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Homework # 2

Sec. 7.4 p 287

Find the rank. Find a basis for the row space. Find a basis for the column space. Hint: Row-reduce the matrix and its transpose.

1. $\begin{bmatrix} 4 & -2 & 6 \\ -2 & 1 & -3 \end{bmatrix} \frac{1}{2} \longrightarrow \begin{bmatrix} 4 & -2 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

$\boxed{\text{rank } A = 1}$, $\boxed{RS_A = \text{span}\{[4, -2, 6]\}}$

$\boxed{CS_A = \text{span}\{[4, -2]^T\}}$

Select $x_2 = s$, $x_3 = t$; we have $x_1 = \frac{1}{2}x_2 - \frac{3}{2}x_3$
$= \frac{1}{2}s - \frac{3}{2}t$

$\vec{x} = [1/2, 1, 0]^T s + [-3/2, 0, 1]^T$

$\boxed{\text{nullity } A = 2}$, $\boxed{N_A = \text{span}\{[1/2, 1, 0]^T, [-3/2, 0, 1]^T\}}$

⑤ $\begin{bmatrix} 0.2 & -0.1 & 0.4 \\ 0 & 1.1 & -0.3 \\ 0.1 & 0 & -2.1 \end{bmatrix} \cdot 10 \rightarrow \begin{bmatrix} 2 & -1 & 4 \\ 0 & 11 & -3 \\ 1 & 0 & -21 \end{bmatrix} -\frac{1}{2}$

$\begin{bmatrix} 2 & -1 & 4 \\ 0 & 11 & -3 \\ 0 & \frac{1}{2} & -23 \end{bmatrix} \frac{-1}{22} \rightarrow \begin{bmatrix} 2 & -1 & 4 \\ 0 & 11 & -3 \\ 0 & 0 & -\frac{503}{22} \end{bmatrix}$

$\boxed{\text{rank } A = 3}$, $\boxed{RS_A = \text{span}\{[2,-1,4],[0,11,-3],[0\ 0\ {-503}/{22}]\}}$

$\boxed{CS_A = \text{span}\{[2,0,1]^T,[-1,11,0]^T,[4,-3,-21]^T\}}$

Since there is no free variables $\boxed{\text{nullity} = 0}$, $\boxed{N_A = \{\phi\}}$

⑦ $\begin{bmatrix} 8 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \\ 4 & 0 & 2 & 0 \end{bmatrix} -\frac{1}{2} \longrightarrow \begin{bmatrix} 8 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\boxed{\text{rank } A = 2}$, $\boxed{RS_A = \text{span}\{[8,0,4,0],[0,2,0,4]\}}$

$\boxed{CS_A = \text{span}\{[8,0,4]^T,[0,2,0]^T\}}$

Set $x_3 = s,\ x_4 = t\ ;\ x_2 = -2x_4 \Rightarrow x_2 = -2t$

$x_1 = -\frac{1}{2}x_3 \Rightarrow x_1 = \frac{-1}{2}s$

We have $\boxed{\text{nullity } A = 2}$, $\boxed{N_A = \text{span}\{[-1/2,0,1,0]^T,[0,-2,0,1]^T\}}$

Sec. 7.7 p 300

④ Expansion Numerically Impractical. Show that the computation of an nth-order determinant involves $n!$ multiplications, which if a multiplication takes 10^{-9} sec would take these times:

n	10	15	20	25
Time	0.004 sec	22 min	77 years	$0.5 \cdot 10^{9}$ years

Look at 2x2 matrix $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$

2 multiplications

$n! = 2! = 2$

Look at 3x3 matrix $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} -$

$a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{11} \cdot (a_{22} \cdot a_{33} - a_{23} \cdot a_{32}) -$

$a_{12} \cdot (a_{21} \cdot a_{33} - a_{23} \cdot a_{31}) +$

$a_{13} \cdot (a_{21} \cdot a_{32} - a_{22} \cdot a_{31})$

6 multiplications

$n! = 3! = 6$

Looking at general matrix

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

where C_{1k} is a cofactor which involves $(n-1)!$ multiplications and we have n cofactors

$$\therefore \boxed{n(n-1)! = n!}$$

Showing the details, evaluate:

7. $$\begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\beta & \cos\beta \end{vmatrix} = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Using trig. identity $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$$\therefore \cos\alpha\cos\beta - \sin\alpha\sin\beta = \boxed{\cos(\alpha+\beta)}$$

12. $$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = a\begin{vmatrix} a & b \\ c & a \end{vmatrix} - b\begin{vmatrix} c & b \\ b & a \end{vmatrix} + c\begin{vmatrix} c & a \\ b & c \end{vmatrix}$$

$$= a(a^2-bc) - b(ca-b^2) + c(c^2-ab)$$

$$= a^3 - abc + b^3 - abc + c^3 - abc$$

$$= \boxed{a^3 + b^3 + c^3 - 3abc}$$

Solve by Cramer's rule. Check by Gauss elimination and back substitution. Show details.

