24) $Q(x) = x^T A x$

$A = X D X^{-1}$

since X is orthonormal $X^{-1} = X^T$

$Q(x) = x^T X D X^T x$

let

$X^T x = y \quad \Rightarrow y^T = x^T X^{T^T} = x^T X$

$x = X y$

$Q(x) = y^T D X^T X y$

$= y^T D y$

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$= \sum_i \lambda_i y_i^2$

since $y_i^2 \geq 0$

$\therefore$ if $\lambda_i > 0$	,	positive definite
if $\lambda_i < 0$	,	Negative definite
if $\lambda_i < 0$ or $\lambda_i > 0$	,	indefinite

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$4x_1^2 + 12x_1x_2 + 13x_2^2 = 16$

$[x_1 \quad x_2]\begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 16$

$A = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix}$

$\lambda I - A = \begin{bmatrix} \lambda-4 & -6 \\ -6 & \lambda-13 \end{bmatrix}$

$\det = (\lambda-4)(\lambda-13) - 36$

$0 = \lambda^2 - 17\lambda + 16$

$0 = (\lambda-16)(\lambda-1)$

$\lambda = 1 \quad , \quad \lambda = 16$

$\therefore$ positive definite

$|4| = 4 > 0$

$\begin{vmatrix} 4 & 6 \\ 6 & 13 \end{vmatrix} = 16 > 0$

$\therefore$ prove

$-11x_1^2 + 84x_1x_2 + 24x_2^2 = 156$

$[x_1 \quad x_2]\begin{bmatrix} -11 & 42 \\ 42 & 24 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 156$

let $B = \begin{bmatrix} -11 & 42 \\ 42 & 24 \end{bmatrix}$

$[\lambda I - B] = \begin{bmatrix} \lambda+11 & -42 \\ -42 & \lambda-24 \end{bmatrix}$

$\det = (\lambda+11)(\lambda-24)-(42)(42)$

$= \lambda^2 - 13\lambda - 2028$

$= (\lambda-52)(\lambda+39)$

$\lambda = 52$ or $\lambda = -39$

$\therefore$ indefinite

$|-11| = -11 < 0$

$\begin{vmatrix} -11 & 42 \\ 42 & 24 \end{vmatrix} = -2028 < 0$

$\therefore$ indefinite (prove)

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12) $y''' + 2y'' - y' - 2y = 0$

let

$y_1 = y$

$y_1' = y' = y_2$

$y_2' = y'' = y_3$

$y_3' = y''' = -2y'' + y' + 2y$

$= -2y_3 + y_2 + 2y_1$

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}$

$[\lambda I - A] = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -2 & -1 & \lambda+2 \end{bmatrix}$

$\det = \lambda[(\lambda)(\lambda+2) - 1] + 1[-2]$

$= \lambda(\lambda^2 + 2\lambda - 1) - 2$

$= \lambda^3 + 2\lambda^2 - \lambda - 2$

$= (\lambda + 1)(\lambda^2 + \lambda - 2)$

$= (\lambda+1)(\lambda+2)(\lambda-1)$

$\lambda = -1, \lambda = 1, \lambda = -2$

when $\lambda = -1$

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ -2 & -1 & 1 \end{bmatrix}$$

$-r_1 \to r_1$, $-r_2 \to r_2$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & -1 & 1 \end{bmatrix}$$

$2r_1 + r_3 \to r_3$
$r_1 - r_2 \to r_1$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$r_2 - r_3 \to r_3$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

when $\lambda = -2$

$$\begin{bmatrix} -2 & -1 & 0 \\ 0 & -2 & -1 \\ -2 & -1 & 0 \end{bmatrix}$$

$r_1 - r_3 \to r_3$
$-\frac{1}{2} r_2 \to r_2$

$$\begin{bmatrix} -2 & -1 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$-1/2\, r_1 \to r_1$

$$\begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x}_2 = \begin{bmatrix} 1/4 \\ -1/2 \\ 1 \end{bmatrix}$$

when $\lambda = 1$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -2 & -1 & 3 \end{bmatrix}$$

