5) $A = \begin{bmatrix} 0.2 & -0.1 & 0.4 \\ 0 & 1.1 & -0.3 \\ 0.1 & 0 & -2.1 \end{bmatrix} \xrightarrow[10R_3]{10R_1,\ 10R_2} \begin{bmatrix} 2 & -1 & 4 \\ 0 & 11 & -3 \\ 1 & 0 & -21 \end{bmatrix}$

$\xrightarrow{R_3 - 1/2 R_1} \begin{bmatrix} 2 & -1 & 4 \\ 0 & 11 & -3 \\ 0 & 1/2 & -23 \end{bmatrix} \xrightarrow[R_2 - 11R_3]{2R_3} \begin{bmatrix} 2 & -1 & 4 \\ 0 & 0 & 250 \\ 0 & 1 & -23 \end{bmatrix}$

# of non-zero rows $= r(A) = 3$

$r(A) = \dim RS_A = \dim CS_A$

$RS_A = \text{span}\left\{ [0.2, -0.1, 0.4], [0, 1.1, -0.3], [0.1, 0, -2.1] \right\}$

$CS_A = \text{span}\left\{ \begin{bmatrix} 0.2 \\ 0 \\ 0.1 \end{bmatrix}, \begin{bmatrix} -0.1 \\ 1.1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.4 \\ -0.3 \\ -2.1 \end{bmatrix} \right\}$

# of columns $= 3$

$\dim N_A = 3 - 3 = 0$

No null space

7) $A = \begin{bmatrix} 8 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \\ 4 & 0 & 2 & 0 \end{bmatrix} \xrightarrow{R_3 - 2R_1} \begin{bmatrix} 8 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow[1/4 R_1]{1/2 R_2} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$r(A) =$ # of non-zero rows $= 2$

$r(A) = \dim RS_A = \dim CS_A$

$RS_A = \text{span}\left\{ [8, 0, 4, 0], [0, 2, 0, 4] \right\}$

$CS_A = \text{span}\left\{ \begin{bmatrix} 8 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \right\}$

$\dim N_A = \#$ of columns $- r(A) = 4 - 2 = 2$

N_A:

$$\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$\Rightarrow \quad 2v_1 + v_3 = 0$

$v_2 + 2v_4 = 0$

$v_3 = t \quad \Rightarrow \quad v_1 = \dfrac{-t}{2}$

$v_2 = s \quad \Rightarrow \quad v_4 = -\dfrac{s}{2}$

$$\vec{v} = \begin{bmatrix} -1/2 \\ 0 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1/2 \end{bmatrix} s$$

$$N_A = \operatorname{span}\left\{ \begin{bmatrix} -1/2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1/2 \end{bmatrix} \right\}$$

7.7 4) Let $A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \underbrace{a_{11}a_{22}}_{\#1} - \underbrace{a_{12}a_{21}}_{\#2} = 2$ multiplications $= 2!$

$$A_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}\underbrace{(a_{22}a_{33} - a_{23}a_{32})}_{\#1,2}$$

$$- \underbrace{a_{12}(a_{21}a_{33} - a_{23}a_{31})}_{\#3,4}$$

$$+ \underbrace{a_{13}(a_{21}a_{32} - a_{22}a_{31})}_{\#5,6}$$

$= 6$ multiplications $= 3!$

From above lets assume $(n-1)\times(n-1)$ matrix has $(n-1)!$ multiplications.

$$A_n = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = a_{11}\underbrace{\begin{bmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & & \\ a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{(n-1)\times(n-1)\text{ matrix}} - a_{12}\underbrace{\begin{bmatrix} a_{21}, a_{23} & \cdots & a_{2n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}}_{(n-1)\times(n-1)\text{ matrix}} + \cdots$$

$$= a_{11}(n-1)! - a_{12}(n-1)! + \cdots a_{1n}(n-1)!$$

$$= n \times (n-1)!$$

$$= n!$$

$\therefore$ It is proved that computation of a n-th order determinant by expansion involves $n!$ multiplications, using Induction.

