\left(\frac{n\pi x}{20}\right)dx \rightarrow \frac{220}{20\sinh(2n\pi)} \frac{20}{n\pi}\left(-\cos\left(\frac{n\pi x}{20}\right)\right)_0^{20}$

$\rightarrow \frac{-220}{n\pi\sinh(2n\pi)}\left((-1)^n - 1\right)$

$\rightarrow A_n^* = \begin{cases} \frac{440}{n\pi\sinh(2n\pi)} & \text{for odd } n \\ 0 & \text{for even } n \end{cases}$

$$\boxed{\therefore u(x,t) = \sum_{n=1}^{\infty} A_n^* \sin\left(\frac{n\pi x}{20}\right)\sinh\left(\frac{n\pi y}{20}\right) \rightarrow \frac{440}{\pi}\left[\frac{1}{\pi\sinh(2\pi)}\sin\left(\frac{\pi x}{20}\right)\sinh\left(\frac{\pi y}{20}\right) + \frac{1}{3\sinh(6\pi)}\sin\left(\frac{3\pi x}{20}\right)\sinh\left(\frac{3\pi y}{20}\right) + \frac{1}{5\sinh(10\pi)}\sin\left(\frac{5\pi x}{20}\right)\sinh\left(\frac{5\pi y}{20}\right) + \dots\right]}$$

Problem # 21: The faces of the thin square plate in figure 297 with side $a = 24$ are perfectly insulated. The upper side is kept at 25°C and the other sides are kept at 0°C. Find the steady state temperature $u(x,y)$ in the plate

from Section 12.6 (18) $\rightarrow$ $u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} F_n(x)G_n(y) = \sum_{n=1}^{\infty} A_n^* \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)$

Applying initial conditions $u(x,b) = f(x) = \sum_{n=1}^{\infty} A_n^* \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi b}{a}\right)$

Where $A^*_n = \frac{2}{a\sinh(n\pi b/a)} \int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx \rightarrow \frac{2}{24\sin(n\pi)} \int_0^{24} (25)\sin\left(\frac{n\pi x}{24}\right)dx$

$\rightarrow \frac{50}{24\sin(n\pi)} \int_0^{24} \sin\left(\frac{n\pi x}{24}\right)dx \rightarrow \frac{50}{24\sin(n\pi)} \frac{24}{n\pi}\left(-\cos\left(\frac{n\pi x}{24}\right)\right)_0^{24}$

$\rightarrow \frac{-50}{n\pi\sin(n\pi)}\left(\cos(n\pi) - 1\right)$

$\rightarrow \frac{-50}{n\pi\sin(n\pi)}\left((-1)^n - 1\right)$

$\rightarrow A_n^* = \begin{cases} \frac{100}{n\pi\sin(n\pi)} & \text{for odd } n \\ 0 & \text{for even } n \end{cases}$

Another way to symbolize on odd $n \rightarrow (2n-1)$

Combine with general solution

$$u(x,t) = \sum_{n=1}^{\infty} A_n^* \sin\left(\frac{n\pi x}{24}\right)\sinh\left(\frac{n\pi y}{24}\right) = \sum_{n=1}^{\infty} \frac{100}{(2n-1)\pi\sin((2n-1)\pi)}\sin\left(\frac{(2n-1)\pi x}{24}\right)\sinh\left(\frac{(2n-1)\pi y}{24}\right)$$

$$\rightarrow \boxed{u(x,t) = \frac{100}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)\sin((2n-1)\pi)}\sin\left(\frac{(2n-1)\pi x}{24}\right)\sinh\left(\frac{(2n-1)\pi x}{24}\right)}$$