$2r_1 + r_3 \to r_3$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -3 & 3 \end{bmatrix}$$

$3r_2 + r_3 \to r_3$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

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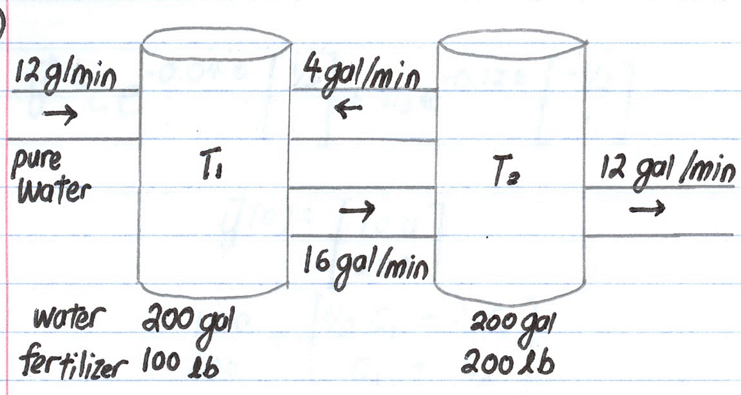
$$\boxed{\vec{y} = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1/4 \\ -1/2 \\ 1 \end{bmatrix} + c_3 e^{t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

18)



$$y_1' = 12\cdot 0 + \frac{4}{200}y_2 - \frac{16}{200}y_1 = 0.02y_2 - 0.08y_1$$

$$y_2' = \frac{16}{200}y_1 - \frac{4}{200}y_2 - \frac{12}{200}y_2 = 0.08y_1 - 0.08y_2$$

$$\begin{matrix} y_1' \\ y_2' \end{matrix} = \begin{bmatrix} -0.08 & 0.02 \\ 0.08 & -0.08 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda + 0.08 & -0.02 \\ -0.08 & \lambda + 0.08 \end{bmatrix}$$

$$\det = (\lambda + 0.08)^2 - 0.0016$$

$$0 = \lambda^2 + 0.16\lambda + 0.0048$$

$$\lambda = \frac{-0.16 \pm \sqrt{0.0064}}{2}$$

$$= \frac{-0.16 \pm 0.08}{2}$$

$\lambda_1 = -0.04$

$$\begin{bmatrix} 0.04 & -0.02 \\ -0.08 & 0.04 \end{bmatrix}$$

$2r_1 + r_2 \to r_2$

$$\begin{bmatrix} 0.04 & -0.02 \\ 0 & 0 \end{bmatrix}$$

$\frac{1}{0.04} r_1 \to r_1$

$$\begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$$

$$\vec{x}_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

$\lambda_2 = -0.12$

$$\begin{bmatrix} -0.04 & -0.02 \\ -0.08 & -0.04 \end{bmatrix}$$

$2r_1 - r_2 \to r_2$

$$\begin{bmatrix} -0.04 & -0.02 \\ 0 & 0 \end{bmatrix}$$

$-\frac{1}{0.04} r_1$

$$\begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}$$

$$\vec{x}_2 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

$$\vec{y} = c_1 e^{-0.04t} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} + c_2 e^{-0.12t} \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

$$\vec{y}(0) = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$$

$$\begin{matrix} 100 \\ 200 \end{matrix} = \begin{bmatrix} 1/2\, c_1 - 1/2\, c_2 \\ c_1 + c_2 \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & -1/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ -1 & 1/2 \end{bmatrix} \begin{bmatrix} 100 \\ 200 \end{bmatrix}$$

$$= \begin{bmatrix} 200 \\ 0 \end{bmatrix}$$

$$\vec{y} = 200\, e^{-0.04t} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

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$$\boxed{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = e^{-0.04} \begin{bmatrix} 100 \\ 200 \end{bmatrix}}$$