7) $\begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\beta & \cos\beta \end{vmatrix} = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \cos(\alpha+\beta)$

12) $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = a(a^2-bc) - b(ac-b^2) + c(c^2-ab)$

$= a^3 - abc - abc + b^3 + c^3 - abc$

$= a^3 + b^3 + c^3 - 3abc$

22) $2x - 4y = -24$

$5x + 2y = 0$

$\Rightarrow \begin{bmatrix} 2 & -4 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -24 \\ 0 \end{bmatrix}$

$D = 4 + 20 = 24$

$D_1 = \begin{vmatrix} -24 & -4 \\ 0 & 2 \end{vmatrix} = -48$

$D_2 = \begin{vmatrix} 2 & -24 \\ 5 & 0 \end{vmatrix} = 120$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -48/24 \\ 120/24 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$

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Gauss Elimination & back substitution:

$\begin{bmatrix} 2 & -4 & -24 \\ 5 & 2 & 0 \end{bmatrix} \xrightarrow{R_2 - \frac{5}{2}R_1} \begin{bmatrix} 2 & -4 & -24 \\ 0 & 12 & 60 \end{bmatrix}$

$\therefore \; 2x - 4y = -24$

$12y = 60$

$\Rightarrow y = 5$

$\therefore \; 2x - 20 = -24 \Rightarrow 2x = -4 \Rightarrow x = -2$

7.8 2) $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix} = A$

$\det(A) = \begin{vmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{vmatrix} = \cos^2 2\theta + \sin^2 2\theta = 1 \neq 0$

$\therefore$ Inverse of A exists.

$[A|I] = \left[\begin{array}{cc|cc} \cos 2\theta & \sin 2\theta & 1 & 0 \\ -\sin 2\theta & \cos 2\theta & 0 & 1 \end{array}\right]$

$\xrightarrow{\cos 2\theta R_1 + (-\sin 2\theta) R_2} \left[\begin{array}{cc|cc} 1 & 0 & \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & \cos 2\theta & 0 & 1 \end{array}\right]$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 5 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3-5R_1]{R_2-2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 4 & 1 & -5 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3-4R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 & -4 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix}$$

Checking using (1),

$$AA^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 2-2+0 & 0+1+0 & 0+0+0 \\ 5-8+3 & 0+4-4 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore AA^{-1} = I$

Hence checked

20) Formula (4) is, $A^{-1} = \dfrac{1}{\det A}[C_{jk}]^T$

Prob. 6 is $\begin{bmatrix} -4 & 0 & 0 \\ 0 & 8 & 13 \\ 0 & 3 & 5 \end{bmatrix}$

$$A = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 8 & 13 \\ 0 & 3 & 5 \end{bmatrix}$$

$$\det(A) = -4(40-39) - 0(0-0) + 0(0-0)$$
$$= -4 \neq 0$$

Inverse of A exists.

$C_{11} = \begin{vmatrix} 8 & 13 \\ 3 & 5 \end{vmatrix} = 40-39 = 1$ $\qquad C_{21} = -\begin{vmatrix} 0 & 0 \\ 3 & 5 \end{vmatrix} = 0$

$C_{12} = -\begin{vmatrix} 0 & 13 \\ 0 & 5 \end{vmatrix} = 0$ $\qquad C_{22} = \begin{vmatrix} -4 & 0 \\ 0 & 5 \end{vmatrix} = -20$

$C_{13} = \begin{vmatrix} 0 & 8 \\ 0 & 3 \end{vmatrix} = 0$ $\qquad C_{23} = -\begin{vmatrix} -4 & 0 \\ 0 & 3 \end{vmatrix} = +12$

$C_{31} = \begin{vmatrix} 0 & 0 \\ 8 & 13 \end{vmatrix} = 0$ $\qquad C_{32} = -\begin{vmatrix} -4 & 0 \\ 0 & 13 \end{vmatrix} = +52$

$C_{33} = \begin{vmatrix} -4 & 0 \\ 0 & 8 \end{vmatrix} = -32$

$$A^{-1} = \frac{1}{\det A}[C_{jk}]^T = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} -1/4 & 0 & 0 \\ 0 & 5 & -13 \\ 0 & -3 & 8 \end{bmatrix}$$

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6) $y(x) = a\cos 2x + b\sin 2x$ with arbitrary constants a and b.

