$\boxed{8.4-1}$

*Typo in book? eigenvector of $\hat{A}$ not P

MA527
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HW 4

$$A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \qquad P = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$$

Verify that similar matrices have equal eigenvalues.

$$\hat{A} = P^{-1}AP \qquad P^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{3}{2} & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{3}{2} & 2 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{3}{2} & 2 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ -25 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} -25 & 12 \\ -50 & 25 \end{bmatrix}$$

$$\det(\hat{A} - \lambda I) = (-25-\lambda)(25-\lambda) + 600 = \lambda^2 - 625 + 600 = \lambda^2 - 25 = (\lambda-5)(\lambda+5)$$

$\lambda = 5, -5$

$$\det(A - \lambda I) = (3-\lambda)(-3-\lambda) - 16 = \lambda^2 - 9 - 16 = (\lambda-5)(\lambda+5)$$

$\lambda = 5, -5$

$P \qquad (-4-\lambda)(-1-\lambda) - 6 = 0 \qquad \lambda^2 + 5\lambda - 2 = 0 \qquad \lambda = -5.3722, 0.3723$

$\lambda = 0.3723 \quad \begin{bmatrix} -4.3723 & 2 \end{bmatrix} \quad x^{(1)} = \begin{bmatrix} 1 \\ 2.1861 \end{bmatrix} \qquad Px = \begin{bmatrix} 0.3722 \\ 0.8139 \end{bmatrix}$

$A \quad \lambda = 5 \quad \begin{bmatrix} -2 & 4 \end{bmatrix} \quad x^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow P^{-1}x = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \qquad \lambda = 5 \text{ in } \hat{A} \quad \begin{bmatrix} -30 & 12 \end{bmatrix} \Rightarrow y = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$\lambda = -5 \quad \begin{bmatrix} 8 & 4 \end{bmatrix} \quad x^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\boxed{8.4-9}$ ✓

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad (1-\lambda)(4-\lambda) - 4 = \lambda^2 - 5\lambda = \lambda(\lambda - 5)$$

$$\lambda = 0, 5$$

$$\lambda = 0 \quad \begin{bmatrix} 1 & 2 \end{bmatrix} \quad x^{(1)} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \qquad \lambda = 5 \quad \begin{bmatrix} -4 & 2 \end{bmatrix} \quad x^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$X = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \qquad X^{-1} = \frac{1}{-5}\begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}$$

$$D = X^{-1} A X = \begin{bmatrix} -\frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 0 & 5 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$$

$\boxed{8.4-10}$ ✓

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

$$(1-\lambda)(-1-\lambda) = 0 \qquad \lambda = 1, -1$$

$$\lambda = 1 \quad \begin{bmatrix} 0 & 0 \end{bmatrix} \qquad x^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -2 \end{bmatrix}$$

$$\lambda = -1 \quad \begin{bmatrix} 2 & 0 \end{bmatrix} \qquad x^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad X^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$D = X^{-1}AX = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\boxed{8.4\text{-}24}$

$Q(x) = x^T A x$

pos. def. if $Q(x) > 0$ for all $x \neq 0$ — λ all $\oplus$

neg. def " $<$ " — λ all $\ominus$

indefinite $Q(x) > 0$ and < 0 — λ $\oplus$ and $\ominus$

$Q = y^T D y = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2$

- If all eigenvalues of A are positive then Q is positive
- " negative " negative
- If A has both pos. and neg. eigenvalues then Q may be either positive or negative depending on value of x

$\boxed{8.4\text{-}25}$ ✓

Prob. 22: $4x_1^2 + 12x_1x_2 + 13x_2^2 = 16$

$$Q = x^T A x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

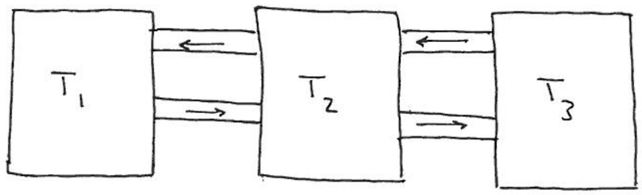
$$\begin{vmatrix} 4 & 6 \\ 6 & 13 \end{vmatrix} = 52 - 36 > 0$$ Positive definite

Prob. 23: $-11x_1^2 + 84x_1x_2 + 24x_2^2 = 156$

$$Q = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} -11 & 42 \\ 42 & 24 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{vmatrix} -11 & 42 \\ 42 & 24 \end{vmatrix} = -11(24) - 42^2 < 0$$ indefinite

