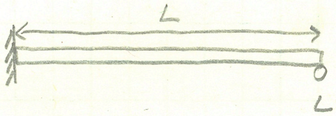
19)



BC's

$$\begin{cases} u(0,t)=0 \\ u_x(L,t)=0 \end{cases} \qquad u_{tt} = c^2 u_{xx} \qquad c^2 = \frac{E}{\rho}$$

Initial Conditions : $\begin{cases} u(x,0)=f(x) \\ u_t(x,0)=0 \end{cases}$

Show that...

$$u = \sum_{n=0}^{\infty} A_n \sin p_n x \cos p_n c t$$

where $A_n = \frac{2}{L}\int_0^L f(x)\sin(p_n x)\,dx \qquad p_n = \frac{(2n+1)\pi}{2L}$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Let $u(x,t) = F(x)G(t)$

$$\frac{\partial^2 u}{\partial t^2} = F\ddot{G} \qquad \frac{\partial^2 u}{\partial x^2} = F''G$$

$$F\ddot{G} = c^2 F''G \qquad \text{so} \quad \frac{F''}{F} = \frac{\ddot{G}}{c^2 G} = k$$

$$F'' - kF = 0$$

$$\ddot{G} - c^2 k G = 0$$

Now for BC's

$u(0,t) = 0 = F(0)G(t) \qquad F(0)=0$

$u_x(L,t) = 0 = F'(L)G(t) \qquad F'(L)=0$

$F(x) = A\cos(px) + B\sin(px)$

$F(0) = A = 0$

$F'(L) = pB\cos(pL) = 0 \qquad pB \neq 0 \text{ so } \quad pL = \cos^{-1}(0) \quad pL = \frac{n\pi}{2} \quad p = \frac{n\pi}{2L}$

$F(x) = \sin\left(\frac{n\pi x}{2L}\right)$

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Now for $G(t)$ $\quad k = -p^2 = -\left(\frac{n\pi}{2L}\right)^2 \quad cp = \lambda_n = \frac{cn\pi}{2L}$

$$\ddot{G} + \lambda_n^2 G = 0$$

$$s^2 + \lambda_n^2 = 0 \qquad s = \pm \lambda_n$$

$$G_n(t) = A_n \cos(\lambda_n t) + B_n \sin(\lambda_n t)$$

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos(\lambda_n t) + B_n \sin(\lambda_n t)\right)\left(\sin \frac{n\pi x}{2L}\right)$$

For initial conditions

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{2L} = f(x)$$

$$\text{so} \quad A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{2L}\right) dx \quad \text{but} \quad p_n = \frac{n\pi}{2L}$$

$$\boxed{\therefore A_n = \frac{2}{L}\int_0^L f(x) \sin(p_n x) dx \ \checkmark}$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \left(\lambda_n A_n \cancelto{0}{\sin(\lambda_n t)} - \lambda_n B_n \cos(\lambda_n t)\right) \sin\left(\frac{n\pi x}{2L}\right)\Bigg|_{t=0} = 0$$

$$\sum_{n=1}^{\infty} -\lambda_n B_n \sin\left(\frac{n\pi x}{2L}\right) = 0 \qquad \therefore \quad B_n = 0$$

$$\boxed{\text{so} \quad u(x,t) = \sum_{n=1}^{\infty} A_n \cos(\lambda_n t) \sin(p_n x) = \sum_{n=1}^{\infty} A_n \cos(c p_n t) \sin(p_n x)}$$

Which is zero for $n = 2, 4, 6, \ldots$ so $p_n = \frac{(2n+1)\pi}{2L}$ for $n = 0, 1, 2, 3$

$$\boxed{\text{so}\ldots \quad u(x,t) = \sum_{n=0}^{\infty} A_n \cos(c p_n t) \sin(p_n x)}$$

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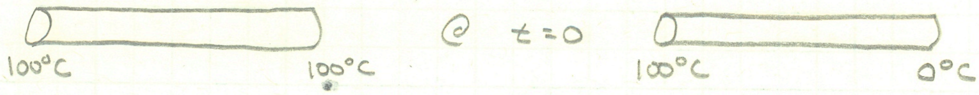
10.)



$u(x,0) = 100°C$ $\quad\quad \lambda_n = 0.01752(n\pi)^2 = \dfrac{c^2\pi^2 n^2}{L^2}$

$u(0,t) = 100°C$

$u(L,t) = 0°C$

Let $u = (0-100)\dfrac{x}{L} + 100 + W(x,t) = 100 - \dfrac{100x}{L} + W(x,t)$

$u(0,t) = 100 + W(x,t) = 100°$ $\quad\quad W(0,t) = 0°C$

$u(L,t) = 0 = W(x,t)$ $\quad\quad W(L,t) = 0°C$

$u(x,0) = 100 = 100 - \dfrac{100x}{L} + W(x,0)$ $\quad\quad W(x,0) = \dfrac{100}{L}x$

$W_t = c^2 W_{xx}$

$$W(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{cn\pi}{L}\right)^2 t}$$

$$B_n = \frac{2}{L}\int_0^L \frac{100}{L} x \sin\left(\frac{n\pi x}{L}\right) dx$$

$$B_n = \frac{2}{\cancel{10}}\int_0^{10} \cancel{10} x \sin\left(\frac{n\pi x}{10}\right) dx \quad\quad u = x \quad v' = \sin\left(\frac{n\pi x}{10}\right)$$