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Problem # 4abc : Series for Dirichlet and Neumann Problems

a) Show that $u_n = r^n\cos(n\theta)$, $u_n = r^n\sin(n\theta)$, $n = 0, 1 \dots$ are solutions of Laplace equation $\nabla^2 u = 0$ with $\nabla^2 u$ given by (5). (what would u_n be in cartesian coordinates? experiment with small n

from Section 12.10 (5) $\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$

For $u_n = r^n\cos(n\theta)$

$\frac{\partial^2 u}{\partial r^2} = n(n-1)r^{n-2}\cos(n\theta)$

$\frac{1}{r}\frac{\partial u}{\partial r} = nr^{(n-1)}\frac{1}{r}\cos(n\theta) = nr^{n-2}\cos(n\theta)$

$\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{r^2}r^n n^2(-\cos(n\theta)) = r^{n-2}n^2(-\cos(n\theta))$

$\nabla^2 u = n(n-1)r^{n-2}\cos(n\theta) + nr^{n-2}\cos(n\theta) - r^{n-2}n^2\cos(n\theta)$

$\rightarrow r^{n-2}\left(n(n-1)\cos(n\theta) + n\cos(n\theta) - n^2\cos(n\theta)\right)$

$\rightarrow r^{n-2}\left(\cancel{n^2\cos(n\theta)} - \cancel{n\cos(n\theta)} + \cancel{n\cos(n\theta)} - \cancel{n^2\cos(n\theta)}\right)$

$\boxed{\therefore \nabla^2 u = 0}$

For $u_n = r^n \sin(n\theta)$

$$\frac{\partial^2 u}{\partial r^2} = n(n-1)\sin(n\theta)\, r^{n-2}$$

$$\frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{r} n r^{n-1}\sin(n\theta) = n r^{n-2}\sin(n\theta)$$

$$\frac{\partial^2 u}{\partial \theta^2}\frac{1}{r^2} = -\frac{1}{r^2} r^n n^2 \sin(n\theta) = -n^2 r^{n-2}\sin(n\theta)$$

$$\nabla^2 u = n(n-1)\sin(n\theta)\, r^{n-2} + n r^{n-2}\sin(n\theta) - n^2 r^{n-2}\sin(n\theta)$$

$$\rightarrow r^{n-2}\left(n(n-1)\sin(n\theta) + n\sin(n\theta) - n^2\sin(n\theta)\right)$$

$$\rightarrow r^{n-2}\left(\cancel{n^2 \sin(n\theta)} - \cancel{n\sin(n\theta)} + \cancel{n\sin(n\theta)} - \cancel{n^2\sin(n\theta)}\right)$$

$$\boxed{\therefore \nabla^2 u = 0}$$

b) Assuming that termwise differentiation is permissible, show that a solution of the Laplace equation in the disk $r < R$ satisfying the boundary condition $u(R,\theta) = f(\theta)$ (R and f given) is

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty}\left[a_n\left(\frac{r}{R}\right)^n \cos(n\theta) + b_n\left(\frac{r}{R}\right)^n \sin(n\theta)\right]$$

where a_n and b_n are the Fourier coeffscients of f

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Substituting $u(r,\theta) = R(r)\Theta(\theta)$

$$\nabla^2 u = R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$ let this be a solution of $u(r,0) = R(r)\Theta(\theta)$

$$\frac{r^2 R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta}$$ by rearranging

Set the equation equal to a constant λ^2

$\rightarrow \Theta'' + \lambda^2\Theta = 0$ (1)

$\rightarrow r^2 R'' + rR' - \lambda^2 R = 0$ (2)

For $K^2 \neq 0$ general solution (1)

$\Theta(\theta) = A\cos(K\theta) + B\sin(K\theta)$

From Boundary conditions $\Theta(0) = \Theta(2\pi)$ and $\Theta'(0) = \Theta'(2\pi)$ gives