4.1-5 ✓



let flow rate $\triangleq r$

tank volume $\triangleq V$

$$y_1' = \frac{r}{V} y_2 - \frac{r}{V} y_1$$

$$y_2' = \underbrace{-\frac{r}{V} y_2}_{\text{to tank 1}} \underbrace{- \frac{r}{V} y_2}_{\text{to tank 3}} + \frac{r}{V} y_1 + \frac{r}{V} y_3$$

$$y_3' = \frac{r}{V} y_2 - \frac{r}{V} y_3$$

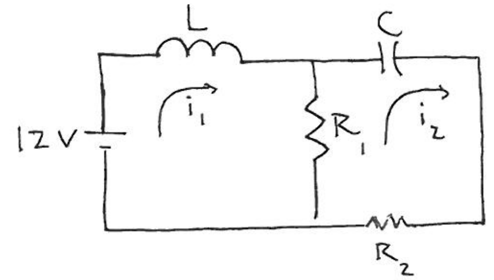
$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} -0.02 & 0.02 & 0 \\ 0.02 & -0.04 & 0.02 \\ 0 & 0.02 & -0.02 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

(symmetric matrix)

$\boxed{4.1\text{-}7}$ ✓

init. cond. $i_1(0) = 0$ A

$i_2(0) = -3$ A



$L = 1$ H

$C = 0.25$ F

$R_1 = 4\ \Omega$

$R_2 = 6\ \Omega$

Setup system of eqns.

1) $12 = Li' + R_1(i_1 - i_2) \implies i' = \frac{12}{L} - \frac{R_1}{L} i_1 + \frac{R_1}{L} i_2$

2) $0 = (i_2 - i_1)R_1 + \frac{1}{C}\int i_2\, dt + R_2 i_2$

take derivative of 2)

$$0 = (i_2' - i_1')R_1 + \frac{1}{C} i_2 + R_2 i_2'$$

sub. i'

$$i_2'(-R_1 - R_2) = \frac{1}{C} i_2 - \left(\frac{12}{L} - \frac{R_1}{L} i_1 + \frac{R_1}{L} i_2\right) R_1$$

$$i_2' = i_2\left(\frac{1}{C} - \frac{R_1^2}{L}\right)\left(\frac{-1}{R_1+R_2}\right) + i_1\left(\frac{R_1^2}{L}\right)\left(\frac{-1}{R_1+R_2}\right) - \frac{12R_1}{L}\left(\frac{-1}{R_1+R_2}\right)$$

sub. parameters and combine

$$\begin{bmatrix} i_1' \\ i_2' \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ -1.6 & 1.2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} 12 \\ 4.8 \end{bmatrix}$$

Agrees w/ page 133

$$J' = AJ + g$$

$\boxed{4.1\text{-}7}$ ✓ (p.2)

Nonhomogeneous. Solve homogeneous system first.

$J' = AJ$ Assume sln. of form $J = xe^{\lambda t}$ $J' = \lambda x e^{\lambda t} = Axe^{\lambda t}$

$$Axe^{\lambda t} = Ax\cancel{e^{\lambda t}} = \lambda x \cancel{e^{\lambda t}}$$

$$\begin{bmatrix} -4-\lambda & 4 \\ -1.6 & 1.2-\lambda \end{bmatrix} \qquad (-4-\lambda)(1.2-\lambda) + 6.4 = 0$$

(see ex. 1 of book)

$$\lambda = -2, -0.8 \qquad x^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad x^{(2)} = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix}$$

$$J_h = c_1 x^{(1)} e^{-2t} + c_2 x^{(2)} e^{-0.8t}$$

Find particular sln.

Try $J_p = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ $J_p' = 0$ $J_p' = AJ_p + g$

$0 = Aa + g$

$$\begin{array}{cc} A & -g \end{array}$$

$$\left[\begin{array}{cc:c} -4 & 4 & -12 \\ -1.6 & 1.2 & -4.8 \end{array}\right] \sim \begin{bmatrix} -1 & 1 & -3 \\ 0 & -0.4 & 0 \end{bmatrix} \qquad \begin{array}{l} a_1 = 3 \\ a_2 = 0 \end{array}$$

$$I_1 = 2c_1 e^{-2t} + c_2 e^{-0.8t} + 3$$

$$I_2 = c_1 e^{-2t} + 0.8 c_2 e^{-0.8t} + 0$$

Apply init. cond. $i_1(0) = 0$ $\quad 0 = 2c_1 + c_2 + 3$

$i_2(0) = -3$ $\quad -3 = c_1 + 0.8c_2 + 0$

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & 0.8 & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & -3 \\ 0 & 0.3 & -1.5 \end{bmatrix} \qquad \boxed{c_2 = -5 \qquad c_1 = 1}$$

$$\boxed{\begin{array}{l} I_1 = 2e^{-2t} + -5e^{-0.8t} + 3 \\ I_2 = e^{-2t} + -4e^{-0.8t} \end{array}}$$

