8.4

verify λs of A = λs of $\hat{A}$

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① $A=\begin{bmatrix}3 & 4\\ 4 & -3\end{bmatrix}$, $P=\begin{bmatrix}-4 & 2\\ 3 & -1\end{bmatrix}$

$\hat{A}=P^{-1}AP$

$\vec{x}=P\vec{y}$ (from Thm. 3: $y=P^{-1}x$ /(·P) $\Rightarrow P\vec{y}=\frac{P}{P}\vec{x}\Rightarrow \vec{x}=P\vec{y}$)

$\begin{vmatrix}3-\lambda & 4\\ 4 & -3-\lambda\end{vmatrix}=0 \Rightarrow [(3-\lambda)(-1)(3+\lambda)]-16=0$

$\Rightarrow -(9-\lambda^2)-16=0 \Rightarrow \lambda^2-25=0 \Rightarrow (\lambda-5)(\lambda+5)=0 \Rightarrow \boxed{\begin{matrix}\lambda_1=-5\\ \lambda_2=5\end{matrix}}$ eigenvalues of A

$P^{-1}=\frac{1}{4-6}\begin{bmatrix}-1 & -2\\ -3 & -4\end{bmatrix} \Rightarrow P^{-1}=\begin{bmatrix}.5 & 1\\ 1.5 & 2\end{bmatrix}$

$\Rightarrow \hat{A}=\begin{bmatrix}.5 & 1\\ 1.5 & 2\end{bmatrix}\begin{bmatrix}3 & 4\\ 4 & -3\end{bmatrix}\begin{bmatrix}-4 & 2\\ 3 & -1\end{bmatrix}=\begin{bmatrix}5.5 & -1\\ 12.5 & 0\end{bmatrix}\begin{bmatrix}-4 & 2\\ 3 & -1\end{bmatrix}=\begin{bmatrix}-25 & 12\\ -50 & 25\end{bmatrix}$

$\begin{vmatrix}-25-\lambda & 12\\ -50 & 25-\lambda\end{vmatrix}=0 \Rightarrow -(25+\lambda)(25-\lambda)+600=0$
$\Rightarrow -625+\lambda^2+600=0 \Rightarrow (\lambda^2-25)=0 \Rightarrow$

$(\lambda-5)(\lambda+5)=0 \Rightarrow \boxed{\begin{matrix}\lambda_1=-5\\ \lambda_2=5\end{matrix}}$ eigenvalues of $\hat{A}$

* $\vec{x}$ for $\lambda_1=-5$: $\begin{bmatrix}3-(-5) & 4\\ 4 & -3-(-5)\end{bmatrix}\vec{x}=\vec{0} \Rightarrow \begin{bmatrix}8 & 4\\ 4 & 2\end{bmatrix}\vec{x}=0$
eigenvectors of A

$\Rightarrow 2x_1+x_2=0 \Rightarrow 2x_1=-x_2 \Rightarrow x_2=-2x_1 \Rightarrow \boxed{\vec{x}_1=\begin{bmatrix}1\\ -2\end{bmatrix}}$ or any non-zero multiple (applicable to all eigenvectors)

$\vec{y}$ for $\lambda_1=-5$: $\begin{bmatrix}-25-(-5) & 12\\ -50 & 25-(-5)\end{bmatrix}\vec{y}=\vec{0} \Rightarrow \begin{bmatrix}-20 & 12\\ -50 & 30\end{bmatrix}\vec{y}=\vec{0}$
eigenvectors of $\hat{A}$

$\Rightarrow -20y_1+12y_2=0$ /÷4 $\Rightarrow -5y_1+3y_2=0 \Rightarrow 5y_1=3y_2 \Rightarrow y_1=\frac{3}{5}y_2$

$\Rightarrow \boxed{\vec{y}_1=\begin{bmatrix}3\\ 5\end{bmatrix}}$. Verify: $\vec{x}_1=P\vec{y}_1=\begin{bmatrix}-4 & 2\\ 3 & -1\end{bmatrix}\begin{bmatrix}3\\ 5\end{bmatrix}=\begin{bmatrix}-2\\ 4\end{bmatrix}=-2\begin{bmatrix}1\\ -2\end{bmatrix}$