$du = dx \quad\quad v = -\dfrac{10}{n\pi}\cos\left(\dfrac{n\pi x}{10}\right)$

$$= 2\left(-\frac{10x}{n\pi}\cos\left(\frac{n\pi x}{10}\right)\Big|_0^{10} + \frac{10}{n\pi}\int_0^{10}\cos\left(\frac{n\pi x}{10}\right)dx\right)$$

$$= 2\left(-\frac{100}{n\pi}\cos(n\pi) + \frac{100}{n^2\pi^2}\left(\cancel{\sin\left(\frac{n\pi x}{10}\right)}^{\,0}\Big|_0^{10}\right)\right)$$

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$$B_n = -\frac{200}{n\pi}\cos(n\pi)$$

$$\boxed{\begin{gathered}\therefore\ u(x,t) = 10(10-x) + \sum_{n=1}^{\infty}\left(-\frac{200}{n\pi}\cos(n\pi)\right) e^{-0.01752(n\pi)^2 t}\sin\left(\frac{n\pi x}{10}\right) \\ \text{at } x = 5 \\ u(5,t) = 50 + \left(\frac{200}{\pi}e^{-0.01752\pi^2 t} - \frac{200}{3\pi}e^{-0.01752\cdot 9\pi^2 t} + \frac{200}{5\pi}e^{-0.01752\cdot 25\pi^2 t} + \ldots\right)\end{gathered}}$$

t =	1	2	3	10	50
u(5,t) =	99°C	94°C	88°C	61°C	50°C

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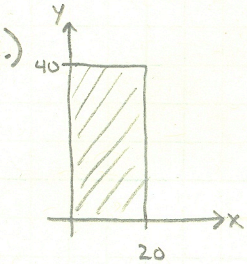
18.)



$u(x,40) = 110\,V \qquad u(0,y) = 0$

$u(x,0) = 0 \qquad u(20,y) = 0$

$$u(x,y) = \sum_{n=1}^{\infty} A_n^{*} \sin\left(\frac{n\pi x}{20}\right) \sinh\left(\frac{n\pi y}{20}\right)$$

$$A_n^{*} = \frac{2\cdot \cancel{110}^{\,11}}{\cancel{20}_{\,10}\sinh(2n\pi)} \int_0^{20} \sin\left(\frac{n\pi x}{20}\right) dx$$

$$= \frac{11}{\sinh(2n\pi)}\ \frac{20}{n\pi}\left(-\cos\left(\frac{n\pi x}{20}\right)\Big|_0^{20}\right)$$

$$= \frac{220}{n\pi \sinh(2n\pi)} \left(-\cos(n\pi) + 1\right)$$

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} \frac{220}{n\pi\sinh(2n\pi)} \left(1-\cos(n\pi)\right) \sin\left(\frac{n\pi x}{20}\right) \sinh\left(\frac{n\pi y}{20}\right)}$$

or

$$\boxed{u(x,y) = \frac{440}{\pi}\left(\frac{\sin\left(\frac{\pi}{20}x\right)\sinh\left(\frac{\pi y}{20}\right)}{\sinh(2\pi)} + \frac{\sin\left(\frac{3\pi x}{20}\right)\sinh\left(\frac{3\pi y}{20}\right)}{3\sinh(6\pi)} + \dots\right)}$$

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7) $f(x,y) = xy$ a and b arbitrary

$$B_{mn} = \frac{4}{ab}\int_0^a\int_0^b x\sin\left(\frac{m\pi x}{a}\right) y\sin\left(\frac{n\pi y}{b}\right)dx\,dy$$

Let $u = x$ $v' = \sin\frac{m\pi x}{a}$

$du = dx$

$v = -\frac{a}{m\pi}\cos\left(\frac{m\pi x}{a}\right)$

$$B_{mn} = \frac{4}{ab}\left(-\frac{ax}{m\pi}\cos\left(\frac{m\pi x}{a}\right)\Big|_0^a + \frac{a}{m\pi}\int_0^a \cos\left(\frac{m\pi x}{a}\right)dx\right)\int_0^b y\sin\left(\frac{n\pi y}{b}\right)dy$$

$$= \frac{4}{\cancel{a}b}\left(-\frac{a^2}{m\pi}\cos(m\pi) + \left(\frac{a}{m\pi}\right)^2 \cancel{\sin\left(\frac{m\pi x}{a}\right)\Big|_0^a}^{\,0}\right)\int_0^b y\sin\left(\frac{n\pi y}{b}\right)dy$$

$$= -\frac{4a}{bm\pi}\cos(m\pi)\left(-\frac{by}{n\pi}\cos\left(\frac{n\pi y}{b}\right)\Big|_0^b + \cancel{\frac{b}{n\pi}\int_0^b \cos\left(\frac{n\pi y}{b}\right)dy}^{\,0}\right)$$

Let $u = y$ $v' = \sin\left(\frac{n\pi y}{b}\right)$

$du = dy$ $v = -\frac{b}{n\pi}\cos\left(\frac{n\pi y}{b}\right)$

$$= \frac{4ab}{mn\pi^2}\underbrace{\cos(m\pi)\cos(n\pi)}$$

m=1	n=1	1
m=1	n=2	-1
m=1	n=3	1
⋮	⋮	⋮

$$\cos(m\pi)\cos(n\pi) = (-1)^{m+n}$$

$$\boxed{B_{mn} = (-1)^{m+n}\frac{4ab}{mn\pi^2}}$$

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so

$$\boxed{u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{m+n}\left(\frac{4ab}{mn\pi^2}\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)}$$