$\boxed{8.1-4}$ ✓

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

Find the eigenvalues $Ax = \lambda x$

$$(A-\lambda I) = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (1-\lambda)(4-\lambda) - 4$$
$$= \lambda^2 - 5\lambda$$

set equal to zero and find roots

$$\lambda(\lambda-5) = 0$$
$$\lambda = 0, 5$$

~~check~~

$\boxed{\lambda_1 = 0}$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \qquad \boxed{x_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

check:
$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = 0\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$\boxed{\lambda_2 = 5}$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \qquad \boxed{x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

check:
$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\boxed{8.1-11}$ ✓

$$A = \begin{bmatrix} 6 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 7 \end{bmatrix} \qquad \lambda = 3$$

$$\det(A - \lambda I) = (6-\lambda)(5-\lambda)(7-\lambda) - 2(2)(7-\lambda) + -2(2)(5-\lambda)$$

$$= (6-\lambda)(\lambda^2 - 12\lambda + 35) - 28 + 4\lambda - 20 + 4\lambda$$

$$= -\lambda^3 + 12\lambda^2 - 35\lambda + 6\lambda^2 - 72\lambda + 210 - 28 + 4\lambda - 20 + 4\lambda$$

$$= -\lambda^3 + 18\lambda^2 - 99\lambda + 162$$

$$\begin{array}{r} 216 \\ -48 \\ \hline 162 \end{array} \qquad \begin{array}{r} 72 \\ +35 \\ \hline 107 \\ -8 \\ \hline 99 \end{array}$$

use root $\lambda = 3$

$$\begin{array}{r} -\lambda^2 - 21\lambda + 36 \\ (\lambda - 3) \overline{) -(\lambda^3 - 18\lambda^2 + 99\lambda - 162)} \\ -(-\lambda^3 + 3\lambda^2) \\ \hline -21\lambda^2 + 99\lambda - 162 \\ -(-21\lambda^2 + 63\lambda) \\ \hline 36\lambda - 162 \\ -(36\lambda \end{array}$$

(error)

$$\begin{array}{r} -\lambda^2 + 15\lambda - 54 \\ (\lambda - 3) \overline{) -\lambda^3 + 18\lambda^2 - 99\lambda + 162} \\ -(-\lambda^3 + 3\lambda^2) \\ \hline 15\lambda^2 - 99\lambda + 162 \\ -(15\lambda^2 - 45\lambda) \\ \hline -54\lambda + 162 \\ -(-54\lambda + 162) \\ \hline 0 \end{array}$$

$$-(\lambda^2 - 15\lambda + 54) = -(\lambda - 9)(\lambda - 6)$$

$\lambda_1 = 3$

$\lambda_2 = 9$

$\lambda_3 = 6$

(see next page)

$\boxed{8.1-11}$ ✓ (p. 2)

$\boxed{\lambda_1 = 3}$

$$\begin{bmatrix} 3 & 2 & -2 \\ 2 & 2 & 0 \\ -2 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & -2 \\ 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & -2 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & -1 & -2 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

let $x_3 = 1$

then: $x_2 = -2$

$x_1 = 2$

$$\boxed{X = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}}$$

$\boxed{\lambda_2 = 9}$

$$\begin{bmatrix} -3 & 2 & -2 \\ 2 & -4 & 0 \\ -2 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} -3 & 2 & -2 \\ 1 & -2 & 0 \\ 0 & -4 & -2 \end{bmatrix} \sim \begin{bmatrix} -3 & 0 & -3 \\ 1 & -2 & 0 \\ 0 & -2 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & -1 \\ 0 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

let $x_3 = 2$

$x_2 = -1$

$x_1 = -2$

$$\boxed{X = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}}$$

$\boxed{\lambda_3 = 6}$

$$\begin{bmatrix} 0 & 2 & -2 \\ 2 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