* $\vec{x}$ for $\lambda_2=5$: $\begin{bmatrix}3-5 & 4\\ 4 & -3-5\end{bmatrix}\vec{x}=\vec{0} \Rightarrow \begin{bmatrix}-2 & 4\\ 4 & -8\end{bmatrix}\vec{x}=\vec{0}$

$\Rightarrow$ 1) $-x_1+2x_2=0$
2) $x_1-2x_2=0$ $\Rightarrow x_1=2x_2 \Rightarrow \vec{x}_2=\begin{bmatrix}2\\ 1\end{bmatrix}$

$\vec{y}$ for $\lambda_2=5$: $\begin{bmatrix}-25-5 & 12\\ -50 & 25-5\end{bmatrix}\vec{y}=\vec{0} \Rightarrow \begin{bmatrix}-30 & 12\\ -50 & 20\end{bmatrix}\vec{y}=\vec{0}$

$\Rightarrow -5y_1=-2y_2 \Rightarrow y_1=+\frac{2}{5}y_2 \Rightarrow \vec{y}_2=\begin{bmatrix}2\\ 5\end{bmatrix}$

Verify: $\vec{x}_2=P\vec{y}_2=\begin{bmatrix}-4 & 2\\ 3 & -1\end{bmatrix}\begin{bmatrix}2\\ 5\end{bmatrix}=\begin{bmatrix}2\\ 1\end{bmatrix}$

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pg. 345 Find an eigenbasis, diagonalize.

9. $A=\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ i) find λs: $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix}=0$

$\Rightarrow (1-\lambda)(4-\lambda)-4=0 \Rightarrow \cancel{4}-4\lambda-\lambda+\lambda^2-\cancel{4}=0 \Rightarrow \lambda^2-5\lambda=0$

$\Rightarrow \lambda(\lambda-5)=0 \Rightarrow \boxed{\lambda_1=0;\ \lambda_2=5}$

ii) find $\vec{x}$: for $\lambda_1=0$: $\begin{bmatrix} 1-0 & 2 \\ 2 & 4-0 \end{bmatrix}\vec{x}=0 \Rightarrow x_1+2x_2=0 \Rightarrow x_1=-2x_2$

$\Rightarrow \vec{x_1}=\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

for $\lambda_2=5$: $\begin{bmatrix} 1-5 & 2 \\ 2 & 4-5 \end{bmatrix}\vec{x}=\vec{0} \Rightarrow \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\vec{x}=\vec{0} \Rightarrow 2x_1=x_2$

$\Rightarrow \vec{x_2}=\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

iii) find X and X^{-1}: $X=\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$; det P $= -4-1=-5$

$\Rightarrow X^{-1}=\frac{1}{-5}\begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}=\begin{bmatrix} -.4 & .2 \\ .2 & .4 \end{bmatrix}$

iv) find eigenbasis and D: eigenbasis: $\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$D=X^{-1}AX=\begin{bmatrix} -.4 & .2 \\ .2 & .4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}=$

$=\begin{bmatrix} -.4 & .2 \\ .2 & .4 \end{bmatrix}\begin{bmatrix} 0 & 5 \\ 0 & 10 \end{bmatrix} \Rightarrow D=\begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$

10. $A=\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$; $\begin{vmatrix} 1-\lambda & 0 \\ 2 & -1-\lambda \end{vmatrix}=0 \Rightarrow -(1+\lambda)(1-\lambda)=0$ $/\cdot(-1)$

$\Rightarrow (1+\lambda)(1-\lambda)=0$

$\Rightarrow \boxed{\begin{matrix}\lambda_1=1 \\ \lambda_2=-1\end{matrix}}$ $\lambda_1=1$: $\begin{bmatrix} 1-1 & 0 \\ 2 & -1-1 \end{bmatrix}\vec{x}=\vec{0} \Rightarrow \begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix}\vec{x}=\vec{0}$

$\Rightarrow x_1=x_2 \Rightarrow \vec{x_1}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2=-1$: $\begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}\vec{x}=\vec{0} \Rightarrow \begin{matrix} 2x_1+0x_2=0 \\ 2x_1=-0x_2 \end{matrix} \Rightarrow x_1=0 \Rightarrow \vec{x_2}=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

eigenbasis: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $X=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow X^{-1}=\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

$D=\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow D=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

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pg. 346 (24). $Q(\vec{x}) = \vec{x}^T A \vec{x}$, A is symmetric $\Rightarrow$ has orthonormal basis of eigen vectors, they form X which is orthogonal

if A is
- positive definite $\Rightarrow Q(\vec{x}) > 0 \ \forall \vec{x} \neq 0$ — prove $\lambda(A) > 0$
- negative definite $\Rightarrow Q(\vec{x}) < 0 \ \forall \vec{x} \neq 0$ — prove $\lambda(A) < 0$
- indefinite $\Rightarrow \exists \vec{x}, \vec{y}$ s.t. $Q(\vec{x}) > 0$, $Q(\vec{y}) < 0$ — prove $\lambda(A) > 0$ & $\lambda(A) < 0$

def.