(22) $2x - 4y = -24$
$5x + 2y = 0$ $\Rightarrow \left[\begin{array}{cc|c} 2 & -4 & -24 \\ 5 & 2 & 0 \end{array}\right]$

$$D = \begin{vmatrix} 2 & -4 \\ 5 & 2 \end{vmatrix} = 2\cdot 2 - (-4)\cdot 5 = 4 + 20 = 24$$

$$D_1 = \begin{vmatrix} -24 & -4 \\ 0 & 2 \end{vmatrix} = -24\cdot 2 - (-4)\cdot 0 = -48$$

$$D_2 = \begin{vmatrix} 2 & -24 \\ 5 & 0 \end{vmatrix} = 2\cdot 0 - (-24)\cdot 5 = 120$$

$$x = D_1/D = -48/24 = -2 \qquad y = D_2/D = 120/24 = 5$$

$$\boxed{x = -2, \quad y = 5}$$

$$\left[\begin{array}{cc|c} 2 & -4 & -24 \\ 5 & 2 & 0 \end{array}\right] \cdot \left(-\frac{5}{2}\right) \longrightarrow \left[\begin{array}{cc|c} 2 & -4 & -24 \\ 0 & 12 & 60 \end{array}\right]$$

$12y = 60 \Rightarrow y = 60/12 = 5 \qquad \underline{y = 5}$

$2x - 4y = -24 \Rightarrow 2x = -24 + 4(5) = -4$

$\underline{x = -2}$

Sec. 7.8 p 308

Find the inverse by Gauss-Jordan (or by (4*) if n = 2).
Check by using (1).

(2.) $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix} \Rightarrow \begin{vmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{vmatrix} =$

$\cos^2 2\theta + \sin^2 2\theta = 1$

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}^{-1} = \boxed{\begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}}$$

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix} \cdot \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} =$$

$$\begin{bmatrix} \cos^2 2\theta + \sin^2 2\theta & -\sin 2\theta \cos 2\theta + \sin 2\theta \cos 2\theta \\ -\sin 2\theta \cos 2\theta + \sin 2\theta \cos 2\theta & \sin^2 2\theta + \cos^2 2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} \cdot \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix} =$$

$$\begin{bmatrix} \cos^2 2\theta + \sin^2 2\theta & \cos 2\theta \sin 2\theta - \cos 2\theta \sin 2\theta \\ \cos 2\theta \sin 2\theta - \cos 2\theta \sin 2\theta & \cos^2 2\theta + \sin^2 2\theta \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

⑤ $$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 4 & 1 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 5 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -2 \\ -5 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 4 & 1 & -5 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -4 \\ \\ \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 & -4 & 1 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix}}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Formula (4) is occasionally needed in theory. To understand it, apply it and check the result by Gauss-Jordan

⑳ In Prob 6. $$\begin{bmatrix} -4 & 0 & 0 \\ 0 & 8 & 13 \\ 0 & 3 & 5 \end{bmatrix} \Rightarrow \begin{vmatrix} -4 & 0 & 0 \\ 0 & 8 & 13 \\ 0 & 3 & 5 \end{vmatrix} =$$

$$-4(8 \cdot 5 - 13 \cdot 3) = -4$$

$$\boxed{A^{-1} = \frac{-1}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -20 & 52 \\ 0 & 12 & -32 \end{bmatrix}}, \quad \left[\begin{array}{ccc|ccc} -4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 8 & 13 & 0 & 1 & 0 \\ 0 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -\frac{3}{8} \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} -4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 8 & 13 & 0 & 1 & 0 \\ 0 & 0 & 1/8 & 0 & -3/8 & 1 \end{array}\right] \begin{array}{l} \\ -104 \\ \\ \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} -4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 8 & 0 & 0 & 40 & -104 \\ 0 & 0 & 1/8 & 0 & -3/8 & 1 \end{array}\right] \Rightarrow \begin{bmatrix} -1/4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & -3 & 8 \end{bmatrix}$$

Sec. 7.9 p 318

Is the given set, taken with the usual addition and scalar multiplication, a vector space? Give reason. If your answer is yes, find the dimension and a basis.

③ All vectors in $\mathbb{R}^3$ satisfying $-v_1 + 2v_2 + 3v_3 = 0, -4v_1 + v_2 + v_3 = 0$.