let $x_3 = 1$

$x_2 = 1$

$x_1 = \frac{1}{2}$

$$\boxed{X = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}}$$

$\boxed{8.1\text{-}12}$ ✓ checked w/ MATLAB

$$A = \begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} 3-\lambda & 5 & 3 \\ 0 & 4-\lambda & 6 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (3-\lambda)(4-\lambda)(1-\lambda) - 5(0) + 3(0)$$

$$\lambda = 3, 4, 1$$

$\boxed{\lambda = 3}$

$$\begin{bmatrix} 0 & 5 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 0 & 5 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad X = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

or just take transpose

$\boxed{\lambda = 4}$

$$\begin{bmatrix} -1 & 5 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} -1 & 5 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \boxed{X = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}}$$

$\boxed{\lambda = 1}$

$$\begin{bmatrix} 2 & 5 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & -7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad X = \begin{bmatrix} 7/2 \\ -2 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 7 \\ -4 \\ 2 \end{bmatrix}}$$

$\boxed{8.1\text{-}24}$ ✓ Show that A^{-1} exists if and only if the eigenvalues $\lambda_1 \ldots \lambda_n$ are all nonzero, and then A^{-1} has the eigenvalues $\frac{1}{\lambda_1} \cdots \frac{1}{\lambda_n}$

- let $\lambda_1 = 0$, then $0 = \det(A - \lambda I) = \det(A)$

  The determinant of A is equal to zero, thus A is singular and does not have an inverse.

- $Ax = \lambda x$

  $A^{-1}Ax = \lambda A^{-1}x$

  $Ix = \lambda A^{-1}x$

  $\boxed{\lambda^{-1}x = A^{-1}x}$

  The eigenvalues of A^{-1} are the inverse of the eigenvalues of A.

$\boxed{8.3\text{-}3}$ ✓

$$A = \begin{bmatrix} 2 & 8 \\ -8 & 2 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & -8 \\ 8 & 2 \end{bmatrix} \qquad \begin{matrix} A^T \neq A \\ A^T \neq -A \end{matrix}$$

$$A^{-1} = \frac{\begin{bmatrix} 2 & -8 \\ 8 & 2 \end{bmatrix}}{4+64} = \frac{1}{34}\begin{bmatrix} 1 & -4 \\ 4 & 1 \end{bmatrix} \qquad A^T \neq A^{-1}$$

Spectrum of A

$$\det(A-\lambda I) = (2-\lambda)^2 + 64 = \lambda^2 - 4\lambda + 68$$

$$\frac{4 \pm \sqrt{16-4\cdot 68}}{2} \qquad \lambda = 2 \pm i\sqrt{256/4} = 2 \pm i\sqrt{64}$$

$4 - 68 = -64$

$\boxed{\lambda = 2 \pm 8i}$

- The eigenvalues are complex, thus by Thm 1 the matrix is not symmetric, nor skew-symmetric.
- The absolute value of the eigenvalue pair is $\sqrt{68}$ $\therefore$ As the absolute value is not 1, by Thm 5 the matrix is not orthogonal.

$\boxed{\begin{array}{l}\text{Not sym.}\\ \text{Not skew-sym.}\\ \text{Not orthg.}\end{array}}$

$\boxed{8.3\text{-}6}$

$$A = \begin{bmatrix} a & k & k \\ k & a & k \\ k & k & a \end{bmatrix} \qquad A^T = A \quad \text{symmetric}$$

$$A - \lambda I = \begin{bmatrix} a-\lambda & k & k \\ k & a-\lambda & k \\ k & k & a-\lambda \end{bmatrix} \sim \begin{bmatrix} a-\lambda & k & k \\ 0 & a-\lambda-k & k-a+\lambda \\ k & k & a-\lambda \end{bmatrix}$$

$$R_1 = R_1 + \frac{-1}{k}(a-\lambda) \sim \begin{bmatrix} 0 & k-a+\lambda & k - \frac{1}{k}(a-\lambda)^2 \\ 0 & a-\lambda-k & k-a+\lambda \\ k & k & a-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = k(k-a+\lambda)^2 - k(a-\lambda-k)\left(k - \frac{1}{k}(a-\lambda)^2\right)$$

$$= k(\lambda^2 + a^2 + k^2 - 2ka + 2\lambda k - 2\lambda a) - (a-\lambda-k)(k^2 - (a-\lambda)^2)$$