$Q = \vec{x}^T A \vec{x} = \vec{x}^T X D X^T \vec{x}$ since $D \overset{def}{=} X^{-1} A X$ (Thm. 4 pg. 341) $\Rightarrow A = X D X^{-1}$

$\left(D = X^{-1} A X \ / \cdot X; \ XD = AX \ / \cdot X^{-1}; \ XDX^{-1} = AXX^{-1}, \ XX^{-1} = I, \ AI = A \right)$

But $X^{-1} = X^T$, A is sym.

$\Rightarrow A = X D X^T$

- Let $X^T \vec{x} = \vec{y} = X^{-1} \vec{x}$; $X^{-1} \vec{x} = \vec{y} \ / \cdot X \Rightarrow \vec{x} = X \vec{y}$
- $\vec{x}^T X = (X^T x)^T = \vec{y}^T$; $X^T \vec{x} = \vec{y}$

$\Rightarrow \boxed{Q = y^T D \vec{y} = \sum_{i=1}^{n} \lambda_i y_i^2}$ canonical form

$y_i^2 > 0$ always

$\Rightarrow Q < 0 \Leftrightarrow \lambda_i < 0$ neg. def.

$Q > 0 \Leftrightarrow \lambda_i > 0$ pos. def

and if λ_i takes positive or neg. values so will $Q \Leftrightarrow$ indef.

(25). Show #22 is positive def.

$Q(x) = \vec{x}^T A \vec{x}$, A symm., show principal minors > 0 $\left(a_{11} > 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} > 0\right)$ for pos. def.

22

$4x_1^2 + 12x_1x_2 + 13x_2^2 = 16$ $\quad 4 > 0$

$\Rightarrow A = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix} \Rightarrow \begin{vmatrix} 4 & 6 \\ 6 & 13 \end{vmatrix} = 4(13) - 36 = 16 > 0 \Rightarrow$ pos. def

23 $\quad -11x_1^2 + 84x_1x_2 + 24x_2^2 = 156 \Rightarrow A = \begin{bmatrix} -11 & 42 \\ 42 & 24 \end{bmatrix}$

Recall from 24, that an indef. has both positive & negative eigen values

$\Rightarrow \begin{vmatrix} -11-\lambda & 42 \\ 42 & 24-\lambda \end{vmatrix} = 0 \Rightarrow (-11-\lambda)(24-\lambda) - 1764 = 0 \Rightarrow -264 - 24\lambda + 11\lambda + \lambda^2 - 1764 = 0$

$\Rightarrow \lambda^2 - 13\lambda - 2028 = 0$; $\lambda = \frac{13 \pm \sqrt{169 + 8112}}{2} = \frac{13 \pm 91}{2}$

$\Rightarrow \lambda_1 = 52, \lambda_2 = -39$

$\Rightarrow$ indefinite

4.1 Write the syst. of ODEs

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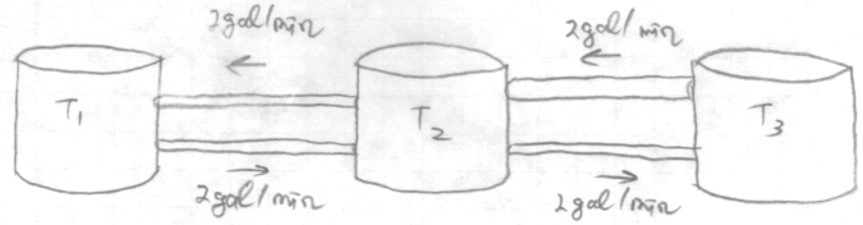
5.



Let $y(t)$ be the amount of fertilizer @ time t ; flow rate is 2gal/min

for T_1 @ $t=0$, 100 gal H_2O, no fert.