If we select vectors satisfying the above we can show that addition and scalar multiplication will satisfy the above criteria.

$$\begin{bmatrix} -1 & 2 & 3 \\ -4 & 1 & 1 \end{bmatrix} -4 \Rightarrow \begin{bmatrix} -1 & 2 & 3 \\ 0 & -7 & -11 \end{bmatrix}$$

$\boxed{\text{rank } A = 2}$, $\boxed{\text{nullity } A = 3 - 2 = 1}$

Setting $x_3 = t$

$-7x_2 = 11x_3 \Rightarrow x_2 = -{}^{11}/_7 t$

$-1x_1 + 2x_2 + 3x_3 \Rightarrow x_1 = 2x_2 + 3x_3$

$= -{}^{22}/_7 t + 3t = -{}^{1}/_7 t$

$$\vec{x} = \begin{bmatrix} -1/7 \\ -11/7 \\ 1 \end{bmatrix} t \qquad \boxed{N_A = \text{span}\{[-1/7, -11/7, 1]\}}$$

4. All skew-symmetric 3x3 matrices.

- Addition:

$$\begin{bmatrix} 0 & a_1 & -a_2 \\ -a_1 & 0 & -a_3 \\ a_2 & a_3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b_1 & -b_2 \\ -b_1 & 0 & -b_3 \\ b_2 & b_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & (a_1+b_1) & -(a_2+b_2) \\ -(a_1+b_1) & 0 & -(a_3+b_3) \\ (a_2+b_2) & (a_3+b_3) & 0 \end{bmatrix}$$

- Scalar multiplication

$$c\begin{bmatrix} 0 & a_1 & -a_2 \\ -a_1 & 0 & -a_3 \\ a_2 & a_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & ca_1 & -ca_2 \\ -ca_1 & 0 & -ca_3 \\ ca_2 & ca_3 & 0 \end{bmatrix}$$

This is a vector space.

This vector space is of $\boxed{\text{dimension } 3}$, because we can form a skew-symmetric matrix with the following 3 bases

$$\boxed{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}}$$

⑥ All functions $y(x) = a\cos 2x + b\sin 2x$ with arbitrary constants a and b

- Addition of two functions

$y(x) = a\cos 2x + b\sin 2x$

$z(x) = c\cos 2x + d\sin 2x$

$y(x) + z(x) = (a+c)\cos 2x + (b+d)\sin 2x$

If $a+c=0$ or $b+d=0$, this will not be a vector space.

⑨ All 2×2 matrices $[a_{jk}]$ with $a_{11} + a_{22} = 0$.

- Addition: $\begin{bmatrix} a_{11} & x \\ x & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & x \\ x & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & x \\ x & a_{22}+b_{22} \end{bmatrix}$

$(a_{11}+b_{11}) + (a_{22}+b_{22}) = (a_{11}+a_{22}) + (b_{11}+b_{22}) = 0$ ✓

- Scalar multiplication: $c\begin{bmatrix} a_{11} & x \\ x & a_{22} \end{bmatrix} = \begin{bmatrix} ca_{11} & x \\ x & ca_{22} \end{bmatrix}$

$ca_{11} + ca_{22} = c(a_{11}+a_{22}) = 0$ ✓

$\boxed{\text{Basis}: \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}$, $\boxed{\text{dimension} = 3}$

Find the inverse transformation. Show the details.

(12) $y_1 = 3x_1 + 2x_2$
$y_2 = 4x_1 + x_2$
$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Using (4*) $\det A = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 3-8 = -5$

$$A^{-1} = \frac{-1}{5} \begin{bmatrix} 1 & -2 \\ -4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{-1}{5} \begin{bmatrix} 1 & -2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{-1}{5} \begin{bmatrix} y_1 - 2y_2 \\ -4y_1 + 3y_2 \end{bmatrix} \quad \boxed{\begin{array}{l} x_1 = {}^{-1}/_5 y_1 + {}^{2}/_5 y_2 \\ x_2 = {}^{4}/_5 y_1 - {}^{3}/_5 y_2 \end{array}}$$

(22) Orthogonality. Find all vectors in $\mathbb{R}^3$ orthogonal to $[2\ 0\ 1]$. Do they form a vector space?

$$\langle [2\ 0\ 1], \vec{x} \rangle = 0$$

$$\begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Set $x_3 = t$, $x_2 = s$

$2x_1 + x_3 = 0 \Rightarrow x_1 = \frac{-1}{2} t$

$$\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix} t$$

Vectors of the form of $\vec{x}$ will be orthogonal to $[2, 0, 1]$. This is a vector space