$\cancel{(k-a+\lambda)}\left[\cancel{k(k-a+\lambda)}\right.$ Next page

$\boxed{8.3\text{-}6}$ (p.2)

$$= \left[k(k-a+\lambda) + k^2 - (a-\lambda)^2\right](k-a+\lambda)$$

$$= (\lambda+k-a)\left[k^2 - ka + k\lambda + k^2 - a^2 - \lambda^2 + 2\lambda a\right]$$

$$= (\lambda+k-a)\left[-\lambda^2 + \lambda(2a+k) + 2k^2 - a^2 - ka\right]$$

$$= (\lambda+k-a) \quad \frac{-(2a+k) \pm \sqrt{(2a+k)^2 + 4(2k^2-a^2-ka)}}{-2}$$

$$\frac{+(2a+k) \pm \sqrt{9k^2}}{2} = \frac{2a+k \pm 3k}{2} = a+2k,\ a-k$$

$$\boxed{\begin{aligned} \lambda_1 &= a-k \\ \lambda_2 &= a+2k \\ \lambda_3 &= a-k \end{aligned}}$$

repeat

checked w/ a few examples in MATLAB.

$$\lambda = a-k \qquad \begin{bmatrix} k & k & k \\ k & k & k \\ k & k & k \end{bmatrix} \qquad \boxed{X = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad X = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}$$

$$\lambda = a+2k \quad \begin{bmatrix} -2k & k & k \\ k & -2k & k \\ k & k & -2k \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & -2 \\ 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -3 \\ 1 & -1 & -2 \\ 0 & 1 & -3 \end{bmatrix}$$

error

$$\boxed{\cancel{X = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}}}$$

↳ obvious?? $\boxed{X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$

$\boxed{8.3\text{-}8}$ ✓

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$$A^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta} \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 & 0 \\ 0 & \cos\theta & -(-\sin\theta) \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$A^T = A^{-1}$ Matrix is orthogonal.

By Thm 5 the eigenvalues are "real or complex conjugates in pairs and have absolute value 1."

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & (\cos\theta)-\lambda & -\sin\theta \\ 0 & \sin\theta & (\cos\theta)-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (1-\lambda)\left[(\cos\theta - \lambda)^2 + \sin^2\theta\right]$$
$$= \left[\cos^2\theta + \sin^2\theta + \lambda^2 - 2\lambda\cos\theta\right](1-\lambda)$$
$$= -(\lambda^2 - 2\lambda\cos\theta + 1)(\lambda - 1)$$

$\cos^2\theta - 1 = -\sin^2\theta$

$\sin^2\theta + \cos^2\theta = 1$

$$\hookrightarrow \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4\cdot 1}}{2\cdot 1} = \cos\theta \pm \underbrace{\sqrt{\cos^2\theta - 1}}_{\sqrt{-\sin^2\theta}}$$

$\cancel{\boxed{\lambda = 1,\ \cos\theta \pm \sqrt{\cos^2\theta - 1}}}$ $\Longrightarrow$ $\boxed{\lambda = 1,\ \cos\theta \pm i\sin\theta}$

if $\theta \neq 0, 2\pi n$

i.e. $1, e^{\pm j\theta}$

rotation

Absolute value $= \sqrt{\cos^2\theta + \sin^2\theta} = 1$

$\boxed{8.5-1}$ ✓

$$A = \begin{bmatrix} 6 & i \\ -i & 6 \end{bmatrix} \qquad \bar{A}^T = \begin{bmatrix} 6 & i \\ -i & 6 \end{bmatrix}$$