T_2 @ $t=0$, 100 gal H_2O, 150 # of fert. dissolved

T_3 @ $t=0$, 100 gal H_2O, no fert.

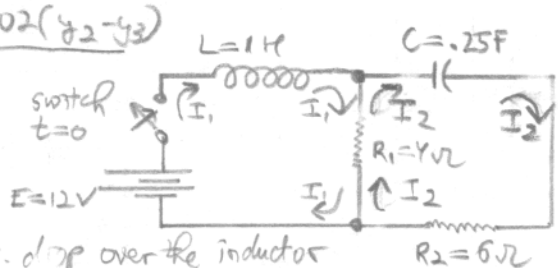
$y'(t) =$ inflow rate − outflow rate

$\Rightarrow y_1'(t) = \frac{2}{100}y_2 - \frac{2}{100}y_1 = \underline{.02(y_2 - y_1)}$

$y_2'(t) = .02y_3 + .02y_1 - .02y_2 - .02y_2 = \underline{.02(y_3 - 2y_2 + y_1)}$

$y_3'(t) = .02y_2 - .02y_3 = \underline{.02(y_2 - y_3)}$

7. $I_1 = 0A$ ↗ , $I_2 = -3A = 3A$ ↺ @ $t=0$



Left loop: $LI_1' = I_1'$ volt. drop over the inductor

$R_1(I_1 - I_2) = 4(I_1 - I_2)$ volt drop over R_1 ; I_1 & I_2 flow in opp. dir.

From Kirchhoff's Volt. Law: $I_1' + 4(I_1 - I_2) = 12$

$\Rightarrow \underline{I_1' = 12 - 4I_1 + 4I_2}$ 1)

assume I_2 is CW, I_1 is CW

Right loop: $R_2 I_2 = 6I_2$ volt. drop over R_2

$R_1(I_2 - I_1) = 4(I_2 - I_1)$ volt. drop over R_1

$\frac{1}{C}\int I_2\,dt = 4\int I_2\,dt$ volt. drop over capacitor

$\Rightarrow 6I_2 + 4(I_2 - I_1) + 4\int I_2\,dt = 0 \Rightarrow 10I_2 - 4I_1 + 4\int I_2\,dt = 0$ / ÷10

$I_2 - .4I_1 + .4\int I_2\,dt = 0$, differentiate $\Rightarrow I_2' - .4I_1' + .4I_2 = 0$ 2)

2) $I_2' = -.4I_2 + .4I_1'$, from 1) $\Rightarrow I_2' = -.4I_2 + .4(12 - 4I_1 + 4I_2)$

$\Rightarrow \underline{I_2' = -1.6I_1 + 1.2I_2 + 4.8}$ 2) or $\underbrace{\vec{J}' = A\vec{J} + \vec{g}}_{\text{nonhomogeneous}}$ where $\vec{J} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$, $A = \begin{bmatrix} -4 & +4 \\ -1.6 & 1.2 \end{bmatrix}$ from 1), from 2)

$\vec{g} = \begin{bmatrix} 12 \\ +4.8 \end{bmatrix}$ 1), 2)

i) gen. soln. for homogeneous sys.:

$\vec{J}' = A\vec{J} \Rightarrow \vec{J}' - A\vec{J} = \vec{0}$, $\vec{J} = \vec{x}e^{\lambda t} \Rightarrow \vec{J}' = \lambda\vec{x}e^{\lambda t} = A\vec{x}e^{\lambda t}$

$\begin{vmatrix} -4-\lambda & +4 \\ -1.6 & 1.2-\lambda \end{vmatrix} = 0 \Rightarrow (-4-\lambda)(1.2-\lambda) + 6.4 = 0 \Rightarrow -4.8 + 4\lambda - 1.2\lambda + \lambda^2 + 6.4 = 0$

$\Rightarrow \lambda^2 + 2.8\lambda + 1.6 = 0 \Rightarrow (\lambda + 2)(\lambda + .8) = 0 \Rightarrow \boxed{\lambda_1 = -2,\ \lambda_2 = -.8}$

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7.

for $\lambda_1 = -2$: $\begin{bmatrix} -4+2 & +4 \\ -1.6 & 1.2+2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0} \Rightarrow -\cancel{2}x_1 = \overset{2}{\cancel{4}}x_2 \Rightarrow x_1 = +2x_2$