For $y(0) = a\cos 0 + b\cancel{\sin 0}^{0}$

$= a$

$\vec{0}$ does not belong to the vectors forming from given function.

$\therefore$ It does not form a vector space.

9) All 2×2 matrices $[a_{jk}]$ with $a_{11} + a_{12} = 0$

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{bmatrix}$ $\quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & -b_{11} \end{bmatrix}$

$A + B = \begin{bmatrix} a_{11}+b_{11} & b_{12}+a_{12} \\ a_{21}+b_{21} & -a_{11}-b_{11} \end{bmatrix} \Rightarrow a_{11}+b_{11}-a_{11}-b_{11} = 0$

Addition closed

$cA = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & -ca_{11} \end{bmatrix} \Rightarrow ca_{11} - ca_{11} = 0$

Scalar multiplication closed.

$\therefore$ They form a vector space.

Dimension $= 3$

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Basis is given by,

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

12) $y_1 = 3x_1 + 2x_2$
$y_2 = 4x_1 + x_2$

$$A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 + 2x_2 \\ 4x_1 + x_2 \end{bmatrix}$$

$\det(A) = 3 - 8 = -5 \neq 0$

$\therefore$ A is non-singular, A^{-1} exists.

By inverse transformation,

$$x = A^{-1} y$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$A^{-1} = [A \mid I] = \left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 4 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - \frac{4}{3}R_1} \left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 0 & -5/3 & -4/3 & 1 \end{array}\right]$$

$$\xrightarrow[-\frac{3}{5}R_2]{1/3 R_1} \left[\begin{array}{cc|cc} 1 & 2/3 & 1/3 & 0 \\ 0 & 1 & 4/5 & -\frac{3}{5} \end{array}\right] \xrightarrow{R_1 - 2/3 R_2} \left[\begin{array}{cc|cc} 1 & 0 & 7/15 & 2/5 \\ 0 & 1 & 4/5 & -3/5 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 7/15 & 2/5 \\ 4/5 & -3/5 \end{bmatrix}$$

$\therefore$
$$\boxed{\begin{aligned} x_1 &= \frac{7}{15} y_1 + \frac{2}{5} y_2 \\ x_2 &= \frac{4}{5} y_1 - \frac{3}{5} y_2 \end{aligned}}$$

22) Inner product $= 0 \Rightarrow$ Orthogonality

Let $\vec{a} = [a_1 \ a_2 \ a_3]^T$ $\quad \vec{b} = [2 \ 0 \ 1]$

$\vec{a} \cdot \vec{b} = 0$

$2a_1 + a_3 = 0 \Rightarrow a_3 = -2a_1$

Let $a_2 = s$ $\Big\}$ s, t are arbitrary
$a_1 = t$

$$\therefore \vec{a} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} s$$

These will give the vectors in R^3 orthogonal to
$[2 \ 0 \ 1]$

Check for vector space:

Let $\vec{a}_x$ and $\vec{a}_y$ be the vectors from $\vec{a}$.

$\vec{a}_x \cdot \vec{b} = 0$

$\vec{a}_y \cdot \vec{b} = 0$

$(\vec{a}_x + \vec{a}_y) \cdot \vec{b} = \vec{a}_x \cdot \vec{b} + \vec{a}_y \cdot \vec{b} = 0$ $\Big\}$ addition closed

$(c\,\vec{a}_x) \cdot \vec{b} = c\,\vec{a}_x \cdot \vec{b} = 0$ $\Big\}$ scalar multiplication closed

$c \to$ scalar.

$\therefore$ The vectors forming from $\vec{a}$, form a vector space.

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