#24

The proof could be divided into the following 3 parts:

(1) If the eigenvalues of A are all nonzero, then:

$\det(A - 0I) \neq 0$

which means $\det(A) \neq 0$.

$\Rightarrow$ A is a nonsingular matrix.

$\Rightarrow$ A ~~to~~ has an inverse matrix A^{-1}.

(2) If A has an inverse matrix A^{-1}, then ~~rank A~~ ~~is~~ rank $A = n$,

thus $\det(A) \neq 0$.

Hence $\det(A - 0I) \neq 0$

~~That~~ This is to say, 0 is not the eigenvalue of A.

Therefore, all eigenvalues of A are nonzero.

(3) If A^{-1} exists and λ_i $(i = 1, 2, \cdots n)$ ~~is~~ are nonzero eigenvalues, then

$A\vec{x_i} = \lambda_i \vec{x_i}$ ($\vec{x_i}$ is the corresponding ~~eigenvalue~~ eigenvector of eigen value ~~is~~ λ_i)

Multiplying $\frac{1}{\lambda_i} A^{-1}$ from the left gives:

10/10

$$\frac{1}{\lambda_i} A^{-1} \cdot A\vec{x_i} = \frac{1}{\lambda_i} A^{-1} \cdot \lambda_i \vec{x_i}$$

$$\Rightarrow \frac{1}{\lambda_i}\vec{x_i} = A^{-1}\vec{x_i}$$

Therefore, $\frac{1}{\lambda_i}$ is the eigenvalue of A^{-1}, and the corresponding eigenvector is $\vec{x_i}$.

That is to say, A^{-1} has eigenvalues $1/\lambda_1$, $1/\lambda_2$, $\cdots$ $1/\lambda_n$.

This completes the proof.

P338.

3. Solution:

The inverse matrix of matrix $A = \begin{bmatrix} 2 & 8 \\ -8 & 2 \end{bmatrix}$ is

8. Solution:

The inverse matrix of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$ is:

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

Hence $A^T = A^{-1}$. Which means this matrix is orthogonal.

Let $\det(A - \lambda I) = 0$, gives:

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & \cos\theta - \lambda & -\sin\theta \\ 0 & \sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$$

10/10

$$\Rightarrow (\lambda - 1)(\lambda^2 - 2\cos\theta\lambda + 1) = 0$$

$$\Rightarrow \lambda_1 = 1, \quad \lambda_2 = \cos\theta + \sin\theta i, \quad \lambda_3 = \cos\theta - \sin\theta i$$

Therefore, the eigenvalues are real or complex conjugates in pairs and have absolute value 1.

This illustrates Theorem 5.

P351.

1. Solution:

$$A = \begin{bmatrix} 6 & i \\ -i & 6 \end{bmatrix}$$

$$\Rightarrow \bar{A}^T = \begin{bmatrix} 6 & i \\ -i & 6 \end{bmatrix} = A$$

Hence this matrix is Hermitian.

Let $\det(A - \lambda I) = 0$, gives:

$$\begin{vmatrix} 6-\lambda & i \\ -i & 6-\lambda \end{vmatrix} = 0$$

$$A^{-1} = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$$

Thus $\bar{A}^T = -A = A^{-1}$, that is to say, A is unitary and skew-Hermitian.

Let $\det(A - \lambda I) = 0$, we have:

$$\begin{vmatrix} i-\lambda & 0 & 0 \\ 0 & -\lambda & i \\ 0 & i & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (i-\lambda)[\lambda^2 - i^2] = 0$$

$$\Rightarrow (\lambda+i)(\lambda-i)^2 = 0$$

$$\Rightarrow \lambda_1 = -i, \ \lambda_2 = \lambda_3 = i$$

(1) For $\lambda_1 = -i$, the characteristic matrix:

$$A + iI = \begin{bmatrix} 2i & 0 & 0 \\ 0 & i & i \\ 0 & i & i \end{bmatrix} \xrightarrow{\text{row-operation}} \begin{bmatrix} 2i & 0 & 0 \\ 0 & i & i \\ 0 & 0 & 0 \end{bmatrix}$$

The rank is 2.

$x_1 = 0$; choosing $x_2 = 1$, we have $x_3 = -1$ from $ix_2 + ix_3 = 0$

$\Rightarrow$ The eigenvector: $[0 \quad 1 \quad -1]^T$

(2) For $\lambda_2 = i$, the characteristic matrix:

$$A - iI = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -i & i \\ 0 & i & -i \end{bmatrix} \xrightarrow{\text{row-operation}} \begin{bmatrix} 0 & -i & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

10/10

The rank is 1.

Choosing $x_1 = 1$, $x_2 = 1$, we have $x_3 = 1$ from $-ix_2 + ix_3 = 0$

Choosing $x_1 = 0$, $x_2 = 1$, we have $x_3 = 1$ from $-ix_2 + ix_3 = 0$.

$\Rightarrow$ The eigenvector: $[1 \quad 1 \quad 1]^T$ and $[0 \quad 1 \quad 1]^T$

In summary, the eigenvalues and eigenvectors are:

① $\lambda_1 = -i$, $[0 \quad 1 \quad -1]^T$

② $\lambda_2 = i$, $[1 \quad 1 \quad 1]^T$, $[0 \quad 1 \quad 1]^T$

13. Proof:

Since A is Hermitian, B is skew-Hermitian, C is unitary, we have:

$$\bar{A}^T = A$$

$$\bar{B}^T = -B$$

$$\bar{C}^T = C^{-1}$$

Assume $D_{n\times n} = A_{n\times n} B_{n\times n}$, then each element in $D_{n\times n}$ can be written as:

$$d_{jk} = \sum_{l=1}^{n} a_{jl} b_{lk}$$

$$= a_{j1}b_{1k} + a_{j2}b_{2k} + \cdots + a_{jn}b_{nk} \quad (j=1,2,\cdots n,\ k=1,\cdots n)$$

Thus:

$$\bar{d}_{jk} = \overline{a_{j1}b_{1k} + a_{j2}b_{2k} + \cdots + a_{jn}b_{nk}}$$

$$= \overline{a_{j1}b_{1k}} + \overline{a_{j2}b_{2k}} + \cdots + \overline{a_{jn}b_{nk}}$$

$$= \bar{a}_{j1}\bar{b}_{1k} + \bar{a}_{j2}\bar{b}_{2k} + \cdots + \bar{a}_{jn}\bar{b}_{nk}$$

That is to say: $\overline{A_{n\times n}B_{n\times n}} = \bar{D}_{n\times n} = \bar{A}_{n\times n}\bar{B}_{n\times n}$.

Therefore: $\overline{ABC} = \bar{A}\bar{B}\bar{C}$

Hence: $(\overline{ABC})^T = (\bar{A}\bar{B}\bar{C})^T$

$$= \bar{C}^T\bar{B}^T\bar{A}^T$$

$$= C^{-1}(-B)A$$

$$= -C^{-1}BA$$

10/10

This completes the proof.