$\Rightarrow \vec{x}_1 = \begin{bmatrix} +2 \\ 1 \end{bmatrix}$;

for $\lambda_2 = -.8$: $\begin{bmatrix} -4+.8 & +4 \\ -1.6 & 1.2+.8 \end{bmatrix}\vec{x} = \vec{0} \Rightarrow -1.6x_1 = -2x_2 \Rightarrow -x_2 = -.8x_1$

$\Rightarrow \vec{x}_2 = \begin{bmatrix} 1 \\ +.8 \end{bmatrix}$

$\Rightarrow \vec{y}_h = c_1 \vec{x}_1 e^{-2t} + c_2 \vec{x}_2 e^{-.8t}$

ii) gen. soln. for particular soln.: $\vec{g}$ is const.

$A = \begin{bmatrix} -4 & +4 \\ -1.6 & 1.2 \end{bmatrix}$

$\vec{g} = \begin{bmatrix} 12 \\ +4.8 \end{bmatrix}$

$\vec{y}_p = \vec{a} \Rightarrow \vec{y}_p' = 0$ from $\vec{y}' = A\vec{y} + \vec{g}$

$\Rightarrow A\vec{a} + \vec{g} = 0$

$\Rightarrow -4a_1 + 4a_2 + 12 = 0$
$-1.6a_1 + 1.2a_2 + 4.8 = 0$
$\Rightarrow \left[\begin{array}{cc|c} -4 & +4 & -12 \\ -1.6 & 1.2 & -4.8 \end{array}\right] \downarrow \times(-.4) \Rightarrow \left[\begin{array}{cc|c} -4 & +4 & -12 \\ 0 & -.4 & 0 \end{array}\right] \div -4$

$\Rightarrow \left[\begin{array}{cc|c} +1 & -1 & 3 \\ & -.4 & 0 \end{array}\right] \Rightarrow \vec{a} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

$\vec{y} = \vec{y}_h + \vec{y}_p = c_1 \begin{bmatrix} +2 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ +.8 \end{bmatrix} e^{-.8t} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

$\Rightarrow I_1 = +2c_1 e^{-2t} + c_2 e^{-.8t} + 3$

$I_2 = c_1 e^{-2t} + .8 c_2 e^{-.8t}$

iii) initial conditions

$I_1(0) = 0 = +2c_1 + c_2 + 3$; $I_2(0) = 3 = c_1 + .8c_2 \Rightarrow c_1 = 3 - .8c_2$

$\Rightarrow +2(3 - .8c_2) + c_2 = -3 \Rightarrow \boxed{c_2 = -5 \Rightarrow c_1 = +1}$

$\Rightarrow \boxed{\begin{aligned} I_1 &= 2e^{-2t} - 5e^{-.8t} + 3 \\ I_2 &= 1e^{-2t} - 4e^{-.8t} \end{aligned}}$

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4.1 Solve by: a) convert to a system b) as given

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(12). $y''' + 2y'' - y' - 2y = 0$

a) $\Rightarrow y''' = 2y + y' - 2y''$ let $y_1 = y$,

1) $\Rightarrow y_2 = y' = y_1'$

2) $\Rightarrow y_3 = y_2' = y''$

3) $\Rightarrow y_3' = y_2'' = y''' = 2y_1 + y_2 - 2y_3$

also $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$\Rightarrow \vec{y}' = A\vec{y} = \begin{array}{c} 1) \\ 2) \\ 3) \end{array}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix}$$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & 1 & -2-\lambda \end{vmatrix} = 0 \Rightarrow -\lambda\left((-\lambda)(-2-\lambda) - 1\right) + 2 = 0$$

$$\Rightarrow -\lambda(2\lambda + \lambda^2 - 1) + 2 = 0 \Rightarrow -\lambda^3 - 2\lambda^2 + \lambda + 2 = 0$$

$$\Rightarrow (\lambda - 1)(\lambda^2 + 3\lambda + 2) = 0 \Rightarrow (\lambda-1)(\lambda+2)(\lambda+1) = 0 \Rightarrow \boxed{\lambda_1 = 1, \lambda_2 = -2, \lambda_3 = -1}$$

I) for $\lambda_1 = 1$: $\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 1 & -3 \end{bmatrix}\vec{x} = \vec{0} \Rightarrow \begin{matrix} x_1 = x_2 \\ x_2 = x_3 \end{matrix} \Rightarrow x_1 = x_2 = x_3$