$\boxed{4.1\text{-}12}$ ✓

$$y''' + 2y'' - y' - 2y = 0$$

a) Convert to a system

$$y^{(3)} = -2y^{(2)} + y^{(1)} + 2y$$

$$y_1 = y \qquad y_2 = y' \qquad y_3 = y''$$

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\det(A - \lambda I) = \cancel{(-\lambda)(-1)(-\lambda)(2\lambda-1)}$$

$$= (-\lambda)(-\lambda(-2-\lambda) - 1) - 1(-2)$$

$$= \lambda^2(-2-\lambda) + \lambda + 2 = -\lambda^3 - 2\lambda^2 + \lambda + 2 = \lambda^3 + 2\lambda^2 - \lambda - 2$$

$$= (\lambda - 1)(\lambda + 1)(\lambda + 2) = 0$$

$$\lambda = 1, -1, -2$$

$\lambda = 1$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ 2 & 0 & -2 \\ 2 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 1 & 0 & -1 \\ 2 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & -3 \\ 0 & 2 & -2 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad x^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\lambda = -1$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad x^{(2)} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\boxed{4.1\text{-}12}$ ✓

$\lambda = -2$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad x^{(3)} = \begin{bmatrix} \frac{1}{2} \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

Gen. sln.

$$y = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} e^{-2t}$$

b) $r^3 + 2r^2 - r - 2 = 0$

roots...

$(r-1)(r+1)(r+2) = 0$

$r = 1, -1, -2$

Gen. sln.

$$y = c_1 e^{t} + c_2 e^{-t} + c_3 e^{-2t}$$

$\boxed{4.3\text{-}3}$ ✓

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \overset{A}{\begin{bmatrix} 1 & 2 \\ \frac{1}{2} & 1 \end{bmatrix}} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\det(A - \lambda I) = (1-\lambda)^2 - 1 = \lambda^2 - 2\lambda = \lambda(\lambda - 2)$$

$$\lambda = 0 \quad [1 \quad 2] \quad x^{(1)} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\lambda = 2 \quad [-1 \quad 2] \quad x^{(2)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\boxed{\begin{aligned} y_1 &= c_1 2e^{2t} - c_2 2 \\ y_2 &= c_1 e^{2t} + c_2 \end{aligned}}$$

$\boxed{4.3\text{-}13}$ ✓

$$y_1' = y_2 \qquad y_1(0) = 0$$
$$y_2' = y_1 \qquad y_2(0) = 2$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \det(A - \lambda I) = (-\lambda)^2 - 1 = \lambda^2 - 1 = (\lambda+1)(\lambda-1)$$

$$\lambda = -1 \qquad [1 \quad 1] \qquad x^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda = 1 \qquad [-1 \quad 1] \qquad x^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y_1 = c_1 e^{-t} + c_2 e^{t}$$
$$y_2 = -c_1 e^{-t} + c_2 e^{t}$$

init. cond. $c_1 + c_2 = 0$

$-c_1 + c_2 = 2$

$c_1 = -1$

$c_2 = 1$

$$\boxed{\begin{aligned} y_1 &= -e^{-t} + e^{t} \\ y_2 &= e^{-t} + e^{t} \end{aligned}} \quad \begin{aligned} &= 2\sinh t \\ &= 2\cosh t \end{aligned}$$

$\boxed{4.3-18}$ ✓

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} \frac{-16}{200} & \frac{4}{200} \\ \frac{16}{200} & \frac{-16}{200} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

fresh water flowing in/out has no fertilizer, so not listed.

$$A = \begin{bmatrix} -0.08 & 0.02 \\ 0.08 & -0.08 \end{bmatrix}$$

$$\det(A - \lambda I) = (-0.08-\lambda)^2 - 0.0016$$

$$\lambda^2 + 0.16\lambda + 0.0048$$

$$(\lambda + 0.12)(\lambda + 0.04)$$

$\lambda = -0.12$ $\quad [0.04 \quad 0.02]$ $\quad x^{(1)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\lambda = -0.04$ $\quad [-0.04 \quad 0.02]$ $\quad x^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$y_1 = c_1 e^{-0.12t} + c_2 e^{-0.04t}$$

$$y_2 = -2c_1 e^{-0.12t} + 2c_2 e^{-0.04t}$$

Apply init. cond.

$y_1(0) = 100 \quad y_2(0) = 200$

$c_1 + c_2 = 100$

$-2c_1 + 2c_2 = 200$

$$\begin{bmatrix} 2 & 2 & 200 \\ -2 & 2 & 200 \end{bmatrix}$$

$c_2 = 100$

$c_1 = 0$

$$\boxed{\begin{aligned} y_1 &= 100e^{-0.04t} \\ y_2 &= 200e^{-0.04t} \end{aligned}}$$