$A(-1 + \cos(2\pi K)) + B\sin(2\pi K) = 0$

$-AK\sin(2\pi K) + BK(-1 + \cos(2\pi K)) = 0$

Non-trivial solution $\lambda_n = n$ with $n = 1, 2, 3 \ldots$

For $\lambda = 0$ the general solution is $\Theta(\theta) = A + B\theta$

Non trivial solution $\Theta(0) = A \quad \therefore \Theta(\theta) = A$

For equation (2) $\lambda \neq 0$

Possible solution $(\alpha^2 - \lambda)r^\alpha = 0$

which implies $\alpha = \pm\lambda = n^2$ for $\lambda = 0$, reduces to $r(rR')' = 0$

$\rightarrow$ general solution $R(r) = C_1 \ln(r) + C_2$

Combine the solutions

$$u(r,\theta) = \cancelto{0}{C_1}\ln(r) + \cancelto{\frac{1}{2}A_0}{C_2} + \sum_{n=1}^{\infty}\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right)r^n + \sum_{n=1}^{\infty}\left(\cancelto{0}{\tilde{A}_n}\cos(n\theta) + \cancelto{0}{\tilde{B}_n}\sin(n\theta)\right)r^{-n}$$

$$\rightarrow u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right)r^n$$

To account for entire range of θ and r, r should be a ratio of the maximum

$$\boxed{\therefore u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left[A_n\cos(n\theta) + B_n\sin(n\theta)\right]\left(\frac{r}{R}\right)^n}$$

c) Solve the problem using (20) if $R = 1$ and the boundary conditions $u(\theta) = -100$ if $-\pi < \theta < 0$, $u(\theta) = 100$ volts if $0 < \theta < \pi$

$$f(\theta) = \begin{cases} -100 & -\pi < \theta < 0 \\ 0 & \theta = 0 \\ 100 & 0 < \theta < \pi \end{cases}$$ $\rightarrow$ From Section 11.1 Example 1 $f(x) = \sum \frac{100(1-(-1)^n)}{n\pi}\sin(nx)]$ for odd n

$$\rightarrow f(\theta) = \sum_{n=1}^{\infty}\frac{100(1-(-1)^n)}{\pi n}\sin(n\theta)$$

Substitute for $u(r,\theta)$

$$u(r,\theta) = \frac{1}{2}A_0 + \sum\left[A_n\cos(n\theta) + B_n\sin(n\theta)\right] (1)$$

$$\rightarrow \boxed{u(r,\theta) = \frac{1}{2}(100) + \sum_{n=1}^{\infty}\frac{100(1-(-1)^n)}{n\pi}\sin(n\theta)} \quad ?$$

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Problem #4: Represent $f(x,y)$ by a series (15) where

$f(x,y) = 1 \quad a=b=1$

$$u(x,y,0) = \sum_m^\infty \sum_n^\infty B_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) = f(x,y)$$

$$B_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy = 4\int_0^1\int_0^1 (1)\sin(m\pi x)\sin(n\pi y)\,dx\,dy$$

$$= \frac{4}{n\pi}\int_0^1 \sin(m\pi x)(-\cos(n\pi y))\Big|_0^1 dx = \frac{4}{n\pi}\int_0^1 \sin(m\pi x)((-1)^n - 1)\,dx$$

$$= \frac{4}{n\pi}((-1)^m - 1)\int_0^1 \sin(m\pi x)\,dx = \frac{4}{mn\pi}((-1)^m-1)\left[-\cos(m\pi x)\right]_0^1$$

$$= \frac{4}{mn\pi}((-1)^m-1)((-1)^n-1)$$

Note only work if m and n are odd

$$B_{mn} = \frac{4}{mn\pi}(2)(2) = \frac{16}{mn\pi} \quad \text{for odd } m \text{ and } n;\ 0 \text{ otherwise}$$

Substitute into $u(x,y,0)$

$$u(x,y,0) = \sum_m^\infty\sum_n^\infty \frac{16}{mn\pi}\sin(m\pi x)\sin(n\pi y)$$

$$\boxed{u(x,y,0) = \frac{16}{\pi}\sum_{n=1}^\infty\sum_{m=1}^\infty \frac{\sin(m\pi x)\sin(n\pi y)}{mn} \quad \text{for } m \text{ and } n \text{ odd} \quad 0 \text{ otherwise}}$$