$\Rightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

II) for $\lambda_2 = -2$: $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}\vec{x} = \vec{0} \Rightarrow \begin{matrix} 2x_1 = -x_2 \\ 2x_2 = -x_3 \end{matrix} \Rightarrow 2x_1 = \frac{1}{2}x_3 \Rightarrow x_3 = 4x_1$

$\Rightarrow x_2 = -\frac{1}{2}x_3 \Rightarrow x_2 = -2x_1$

$\Rightarrow \vec{x}_2 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$

III) for $\lambda_3 = -1$: $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix}\vec{x} = \vec{0} \Rightarrow \begin{matrix} x_1 = -x_2 \\ x_2 = -x_3 \end{matrix} \Rightarrow -x_2 = x_3$

$\Rightarrow \vec{x}_3 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

a) $\Rightarrow \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = c_1\vec{x}_1 e^{t} + c_2\vec{x}_2 e^{-2t} + c_3\vec{x}_3 e^{-t} =$

$= c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}e^{t} + c_2\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}e^{-2t} + c_3\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}e^{-t}$ gen. soln.

$\Rightarrow y_1 = c_1 e^{t} + c_2 e^{-2t} - c_3 e^{-t}$

$y_2 = c_1 e^{t} - 2c_2 e^{-2t} + c_3 e^{-t}$

$y_3 = c_1 e^{t} + 4c_2 e^{-2t} - c_3 e^{-t}$

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(12) 8) $y''' + 2y'' - y' - 2y = 0$

$\Rightarrow Y^3 + 2Y^2 - Y - 2 = 0 \Rightarrow (Y-1)(Y+2)(Y+1) = 0$

$\Rightarrow Y_1 = 1;\ Y_2 = -2;\ Y_3 = -1$

$\Rightarrow y(t) = c_1 e^{t} + c_2 e^{-2t} + c_3 e^{-t}$ gen. soln.

4.3

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(3). $y_1' = y_1 + 2y_2$ $\quad \vec{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$, $\vec{y}' = A\vec{y} \Rightarrow A = \begin{bmatrix} 1 & 2 \\ .5 & 1 \end{bmatrix}$

$y_2' = \frac{1}{2} y_1 + y_2$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ .5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} \Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ .5 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 - 1 = 0$

$\Rightarrow \cancel{1} - 2\lambda + \lambda^2 \cancel{-1} = 0 \Rightarrow \lambda(\lambda - 2) = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = 2$

$\lambda_1 = 0$: $\begin{bmatrix} 1 & 2 \\ .5 & 1 \end{bmatrix} \vec{x} = \vec{0} \Rightarrow x_1 = -2x_2 \Rightarrow \vec{x}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\lambda_2 = 2$: $\begin{bmatrix} -1 & 2 \\ .5 & -1 \end{bmatrix} \vec{x} = \vec{0} \Rightarrow -x_1 = -2x_2 \Rightarrow \vec{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\Rightarrow \vec{y} = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{0(t)} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} \Rightarrow \boxed{\begin{matrix} y_1 = -2c_1 + 2c_2 e^{2t} \\ y_2 = c_1 + c_2 e^{2t} \end{matrix}}$ gen. soln.

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(13). $y_1' = y_2$ $\quad y_1(0) = 0$ $\quad A\vec{y} = \vec{y}'$ $\quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \vec{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$

$y_2' = y_1$ $\quad y_2(0) = 2$

$\Rightarrow A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$

$\Rightarrow \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow (\lambda - 1)(\lambda + 1) = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = -1$

$\lambda_1 = 1$: $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \vec{x} = \vec{0} \Rightarrow x_1 = x_2 \Rightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -1$: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x} = \vec{0} \Rightarrow x_1 = -x_2 \Rightarrow \vec{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

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4.3

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(13). $\Rightarrow \vec{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$ $\quad y_1(0)=0$, $y_2(0)=2$

$\Rightarrow y_1 = c_1 e^{t} - c_2 e^{-t} \Rightarrow y_1(0) = 0 = c_1 - c_2 \Rightarrow c_1 = c_2$

$y_2 = c_1 e^{t} + c_2 e^{-t} \Rightarrow y_2(0) = 2 = c_1 + c_2 \Rightarrow \underline{c_1 = c_2 = 1}$

$$\boxed{\begin{aligned} \Rightarrow y_1 &= e^{t} - e^{-t} = 2\sinh t \\ y_2 &= e^{t} + e^{-t} = 2\cosh t \end{aligned}}$$