$\bar{A}^T = A$ The matrix is Hermitian.

$$(A - \lambda I) = \begin{bmatrix} 6-\lambda & i \\ -i & 6-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (6-\lambda)^2 + i^2 = \lambda^2 - 12\lambda + 35 = (\lambda - 7)(\lambda - 5)$$

$\lambda_1 = 7$

$$\begin{bmatrix} -1 & i \end{bmatrix} \qquad -i_1 + i_1 = 0$$

$$X_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$\lambda_2 = 5$

$$\begin{bmatrix} 1 & i \end{bmatrix}$$

$$X_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$\boxed{8.5-2}$ ✓

$$A = \begin{bmatrix} i & 1+i \\ -1+i & 0 \end{bmatrix} \qquad \bar{A}^T = \begin{bmatrix} -i & -1-i \\ 1-i & 0 \end{bmatrix}$$

$\bar{A}^T = -A$ $\quad A$ is skew-Hermitian.

$$\det(A - \lambda I) = (i-\lambda)(-\lambda) - (1+i)(-1+i)$$

$$= \lambda^2 - i\lambda + 1 + 1 = (\lambda - 2i)(\lambda + i)$$

purely imaginary roots as expected

$\lambda_1 = 2i$

$$\begin{bmatrix} -i & 1+i \end{bmatrix} \qquad -ix_1 + x_2(1+i) = 0$$

~~let $x_1 = 0$~~ ~~you cannot~~

$x_2 = 1+i \Rightarrow -ix_1 + (1+i)(1+i) = 0$

$-ix_1 + 1 + 2i - 1 = 0$

$x_1 = 2$

$$X_1 = \begin{bmatrix} 2 \\ 1+i \end{bmatrix}$$

$\lambda_1 = -i$

$$2ix_1 + (1+i)x_2 = 0$$
$$(-1+i)x_1 + -ix_2 = 0$$

$2ix_1 + (1+i)(1+i) = 0$

$i2x_1 + 1 + 2i - 1 = 0$

$x_1 = -1$

$$X_2 = \begin{bmatrix} -1 \\ 1+i \end{bmatrix}$$

$\boxed{8.5-5}$ ✓

$$A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix} \qquad \bar{A}^T = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$$

$\bar{A}^T = -A \implies$ skew-Hermitian

$$A^{-1} = \frac{1}{-i(i^2)} \begin{bmatrix} -i^2 & 0 & 0 \\ 0 & 0 & -i^2 \\ 0 & -i^2 & \end{bmatrix} = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix} \qquad \bar{A}^T = A^{-1} \quad \text{unitary}$$

$$A - \lambda I = \begin{bmatrix} i-\lambda & 0 & 0 \\ 0 & -\lambda & i \\ 0 & i & -\lambda \end{bmatrix} \qquad \det(A-\lambda I) = (i-\lambda)(\lambda^2 - i^2)$$

$$= (\lambda^2+1)(i-\lambda)$$

$$\lambda = i, -i$$

$\boxed{\lambda = i}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -i & i \\ 0 & i & -i \end{bmatrix} \sim \begin{bmatrix} 0 & i & -i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \boxed{X = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \text{ and } X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

$\boxed{\lambda = -i}$

$$\begin{bmatrix} 2i & 0 & 0 \\ 0 & i & i \\ 0 & i & i \end{bmatrix} \sim \begin{bmatrix} 2i & 0 & 0 \\ 0 & i & i \\ 0 & 0 & 0 \end{bmatrix} \qquad \boxed{X = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}$$

$\boxed{8.5-13}$ ✓

Show that $\overline{(ABC)}^T = -C^{-1}BA$ for any $n \times n$ Hermitian A, skew-Hermitian B, and unitary C.

$$\overline{(ABC)}^T = \bar{C}^T \bar{B}^T \bar{A}^T = C^{-1}(-B)A = -C^{-1}BA$$ ✓