Problem #5: See Problem #4 description

$f(x,y) = y \quad a=b=1$

$$u(x,y,0) = \sum\sum B_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$B_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi x}{b}\right)dx\,dy = 4\int_0^1\int_0^1 (y)\sin(m\pi x)\sin(n\pi y)\,dx\,dy$$

$$= \frac{4}{m\pi}\int_0^1 (-\cos(m\pi x))\Big|_0^1 (y)\sin(n\pi y)\,dy = \frac{4}{m\pi}((-1)^m-1)\int_0^1 y\sin(n\pi y)\,dy$$

$$= \frac{4}{m\pi}((-1)^m-1)\left[\frac{\sin(n\pi y) - n\pi\cos(n\pi y)}{(n\pi)^2}\right]_0^1 = \frac{4}{m\pi^3 n^2}((-1)^m-1)\left[\frac{(0) - n\pi((-1)^n-1)}{1}\right]$$

$$= \frac{4}{mn\pi^2}((-1)^m-1)(-(-1)^n+1) = \frac{8}{mn\pi^2}((-1)^{n+1}+1)$$

$$B_{mn} = \frac{16}{mn\pi^2} \quad \text{when } m \text{ is odd and } n \text{ is even} \quad \text{else } 0$$

Substitute into $u(x,y,0)$

$$\boxed{u(x,y,0) = \begin{cases} \dfrac{16}{\pi^2}\displaystyle\sum_{n=1}^\infty\sum_{m=1}^\infty \frac{\sin(m\pi x)\sin(n\pi y)}{mn} & \text{for } m \text{ being odd and } n \text{ being even} \\ 0 & \text{otherwise}\end{cases}}$$

Problem #7: See problem #4 description

$f(x,y) = xy \quad a$ and b are arbitrary

$$B_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy = \frac{4}{ab}\int_0^b\int_0^a (xy)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy$$

$$= \frac{4}{ab}\int_0^b\left[\frac{\sin(m\pi x/a) - m\pi\cos(m\pi x/a)}{(m\pi/a)^2}\right]_0^a y\sin\left(\frac{n\pi y}{b}\right)dy$$

$$= \frac{4a}{m\pi b}\int_0^b -((-1)^m+1)y\sin\left(\frac{n\pi x}{b}\right)dy = \frac{4}{m\pi b}(-((-1)^m+1))\left[\frac{\sin(n\pi y/b) - m\pi/b\cos(m\pi y/b)}{(n\pi/b)^2}\right]_0^b$$

$$= \frac{4ab}{m\pi^2 n}(-((-1)^m+1))(-((-1)^n+1))$$

$$B_{mn} = \frac{4ab}{m\pi^2 n}(-1)^{m+n}$$

Substitute into $u(x,y,0)$

$$\boxed{u(x,y,0 = \frac{4ab}{\pi^2}\sum_{m=1}^\infty\sum_{n=1}^\infty \frac{\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)}{mn}}$$

Problem #18: Show that among all rect[illegible]lar membranes of the same area $A = ab$ and the same c the square membrane is that for which u_{11} has the lowest frequency

Let $A = ab \to b = A/a$

Take $\lambda_{mn} = \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} = 0$

Lowest frequency and the first derivative must be zero, substitute for b

$$\lambda_{mn} = \sqrt{m^2/a^2 + n^2a^2/A^2} = 0 \quad \text{and} \quad m^2(-2)a^{-3} + n^2(2)aA^{-2} = 0$$

$$\to m^2a^{-3} = n^2 a A^{-2} \to \frac{m^2}{n^2} = a^4/A^2 = a^2/b^2 \to \frac{m}{n} = a/b \to \text{for } m=1, n=1 \quad a/b = 1$$

$$\boxed{\text{frequency} = \lambda/2\pi \quad \text{Thus } u_{11} \text{ must have the lowest frequency.}}$$