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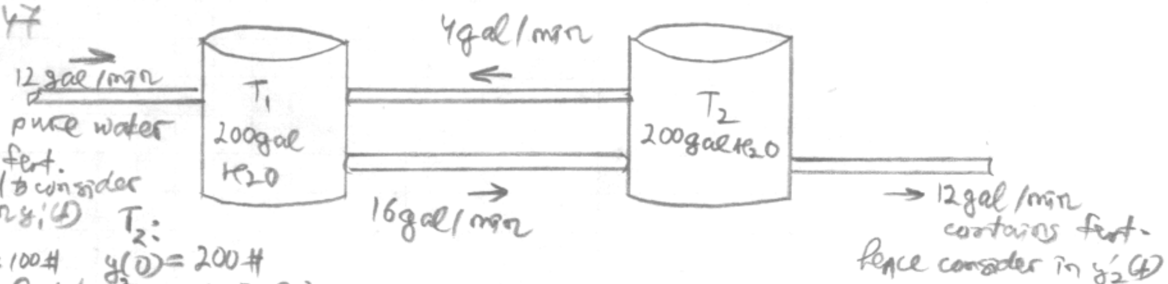
(18).



$\Rightarrow$ no fert.
no need to consider in $y_1(t)$

T_1: $y_1(0) = 100\#$ $\quad T_2$: $y_2(0) = 200\#$
fertilizer content @ $t=0$

$y'(t)$ = inflow − outflow rate

$y_1'(t) = +\frac{4}{200} y_2 - \frac{16}{200} y_1 = -.08 y_1 + .02 y_2$

$y_2'(t) = +\frac{16}{200} y_1 - \frac{4}{200} y_2 - \frac{12}{200} y_2 = .08 y_1 - .08 y_2$

$A\vec{y} = \vec{y}' \Rightarrow \begin{bmatrix} -.08 & .02 \\ .08 & -.08 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$

$\begin{vmatrix} -.08-\lambda & .02 \\ .08 & -.08-\lambda \end{vmatrix} = 0 \Rightarrow (-.08-\lambda)^2 - .0016 = 0 \Rightarrow .0064 + .16\lambda + \lambda^2 - .0016 = 0$
$\Rightarrow \lambda^2 + .16\lambda + .0048 = 0$

$\Rightarrow (\lambda + .12)(\lambda + .04) = 0 \quad \Rightarrow \underline{\lambda_1 = -.12,\ \lambda_2 = -.04}$

- for $\underline{\lambda_1 = -.12}$: $\begin{bmatrix} -.08+.12 & .02 \\ .08 & .04 \end{bmatrix} \vec{x} = \vec{0} \Rightarrow .08x_1 = -.04x_2 \quad / \div .04$
$\Rightarrow 2x_1 = -x_2 \Rightarrow \underline{-2x_1 = x_2}$

  $\Rightarrow \underline{\vec{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}}$

- for $\underline{\lambda_2 = -.04}$: $\begin{bmatrix} -.08+.04 & .02 \\ .08 & -.04 \end{bmatrix} \vec{x} = \vec{0} \Rightarrow .08x_1 = .04x_2$
$\Rightarrow 2x_1 = x_2$

  $\Rightarrow \underline{\vec{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

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18. $\Rightarrow \vec{y} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-.12t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-.04t}$ $\quad y_1(0) = 100$, $y_2(0) = 200$

$\Rightarrow y_1 = c_1 e^{-.12t} + c_2 e^{-.04t} \Rightarrow c_1 + c_2 = 100 \Rightarrow c_1 = 100 - c_2$

$y_2 = -2c_1 e^{-.12t} + 2c_2 e^{-.04t} \Rightarrow -\cancel{2}c_1 + \cancel{2}c_2 = \cancel{200}\ 100 \Rightarrow -100 + c_2 + c_2 = 100$

$\Rightarrow 2c_2 = 200 \Rightarrow \underline{c_2 = 100}$

$\Rightarrow \underline{c_1 = 0}$

$$\Rightarrow \boxed{\begin{aligned} y_1 &= 100e^{-.04t} \\ y_2 &= 200e^{-.04t} \end{aligned}}$$